On an estimate for the curvature of minimal surfaces $z = z(x, y)^$*

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Let the function $z = z(x, y)$ be of class C^2 and a solution of the minimal surface equation in the disc $x^2 + y^2 < R^2$. Introduce the isothermic parameters

$$(1) \quad \begin{aligned} \xi &= x + \int_{(0,0)}^{(x,y)} W^{-1}[(1 + p^2) dx + pq dy] \equiv x + A(x, y), \\ \eta &= y + \int_{(0,0)}^{(x,y)} W^{-1}[pq dx + (1 + q^2) dy] \equiv y + B(x, y). \end{aligned}$$

Here $W = (1 + p^2 + q^2)^{1/2}$. Put $\xi + i\eta = \zeta = \zeta(x, y)$ and $\rho = \min |\zeta|$ for $x^2 + y^2 = R_\epsilon^2$ ($R_\epsilon = R - \epsilon$). Without loss of generality for the following it may be assumed that $|\zeta(R_\epsilon, 0)| = \rho$.¹ It has been shown in [6] that the mapping $x, y \rightarrow \zeta$ enlarges distances. Hence $\rho > R_\epsilon$. The function

$$(2) \quad F(\zeta) = \frac{p - iq}{1 + W} = \frac{d}{d\zeta} \left[z(x, y) + i \int_{(0,0)}^{(x,y)} W^{-1}(p dy - q dx) \right]$$

is a regular-analytic function of ζ ,² and $|F(\zeta)| < 1$. For the Gaussian curvature K one finds ([6], p. 418)

$$(3) \quad K = -|F'|^2 \left(1 + \frac{1}{W} \right)^4.$$

The application, as in [4], of PICK's lemma (see [2], p. 14) yields

$$(4) \quad |F'(0)| \leq \frac{1}{\rho} (1 - |F(0)|^2) = \frac{2}{\rho(1 + W_0)}.$$

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¹ A rotation in the x, y -plane results in the same rotation in the ζ -plane which does not change distances.

² See for instance L. BERS [1], p. 367; [6].

In order to obtain an estimate for ρ observe that the function $G(\zeta) = A(x, y) + iy$ is regular-analytic and univalent in $|\zeta| \leq \rho$ (see [8]; [9], II.15³). Therefore, on $|\zeta| = \rho$,

$$(5) \quad |G(\zeta)| \geq \frac{\rho}{4} |G'(0)| = \frac{\rho}{4} \frac{\sqrt{1+p_0^2}}{1+W_0} \geq \frac{\rho}{4(1+W_0)}$$

(see [2], p. 63) and, in particular, since $A(R_\epsilon, 0) > 0$,

$$(6) \quad |G(\zeta(R_\epsilon, 0))| = A(R_\epsilon, 0) \geq \frac{\rho}{4(1+W_0)}.$$

But

$$(7) \quad \rho = |\zeta(R_\epsilon, 0)| \geq \xi(R_\epsilon, 0) = R_\epsilon + A(R_\epsilon, 0) \geq R_\epsilon + \frac{\rho}{4(1+W_0)}.$$

Hence

$$(8) \quad \rho \geq \frac{R_\epsilon}{1 - \frac{1}{4(1+W_0)}}.$$

Substituting (4) and (8) into (3) one obtains

$$(9) \quad |K_0| \leq \frac{4}{(1+W_0)^2} \frac{1}{R_\epsilon^2} \left(1 - \frac{1}{4(1+W_0)}\right)^2 \left(1 + \frac{1}{W_0}\right)^4 \\ = \frac{4}{W_0^2 R_\epsilon^2} \left(1 + \frac{3}{4W_0}\right)^2 \leq \frac{49}{4} \cdot \frac{1}{W_0^2 R_\epsilon^2},$$

and, letting ϵ tend to zero,

$$(10) \quad |K_0| \leq \frac{49}{4} \cdot \frac{1}{W_0^2 R^2} \leq \frac{49}{4} \cdot \frac{1}{R^2}.$$

This inequality includes the inequality originally proved by E. HEINZ [3] and is similar to the one derived by E. HOPF [5]. The constant in HOPF's inequality is⁴ 16. The best value of the constant is not known.

LITERATURE

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³ At these places T. RADÓ considers the other function $x + iB(x, y)$.

⁴ Using the better value $\mu = \frac{1}{2}$ for HEINZ's constant which is now known, see [7].

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