*Solution of a Time-Dependent, One-Dimensional Neutron Transport Problem**

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1. The fundamental equation. In [1] the following simple model of neutron transport is considered: Let there be a rod of length x made of fissionable material. A neutron moving through the rod at constant speed c has a probability $\lambda^{-1}y + o(y)$ of causing a fission in a distance y . A fission results in the disappearance of the original neutron and the emergence of two similar neutrons, one going to the left, the other to the right. We ask: If a neutron is introduced into one end of a rod of length x at time $t = 0$, what is the expected number $U(x, t)$ of neutrons emerging from that end in the time interval $[0, t]$?

The equation for $U(x, t)$ is derived in [1]. We find

$$(1.1a) \quad \frac{\partial U}{\partial x} + \frac{2}{c} \frac{\partial U}{\partial t} = \lambda^{-1} \int_0^t u(x, z) U(x, t - z) dz + \lambda^{-1}, \quad x \geq 0, t \geq 0,$$

where

$$(1.1b) \quad u(x, t) = \frac{\partial U}{\partial t}(x, t).$$

The function $U(x, t)$ must satisfy the conditions

$$(1.1c) \quad U(x, 0) = 0, \quad U(0, t) = 0.$$

It is convenient to change both time and distance scales by the transformation

$$(1.2) \quad x' = \lambda^{-1}x, \quad t' = c\lambda^{-1}t.$$

Then, on suppressing the primes and using the same functional notation, we get

$$(1.3) \quad \frac{\partial U}{\partial x} + 2 \frac{\partial U}{\partial t} = 1 + \int_0^t u(x, z) U(x, t - z) dz, \quad U(x, 0) = 0, \quad U(0, t) = 0.$$

We shall further assume that $U(x, t)$ is sufficiently well behaved to differentiate (1.3) to obtain

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$$(1.4a) \quad \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial t} = \int_0^t u(x, z) u(x, t - z) dz, \quad x \geq 0, \quad t \geq 0.$$

(This may be regarded as a physical assumption.) From (1.3) we find

$$(1.4b) \quad u(0, t) = 0, \quad u(x, 0) = \frac{1}{2}.$$

The solution of the system (1.4) will now be investigated in detail.

2. The integral equation. If we set

$$(2.1) \quad x = s, \quad t = 2s + \tau, \quad \tau > 0, \quad s \geq 0,$$

we may write (1.4a) in the form

$$(2.2) \quad \frac{d}{ds} u(s, 2s + \tau) = \int_0^{2s + \tau} u(s, z) u(s, 2s + \tau - z) dz.$$

Hence

$$(2.3) \quad u(s, 2s + \tau) = \int_0^s ds' \int_0^{2s' + \tau} u(s', z) u(s', 2s' + \tau - z) dz + c$$

or

$$(2.4) \quad u(x, t) = \int_0^x ds' \int_0^{2s' + t - 2x} u(s', z) u(s', 2s' + t - 2x - z) dz,$$

the constant having been chosen so that $u(0, t) = 0$. This equation holds in the region $t > 2x$.

Similar considerations yield an integral equation for $t < 2x$, namely

$$(2.5) \quad u(x, t) = \frac{1}{2} + \int_0^{\frac{1}{2}t} ds' \int_0^{2s'} u(s' + x - \frac{1}{2}t, z) u(s' + x - \frac{1}{2}t, 2s' - w) dw.$$

This is readily transformed into

$$(2.6) \quad u(x, t) = \frac{1}{2} + \int_{x - \frac{1}{2}t}^x ds'' \int_0^{2(s'' + \frac{1}{2}t - x)} u(s'', w) u(s'', 2[s'' + \frac{1}{2}t - x] - w) dw.$$

If we now define $u(x, t) = 0$ for $x < 0, t < 0$ then (2.4) and (2.6) may be combined to read

$$(2.7) \quad u(x, t) = H(\tfrac{1}{2}t - x) + \int_0^x ds \int_0^{2s + t - 2x} u(s, z) u(s, 2s + t - 2x - z) dz,$$

$$H(y) = \begin{cases} \frac{1}{2}, & y < 0 \\ 0, & y > 0. \end{cases}$$

We fix our attention at present on (2.7) and propose to study its solution.

3. The solution of (2.7). We now introduce an iteration procedure. Let

$$u_0(x, t) \equiv 0,$$

$$(3.1) \quad u_{n+1}(x, t) = H(\tfrac{1}{2}t - x) + \int_0^x ds \int_0^{2s+t-2x} u_n(s, z) u_n(s, 2s+t-2x-z) dz$$

for $x > 0, \quad t > 0;$

$$u_n(x, t) = 0 \quad \text{for } x < 0, \quad t < 0, \quad n = 0, 1, 2, \dots$$

We shall prove that $\lim_{n \rightarrow \infty} u_n(x, t) = u(x, t)$ exists and is, indeed, a solution of (2.7).

First, it is necessary to majorize the functions $u_n(x, t)$. We conjecture that there exists a positive increasing function $f(t)$ such that

$$(3.2) \quad |u_n(x, t)| \leq f(t) \quad \text{for all } x, t \text{ and all } n.$$

We arrive at $f(t)$ in the following manner. If $f(t)$ exists, then from (3.1) and (3.2) we get

$$(3.3) \quad |u_n(x, t)| \leq \frac{1}{2} + \int_0^x ds \int_0^{2s+t-2x} f(z) f(2s+t-2x-z) dz.$$

If

$$(3.4) \quad \begin{aligned} f'(w) &= \frac{1}{2} \int_0^w f(z) f(w-z) dz, \quad w > 0, \\ f(w) &= \frac{1}{2} \quad \text{for } w \leq 0, \end{aligned}$$

then, from (3.3),

$$(3.5) \quad |u_n(x, t)| \leq \frac{1}{2} + 2 \int_0^x f'(2s+t-2x) ds = \frac{1}{2} + f(t) - f(t-2x) \leq f(t),$$

the last inequality holding since $f(t)$ is supposedly increasing.

Clearly, we require the solution of (3.4). It is readily obtained by Laplace transform methods and one finds

$$(3.6) \quad f(t) = t^{-1} I_1(t), \quad t \geq 0,$$

where $I_1(t)$ is the Hankel function of the first kind. (This may be recognized as a result of KAPTEYN (see [2, p. 380]).)

The function $t^{-1} I_1(t)$ is, indeed, positive and increasing. Hence $f(t)$ as given by (3.6) has the properties required. We are thus led to the following lemma:

Lemma 3.1. *The functions $u_n(x, t)$ defined by (3.1) satisfy*

$$(3.7) \quad |u_n(x, t)| \leq t^{-1} I_1(t) \quad \text{for all } t \geq 0 \quad \text{and } n = 0, 1, 2, \dots$$

The next result is proved trivially and is used frequently in what follows.

Lemma 3.2. *For any two integrable functions h and g we have the relation*

$$(3.8) \quad \begin{aligned} \int_0^t \{h(w)h(t-w) - g(w)g(t-w)\} dw \\ = \int_0^t \{h(w) - g(w)\} \{h(t-w) + g(t-w)\} dw. \end{aligned}$$

We omit the proof.

The convergence of the sequence $u_n(x, t)$ can now be investigated.

Lemma 3.3. *Let $X > 0$ and $T > 0$ be fixed but arbitrary. Then for x and t such that $0 \leq x \leq X$, $0 \leq t \leq T$, there exists a constant A (dependent upon X and T) such that*

$$(3.9) \quad |u_n(x, t) - u_{n-1}(x, t)| \leq \frac{A^{n-1}(xt)^{n-1}}{[(n-1)!]^2}, \quad n = 1, 2, 3, \dots$$

Proof. Consider the case $n = 1$. Since $u_0(x, t) \equiv 0$, we have, from (3.1),

$$|u_1(x, t) - u_0(x, t)| \leq \frac{1}{2} < 1.$$

Now suppose (3.9) holds up to n . We prove that it holds for $n + 1$. Clearly, from (3.1) and Lemma 3.2,

$$(3.10) \quad \begin{aligned} u_{n+1}(x, t) - u_n(x, t) &= \int_0^x ds \int_0^{2s+t-2x} [u_n(s, z)u_n(s, 2s+t-2x-z) \\ &\quad - u_{n-1}(s, z)u_{n-1}(s, 2s+t-2x-z)] dz \\ &= \int_0^x ds \int_0^{2s+t-2x} \{u_n(s, z) - u_{n-1}(s, z)\} \{u_n(s, 2s+t-2x-z) \\ &\quad + u_{n-1}(s, 2s+t-2x-z)\} dz. \end{aligned}$$

By Lemma 3.1 we have

$$(3.11) \quad |u_n(s, 2s+t-2x-z) + u_{n-1}(s, 2s+t-2x-z)| \leq A, \\ 0 \leq s \leq x \leq X, \quad 0 \leq t \leq T.$$

Using (3.11) and (3.9) we get from (3.10)

$$(3.12) \quad \begin{aligned} |u_{n+1}(x, t) - u_n(x, t)| &\leq \frac{A^n}{[(n-1)!]^2} \int_0^x ds \int_0^{2s+t-2x} (sz)^{n-1} dz \\ &\leq \frac{A^n}{[(n-1)!]^2} \int_0^x ds \int_0^t (sz)^{n-1} dz = \frac{A^n}{(n!)^2} (xt)^n. \end{aligned}$$

This completes the induction and the lemma is proved.

Thus

$$(3.13) \quad |u_{n+m}(x, t) - u_n(x, t)| \leq \sum_{k=n}^{n+m-1} \frac{A^k (XT)^k}{(k!)^2}, \quad 0 \leq x \leq X, \quad 0 \leq t \leq T,$$

and so $\lim_{n \rightarrow \infty} u_n(x, t) = u(x, t)$ exists *uniformly* for $0 \leq x \leq X$, $0 \leq t \leq T$. Since the functions $u_n(x, t)$ are continuous for $t < 2x$ and for $t > 2x$ it follows that $u(x, t)$ is continuous in these regions. We therefore have the following theorem:

Theorem I. *The integral equation (2.7) possesses a solution $u(x, t)$ continuous for $0 \leq 2x < t$ and for $0 \leq t < 2x$. Further, $|u(x, t)| \leq t^{-1}I_1(t)$ for all t .*

4. The solution of (1.4). It will now be established that the function $u(x, t)$ of Theorem I provides a unique solution of the system (1.4).

If we differentiate (3.1) with respect to t we get

$$(4.1) \quad \frac{\partial u_{n+1}}{\partial t} = \frac{1}{2} \int_0^x u_n(s, 2s + t - 2x) ds \\ + \int_0^x ds \int_0^{2s+t-2x} u_n(s, z) u_n^{(2)}(s, 2s + t - 2x - z) dz,$$

where

$$u_n^{(2)}(x, t) = \frac{\partial u_n}{\partial t}(x, t).$$

From this it is readily proved by induction that for $0 \leq x \leq X$, $0 \leq t \leq T$,

$$(4.2) \quad |u_{n+1}^{(2)}(x, t) - u_n^{(2)}(x, t)| < \frac{B^n t^n}{n!},$$

where B is a constant dependent upon X and T . Hence $\{u_n^{(2)}(x, t)\}$ converges uniformly to $\partial u / \partial t$.

Finally, since

$$(4.3) \quad \frac{\partial u_n}{\partial t} + 2 \frac{\partial u_n}{\partial x} = \int_0^t u_{n-1}(x, z) u_{n-1}(x, t - z) dz,$$

we see that $\lim_{n \rightarrow \infty} \partial u_n / \partial x = \partial u / \partial x$. Also, it is clear that $u_n(x, 0) = \frac{1}{2}$, $u_n(0, t) = 0$ for $n \geq 1$ so that $u(x, t)$ satisfies (1.4b).

To establish the uniqueness of the solution $u(x, t)$ of (1.4) suppose $v(x, t)$ is also a solution, and that $v(x, t)$ is bounded in $0 \leq x \leq X$, $0 \leq t \leq T$. Then $v(x, t)$ also satisfies (2.7) and

$$(4.4) \quad |u_{n+1}(x, t) - v(x, t)| \leq \int_0^x ds \int_0^{2s+t-2x} |u_n(s, z) - v(s, z)| \\ \cdot |u_n(s, 2s + t - 2x - z) + v(s, 2s + t - 2x - z)| dz.$$

Again it may be shown by induction that

$$(4.5) \quad |u_{n+1}(x, t) - v(x, t)| \leq \frac{C^n (xt)^n}{(n!)^2}$$

and so $\lim_{n \rightarrow \infty} u_n(x, t) = u(x, t) = v(x, t)$.

We summarize our results in a theorem.

Theorem II. *The only solution of (1.4) which is bounded in any finite rectangle in the (x, t) plane is the function $u(x, t)$ of Theorem I.*

5. The behavior of $U(x, t) = \int_0^t u(x, z) dz$ for large t . It is of considerable physical interest to consider the asymptotic behavior of

$$(5.1) \quad U(x, t) = \int_0^t u(x, z) dz$$

for large t , since that behavior gives us a measure of the critical condition of the rod (*cf.* [3]). We ask here for information regarding U rather than u partly because the analysis for u is somewhat more delicate, but mainly because the physical interest in U is greater. The device used in the investigation is the Laplace transform. To avoid the necessity of validating various changes of order of integration and differentiation we work with (2.7) instead of (1.4). The known properties of $u(x, t)$ justify all of the operations in what follows.

Let

$$(5.2) \quad F(x, p) = \int_0^\infty e^{-pt} u(x, t) dt.$$

This integral converges for $R(p) > 1$, since $|u(x, t)| \leq t^{-1} I_1(t) = O(e^t)$ for large t . Taking the transform of (2.7) and recalling the convolution theorem ([4, p. 91]) we get

$$(5.3) \quad F(x, p) = \frac{e^{-2px} - 1}{-2p} + e^{-2px} \int_0^x e^{2ps} F^2(s, p) ds.$$

This form makes it clear that $F(x, p)$ is continuous and differentiable with respect to x . We find, after some manipulation,

$$(5.4) \quad \frac{dF}{dx} = F^2 - 2pF + 1, \quad F(0, p) = 0.$$

The system (5.4) is readily solved:

$$(5.5) \quad F(x, p) = \frac{\tan(x\sqrt{1-p^2})}{\sqrt{1-p^2} + p \tan(x\sqrt{1-p^2})}.$$

We recall that the Laplace transform of the function $U(x, t)$ of interest is given by

$$(5.6) \quad L\{U(x, t)\} = \frac{F(x, p)}{p}.$$

In view of the known properties of $U(x, t)$ and $u(x, t)$ we may write ([4, p. 69])

$$(5.7) \quad U(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{pt} \sin(x\sqrt{1-p^2}) dp}{p\{\sqrt{1-p^2} \cos(x\sqrt{1-p^2}) + p \sin(x\sqrt{1-p^2})\}}, \quad c > 1.$$

To determine the asymptotic behavior of $U(x, t)$ we must locate the singularities of the integrand in (5.7) and then show that the line of integration may be shifted to the left of certain of these singularities. We shall now prove several lemmas concerning properties of this integrand. For convenience, we write

$$(5.8) \quad I(p; x, t) = \frac{e^{pt} \sin(x\sqrt{1-p^2})}{p\{\sqrt{1-p^2} \cos(x\sqrt{1-p^2}) + p \sin(x\sqrt{1-p^2})\}},$$

$$(5.9) \quad K(p; x) = \sqrt{1-p^2} \cos(x\sqrt{1-p^2}) + p \sin(x\sqrt{1-p^2}).$$

Despite the presence of the term $\sqrt{1-p^2}$, $I(p; x, t)$ is a meromorphic function of p with poles at $p = 0$ and at the zeros of $K(p; x)$.

Lemma 5.1. *The only zeros with positive real parts of $K(p; x)$ are real, and none exceeds 1 in value.*

Proof. If $K = 0$ then

$$(5.10) \quad \sin(x\sqrt{1-p^2}) = \pm\sqrt{1-p^2}, \quad \cos(x\sqrt{1-p^2}) = \mp p.$$

Write $\sqrt{1-p^2} = \beta + i\gamma$, $p = \sigma + i\tau$. Then (5.10) becomes

$$(5.11) \quad \begin{aligned} (a) \quad & \sin x\beta \cosh x\gamma = \pm\beta, \\ (b) \quad & \cos x\beta \sinh x\gamma = \pm\gamma, \\ (c) \quad & \cos x\beta \cosh x\gamma = \mp\sigma, \\ (d) \quad & \sin x\beta \sinh x\gamma = \pm\tau. \end{aligned}$$

Dividing (5.11b) by (5.11c) we get

$$(5.12) \quad \frac{\tanh x\gamma}{\gamma} = -\frac{1}{\sigma}.$$

For $x > 0$ and $\gamma \neq 0$ the left side is positive. Therefore $\sigma < 0$. Hence, for $\tau \neq 0$ and $\sigma > 0$ or for $\tau = 0$ and $\sigma > 1$, we cannot have $K = 0$.

Lemma 5.2. *If $x > \frac{1}{2}\pi$ there are a finite number of real positive roots of K , $0 < p_1 < p_2 < \dots < p_n < 1$ (dependent upon x in position and number). If $x \leq \frac{1}{2}\pi$, K has no such roots. In both cases $p = 1$ is a root.*

Proof. For $0 < p < 1$ write $\sqrt{1-p^2} = \beta$. Then $K = 0$ is equivalent to

$$(5.13) \quad \tan x\beta = \frac{-\beta}{\sqrt{1-\beta^2}}.$$

A graph readily shows that (5.13) has solutions β if $x > \frac{1}{2}\pi$, none if $x \leq \frac{1}{2}\pi$.

Direct substitution in K verifies that $p = 1$ is a root.

Lemma 5.3. *There are no roots of K of the form $p = i\tau$, $\tau \neq 0$, and $p = 0$ is a root only if $x = \frac{1}{2}(2n+1)\pi$, $n = 0, 1, 2, \dots$.*

Proof. Consider the equations (5.11) and set $\sigma = 0$, $\gamma = 0$. These equations reduce to

$$(5.14) \quad \sin(x\beta) = \pm\beta, \quad \cos(x\beta) = 0.$$

These clearly imply $\beta = \pm 1$, $p = 0$, $x = \frac{1}{2}(2n+1)\pi$.

It is now possible to determine the residues at the poles of $I(p; x, t)$ thus far discovered. It should be noted that while $p = 1$ is a zero of K , it is a regular point of I .

Case I. $x > \frac{1}{2}\pi$, $p = p_i$.

Residue of I (at $p = p_i$) = $\text{Res } I|_{p=p_i}$

$$(5.15) \quad = \lim_{p \rightarrow p_i} \frac{(p - p_i)e^{p_i t}}{p\{\sqrt{1 - p^2} \cot(x\sqrt{1 - p^2}) + p\}} = \frac{e^{p_i t}}{p_i} \frac{1 - p_i^2}{1 + xp_i}.$$

Case II. $x \neq \frac{1}{2}(2n + 1)\pi$, $n = 0, 1, 2, \dots$; $p = 0$.

$$(5.16) \quad \text{Res } I|_{p=0} = \lim_{p \rightarrow 0} \frac{e^{p t}}{\sqrt{1 - p^2} \cot(x\sqrt{1 - p^2}) + p} = \tan x.$$

Case III. $x = \frac{1}{2}(2n + 1)\pi$, $n = 0, 1, 2, \dots$; $p = 0$.

The result of Lemma 5.3 shows that $p = 0$ is a pole of I of order greater than one for these exceptional values of x . If we expand I by Laurent's Theorem about $p = 0$ we find, after considerable calculation,

$$I = \frac{1}{p^2} + \frac{t - \frac{1}{2}(n + \frac{1}{2})\pi}{p} + \dots$$

Thus

$$(5.17) \quad \text{Res } I|_{p=0} = t - \frac{1}{2}(n + \frac{1}{2})\pi.$$

We require two more lemmas.

Lemma 5.4. *There exists for each $x > 0$ a number $\epsilon(x) > 0$ such that, for $R(p) \geq -\epsilon(x)$, $I(p; x, t)$ has no singularities other than those already discussed in Lemmas 5.1, 5.2, and 5.3.*

Proof. It is clear that it is only necessary to show that the zeros of K cannot accumulate at $\pm i\infty$. Assume that there is a sequence of zeros of K , $p_n^* = \sigma_n + i\tau_n$, $\sigma_n < 0$, $\lim_{n \rightarrow \infty} \sigma_n = 0$, $\lim_{n \rightarrow \infty} \tau_n = +\infty$ (or $-\infty$). Consider (5.11a) at these points:

$$(5.18) \quad \frac{\sin x\beta_n}{\beta_n} = \pm \frac{1}{\cosh x\gamma_n}.$$

Since

$$\begin{aligned} \beta_n + i\gamma_n &= \sqrt{1 - (p_n^*)^2} = \pm ip_n^* \left\{ 1 + O\left(\frac{1}{|p_n^*|^2}\right) \right\} \\ &= \pm i(\sigma_n + i\tau_n) \left\{ 1 + O\left(\frac{1}{|p_n^*|^2}\right) \right\}, \end{aligned}$$

it is clear that $\lim_{n \rightarrow \infty} \gamma_n = 0$, $\lim_{n \rightarrow \infty} |\beta_n| = +\infty$. Hence (5.18) cannot hold.

Lemma 5.5. *There exist positive constants R and M , dependent on x , such that for $|p| > R$, $-\epsilon(x) \leq R(p) \leq 2$,*

$$|I(p; x, t)| \leq \frac{M |e^{p t}|}{|p|^2}.$$

Proof. Write

$$(5.19) \quad I(p; x, t) = \frac{e^{pt}}{p\sqrt{1-p^2} \left\{ \cot(x\sqrt{1-p^2}) + \frac{p}{\sqrt{1-p^2}} \right\}}.$$

The bracketed part of the denominator may be written

$$(5.20) \quad \begin{aligned} D &= i \left(\frac{e^{2ix\sqrt{1-p^2}} + 1}{e^{2ix\sqrt{1-p^2}} - 1} \right) + \frac{1}{i\sqrt{1-\frac{1}{p^2}}} \\ &= \frac{2i}{e^{2ix\sqrt{1-p^2}} - 1} + O\left(\frac{1}{|p^2|}\right). \end{aligned}$$

Now

$$|e^{2ix\sqrt{1-p^2}} - 1| \leq e^{2xR(\sqrt{p^2-1})} + 1 \leq m \quad \text{for} \quad -\epsilon(x) \leq R(p) \leq 2.$$

Hence $|D| > 3/m$ for large enough $|p|$. The result is now immediate, and we are in a position to prove the final theorem.

Theorem III. *The function $U(x, t)$ may be written in one of the following forms:*

$$(a) \quad \text{For } x > \frac{1}{2}\pi, \quad x \neq \frac{1}{2}(2k+1)\pi, \quad k = 0, 1, 2, \dots,$$

$$U(x, t) = \sum_{i=1}^n \frac{e^{p_i t}}{p_i} \frac{1 - p_i^2}{1 + xp_i} + \tan x + R(x, t);$$

$$(b) \quad \text{For } x > \frac{1}{2}\pi, \quad x = \frac{1}{2}(2k+1)\pi, \quad k = 1, 2, \dots,$$

$$U(x, t) = \sum_{i=1}^n \frac{e^{p_i t}}{p_i} \frac{1 - p_i^2}{1 + xp_i} + [t - \frac{1}{2}(k + \frac{1}{2})\pi] + R(\frac{1}{2}(2k+1)\pi, t);$$

$$(c) \quad \text{For } x = \frac{1}{2}\pi,$$

$$U(x, t) = t - \frac{1}{4}\pi + R(\frac{1}{2}\pi, t);$$

$$(d) \quad \text{For } 0 < x < \frac{1}{2}\pi,$$

$$U(x, t) = \tan x + R(x, t).$$

In all cases $R(x, t) = O(e^{-\epsilon(x)t})$.

Proof. The theorem obviously results from moving the line of integration in (5.7) from $R(p) = c$ to $R(p) = -\epsilon(x)$. That the contribution across the horizontal paths (symbolically, $\sigma \pm i\infty$) vanishes is clear from Lemma 5.5. The same lemma gives immediately the estimate of the remaining integral.

6. Remarks. In [3] it is shown that the expected number $E(x)$ of neutrons emergent in all time from a rod of length x of the sort described in §1 is

$$E(x) = \tan x, \quad 0 < x < \frac{1}{2}\pi.$$

This is in agreement with Theorem III, part (d). The time-independent case thus serves to establish the critical length of such a rod. It gives no information about the way in which the neutron population grows in the case of a rod of length exceeding the critical length. Theorem III provides this information.

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