Degenerate Waves in Unsteady Gas Flow

D. NAYLOR

Communicated by G. TEMPLE

1. Introduction. In a previous paper [3], it was shown how to construct exact solutions of the quasi-linear partial differential equation for the velocity potential $\phi(x, y, t)$ in the unsteady plane flow of a gas by concentrating on degenerate flows of the type $\mathbf{q}(\lambda, t)$, $\lambda(x, y, t)$ being an unspecified variable. The method proposed for the integration of the equations of flow rested on the application of a contact transformation, with respect to the variables (x, y) , to the potential ϕ . The main results of this theory were that, in the physical plane, the curves of constant speed reduced to a system of straight lines; in general these lines were generated as tangents to an envelope which was not permitted to degenerate into a point. Therefore, for flows of the type considered, there existed no "simple centered wave" in the unsteady plane flow of a gas.

The purpose of the present paper is to develop an analogous theory of waves of the type $\mathbf{q}(\lambda, \mu)$, $\lambda(x, y, z, t)$, $\mu(x, y, z, t)$ in the unsteady spatial flow of a gas. In general, the acoustic speed does not reduce to a function only of the variables λ and μ ; in fact, it is shown in §2 that in general $a = a(\lambda, \mu, t)$. If, in the representation (λ, μ, t) , the acoustic speed becomes independent of t , then it is demonstrated in §10 that flows result from which may be derived "progressive plane waves" for which the velocity depends only on the quantities x/t , y/t , and z/t . This is briefly taken up in §§9, 10. GIESE [2] develops a similar theory for which $\mathbf{q} = \mathbf{q}(\lambda, \mu)$, $a = a(\lambda, \mu)$ without assuming isentropic conditions, referring to such flows as "double waves." Flows for which $\mathbf{q}$ and a reduce to functions of three variables (λ, μ, ν) are referred to as "triple waves." Flows of the type $\mathbf{q}(\lambda, \mu)$, $a(\lambda, \mu, t)$ which are the subject of the present analysis therefore occupy an intermediate position between the double and triple waves of GIESE's theory. However, the title double waves is used for the flows here analyzed. A number of results connected with the properties of the hodograph and characteristic surfaces may be obtained for double waves, in GIESE's terminology; for details of these investigations we refer to GIESE's papers [1], [2].

It is well known that the Legendre transformation

$$\Phi = \phi - \mathbf{r} \cdot \mathbf{q}$$

is well adapted to the study of gas flows in the hodograph space, and especially to the degenerate flows for which the hodograph collapses into a surface or a curve. The linearisation of the equations of motion effective by the use of this transformation in steady flow is not, however, achieved in the case of unsteady flow. However, the reduction in the number of independent variables from four to three in the case of the unsteady motion seems to be best carried out in terms of Φ , and this procedure is adopted here. The method employed leads to equations of the Monge-Ampère type for the velocity and the function Φ , and one of these equations immediately provides an integral relation connecting u , v , w and a fourth variable λ . The Monge-Ampère equation is then considerably simplified when the unspecified variable λ is taken as an independent variable, which is in fact a characteristic variable for this equation. Finally by a further change of variable, in which essentially the acoustic speed is introduced as a fundamental variable, the problem is reduced to the integration of a linear equation of the Euler-Poisson type. The general integral of this equation may be written down explicitly. In §6 it is deduced that the flow problem in each surface $\lambda(x, y, z, t) = \text{constant}$ is essentially equivalent to the Riemann theory of unsteady one-dimensional gas flow expressed in terms of the invariants

$$r = q + \frac{2a}{\gamma - 1}, \quad s = q - \frac{2a}{\gamma - 1}$$

where now q refers to the total fluid speed.

Finally it is proved that no centered wave of the type $q(\lambda, \mu)$ is possible in unsteady gas motion.

2. General Equations for q . The equations satisfied by q, ϕ in unsteady potential flow may be written [3]

$$(1) \quad a^2 \operatorname{div} \mathbf{q} = q\mathbf{q} \cdot \operatorname{grad} q + 2qq_t + \phi_{tt},$$

$$(2) \quad \frac{q^2}{2} + \frac{a^2}{\gamma - 1} + \phi_t = \frac{c^2}{2},$$

c denoting a constant and $a = (\gamma p / \rho)^{\frac{1}{2}}$ the acoustic speed. Now introduce Φ , the Legendre transform of ϕ , defined by

$$(3) \quad \Phi = \phi - \mathbf{r} \cdot \mathbf{q},$$

so that, since $\mathbf{q} = \mathbf{q}(\lambda, \mu)$,

$$d\Phi = -(\mathbf{r} \cdot \mathbf{q}_\lambda) d\lambda - (\mathbf{r} \cdot \mathbf{q}_\mu) d\mu.$$

Therefore $\Phi = \Phi(\lambda, \mu, t)$ and

$$(4) \quad \Phi_\lambda + \mathbf{r} \cdot \mathbf{q}_\lambda = 0, \quad \Phi_\mu + \mathbf{r} \cdot \mathbf{q}_\mu = 0.$$

Since Φ reduces to a function of (λ, μ, t) only, the equations (4) represent for fixed t a two-parameter system of straight lines along each of which $\mathbf{q}(\lambda, \mu)$

is constant. Therefore in the physical space the curves of constant speed comprise a system of straight lines. By differentiation of (3), (4) the following relations are obtained:

$$\begin{aligned}
 (5) \quad & \phi_t = \Phi_t, \\
 & (\Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda}) \text{grad } \lambda + (\Phi_{\lambda\mu} + \mathbf{r} \cdot \mathbf{q}_{\lambda\mu}) \text{grad } \mu + \mathbf{q}_\lambda = 0, \\
 (6) \quad & (\Phi_{\lambda\mu} + \mathbf{r} \cdot \mathbf{q}_{\lambda\mu}) \text{grad } \lambda + (\Phi_{\mu\mu} + \mathbf{r} \cdot \mathbf{q}_{\mu\mu}) \text{grad } \mu + \mathbf{q}_\mu = 0, \\
 & (\Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda})\lambda_t + (\Phi_{\lambda\mu} + \mathbf{r} \cdot \mathbf{q}_{\lambda\mu})\mu_t + \Phi_{\lambda t} = 0, \\
 & (\Phi_{\lambda\mu} + \mathbf{r} \cdot \mathbf{q}_{\lambda\mu})\lambda_t + (\Phi_{\mu\mu} + \mathbf{r} \cdot \mathbf{q}_{\mu\mu})\mu_t + \Phi_{\mu t} = 0, \\
 (7) \quad & \phi_{tt} = \Phi_{tt} + \Phi_{\lambda t}\lambda_t + \Phi_{\mu t}\mu_t.
 \end{aligned}$$

Solving the equations (6) for $\text{grad } \lambda$, $\text{grad } \mu$, λ_t and μ_t gives

$$\begin{aligned}
 D \text{grad } \lambda &= B_2 \mathbf{q}_\mu - B_3 \mathbf{q}_\lambda, & D \text{grad } \mu &= B_2 \mathbf{q}_\lambda - B_1 \mathbf{q}_\mu, \\
 D\lambda_t &= B_2 \Phi_{\mu t} - B_3 \Phi_{\lambda t}, & D\mu_t &= B_2 \Phi_{\lambda t} - B_1 \Phi_{\mu t}
 \end{aligned}$$

in which

$$(8) \quad B_1 = \Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda}, \quad B_3 = \Phi_{\mu\mu} + \mathbf{r} \cdot \mathbf{q}_{\mu\mu}, \quad B_2 = \Phi_{\lambda\mu} + \mathbf{r} \cdot \mathbf{q}_{\lambda\mu},$$

and

$$D = B_1 B_3 - B_2^2.$$

The general equation (1) for the velocity may be written, for flows of the type considered,

$$\begin{aligned}
 (a^2 \mathbf{q}_\lambda - qq_\lambda \mathbf{q}) \cdot \text{grad } \lambda + (a^2 \mathbf{q}_\mu - qq_\mu \mathbf{q}) \cdot \text{grad } \mu \\
 = \Phi_{tt} + (\Phi_{\lambda t} + 2qq_\lambda)\lambda_t + (\Phi_{\mu t} + 2qq_\mu)\mu_t.
 \end{aligned}$$

Substituting the above expressions for the derivatives of λ and μ , this becomes

$$\begin{aligned}
 (9) \quad a^2(B_1 \mathbf{q}_\mu^2 - 2B_2 \mathbf{q}_\lambda \cdot \mathbf{q}_\mu + B_3 \mathbf{q}_\lambda^2) &= -D\Phi_{tt} + B_1(qq_\mu + \Phi_{\mu t})^2 \\
 &\quad - 2B_2(qq_\lambda + \Phi_{\lambda t})(qq_\mu + \Phi_{\mu t}) + B_3(qq_\lambda + \Phi_{\lambda t})^2.
 \end{aligned}$$

From (2), (5) it follows that a^2 must be obtained from the equation

$$(10) \quad \frac{q^2}{2} + \frac{a^2}{\gamma - 1} + \Phi_t = \frac{c^2}{2}$$

so that $a = a(\lambda, \mu, t)$.

From (3), (4) it is possible to express $\mathbf{r}$ in terms of (λ, μ, t) and a third variable θ in the form

$$(11) \quad \mathbf{r} = \boldsymbol{\varrho}(\lambda, \mu, t) + \theta \mathbf{n}(\lambda, \mu)$$

in which $\boldsymbol{\varrho}$ is any vector satisfying

$$(12) \quad \Phi_\lambda + \boldsymbol{\varrho} \cdot \mathbf{q}_\lambda = 0, \quad \Phi_\mu + \boldsymbol{\varrho} \cdot \mathbf{q}_\mu = 0$$

and $\theta(\lambda, \mu, \phi, t)$ satisfies

$$(13) \quad \theta \mathbf{n} \cdot \mathbf{q} = \phi - \Phi - \varrho \cdot \mathbf{q}$$

where

$$(14) \quad \mathbf{n}(\lambda, \mu) = \mathbf{q}_\lambda \times \mathbf{q}_\mu.$$

The equation (13) fails to determine ϕ in terms of θ when $\mathbf{n} \cdot \mathbf{q} = 0$ which according to (14) implies that the vectors $\mathbf{q}$, $\mathbf{q}_\lambda$, $\mathbf{q}_\mu$ are coplanar. In this case it follows from (6) that $\text{grad } \phi$, $\text{grad } \lambda$ and $\text{grad } \mu$ are coplanar so that ϕ is expressible in terms of (λ, μ, t) only and may not be used as an independent variable along with λ and μ . However, irrespective of this possibility it is manifest from (11) that θ quite generally may be considered as being independent of λ and μ and the subsequent analysis based on this result will be applicable in either instance. The equations (8) now give, on substituting $\mathbf{r}$ from (11),

$$(15) \quad \begin{aligned} B_1 &= \Phi_{\lambda\lambda} + \varrho \cdot \mathbf{q}_{\lambda\lambda} + \theta \mathbf{n} \cdot \mathbf{q}_{\lambda\lambda}, \\ B_2 &= \Phi_{\lambda\mu} + \varrho \cdot \mathbf{q}_{\lambda\mu} + \theta \mathbf{n} \cdot \mathbf{q}_{\lambda\mu}, \\ B_3 &= \Phi_{\mu\mu} + \varrho \cdot \mathbf{q}_{\mu\mu} + \theta \mathbf{n} \cdot \mathbf{q}_{\mu\mu}. \end{aligned}$$

When these relations are substituted in (9) there results a quadratic expression in θ , the coefficients of which being functions of (λ, μ, t) only because, as a consequence of the fundamental assumption, Φ , a^2 and $\mathbf{q}$ are independent of θ in the representation $(\lambda, \mu, t, \theta)$. As the equation is to be satisfied identically it must be evanescent in θ . This will lead to three relations involving Φ and $\mathbf{q}$ which will be obtained in §3 after a suitable choice of the arbitrary variables (λ, μ) . It is proved directly however that the surface $\mathbf{q} = \mathbf{q}(\lambda, \mu)$ in the hodograph space is developable. This follows immediately when the term in θ^2 is considered in the expression obtained by inserting the functions B_i which are linear in θ in (9) which leads to

$$(16) \quad \Phi_{,tt}[(\mathbf{n} \cdot \mathbf{q}_{\lambda\lambda})(\mathbf{n} \cdot \mathbf{q}_{\mu\mu}) - (\mathbf{n} \cdot \mathbf{q}_{\lambda\mu})^2] = 0.$$

Now the vector $\mathbf{n}$ as defined by (14) is directed along the normal to the surface $\mathbf{q}(\lambda, \mu)$ in the hodograph space. The condition (16) is then recognised as the condition that the hodograph is a developable surface. (See for example [4].) We have temporarily put to one side the possibility that $\Phi_{,tt}$ vanishes; this reduces the theory to the theory of "double waves" developed by GIESE which is discussed briefly in §7.

3. Equations for Φ . It is convenient to select $\lambda = u$, $\mu = v$ so that, by the fundamental hypothesis, w is now to be regarded as a function of u and v only, while Φ becomes a function of u , v and t . Writing

$$\mathbf{q} = [u, v, w(u, v)],$$

so that

$$\begin{aligned} \mathbf{q}_u &= (1, 0, w_u), & \mathbf{q}_v &= (0, 1, w_v), \\ \mathbf{q}_{uu} &= (0, 0, w_{uu}), & \mathbf{q}_{uv} &= (0, 0, w_{uv}), & \mathbf{q}_{vv} &= (0, 0, w_{vv}), \\ \mathbf{n} &= (-w_u, -w_v, 1), \end{aligned}$$

then

$$\mathbf{n} \cdot \mathbf{q}_{uu} = w_{uu}, \quad \mathbf{n} \cdot \mathbf{q}_{uv} = w_{uv}, \quad \mathbf{n} \cdot \mathbf{q}_{vv} = w_{vv}.$$

The vector $\mathbf{p} = (\rho_1, \rho_2, \rho_3)$ is now any solution of the system

$$\Phi_u + \rho_1 + \rho_3 w_u = 0, \quad \Phi_v + \rho_2 + \rho_3 w_v = 0;$$

and such a solution is clearly

$$\mathbf{p} = (-\Phi_u, -\Phi_v, 0).$$

The expressions (15) now give

$$B_1 = \Phi_{uu} + \theta w_{uu}, \quad B_3 = \Phi_{vv} + \theta w_{vv}, \quad B_2 = \Phi_{uv} + \theta w_{uv}.$$

Substituting these expressions in (9) and imposing the condition that this equation fails to determine θ yields the following three Monge-Ampère equations of mixed type for $w(u, v)$, $\Phi(u, v, t)$:

$$(17) \quad \Phi_{tt}(w_{uu}w_{vv} - w_{uv}^2) = 0,$$

$$(18) \quad \begin{aligned} a^2[(1 + w_u^2)w_{vv} - 2w_uw_vw_{uv} + (1 + w_v^2)w_{uu}] \\ = -\Phi_{tt}[w_{uu}\Phi_{vv} - 2w_{uv}\Phi_{uv} + w_{vv}\Phi_{uu}] + w_{uv}(\Phi_{vt} + qq_v)^2 \\ - 2w_{uv}(\Phi_{ut} + qq_u)(\Phi_{vt} + qq_v) + w_{vv}(\Phi_{ut} + qq_u)^2, \end{aligned}$$

$$(19) \quad \begin{aligned} a^2[(1 + w_u^2)\Phi_{vv} - 2w_uw_v\Phi_{uv} + (1 + w_v^2)\Phi_{uu}] \\ = \Phi_{tt}(\Phi_{uv}^2 - \Phi_{uu}\Phi_{vv}) + \Phi_{uu}(\Phi_{vt} + qq_v)^2 \\ - 2\Phi_{uv}(\Phi_{ut} + qq_u)(\Phi_{vt} + qq_v) + \Phi_{vv}(\Phi_{ut} + qq_u)^2. \end{aligned}$$

As indicated in §2 it is assumed that Φ_{tt} does not vanish identically so that to satisfy (17) the equation of developable surfaces follows,

$$(20) \quad w_{uu}w_{vv} - w_{uv}^2 = 0,$$

for which a general integral is expressible in the form

$$(21) \quad w = uf(\lambda) + vg(\lambda) + h(\lambda)$$

where

$$(22) \quad 0 = uf'(\lambda) + vg'(\lambda) + h'(\lambda),$$

the functions $f(\lambda)$, $g(\lambda)$ and $h(\lambda)$ being arbitrary. The variable λ here invoked may be regarded as a function of (u, v) given implicitly by equation (22). For

future use it is convenient to collect here the expressions for the various derivatives of the functions $w(u, v)$ and $\lambda(u, v)$.

From (21), (22) it follows that

$$(23) \quad w_u = f, \quad w_v = g.$$

Also, differentiation of (22) leads to

$$(24) \quad A\lambda_u + f_\lambda = 0, \quad A\lambda_v + g_\lambda = 0$$

where

$$A(u, v) = uf_{\lambda\lambda} + vg_{\lambda\lambda} + h_{\lambda\lambda}.$$

Differentiation of equations (23) gives

$$w_{uu} = f_\lambda\lambda_u, \quad w_{uv} = f_\lambda\lambda_v = g_\lambda\lambda_u, \quad w_{vv} = g_\lambda\lambda_v,$$

so that, using (24),

$$(25) \quad w_{uu} = -A\lambda_u^2, \quad w_{uv} = -A\lambda_u\lambda_v, \quad w_{vv} = -A\lambda_v^2.$$

The following inversion formulas also exist

$$(26) \quad Ju_\lambda = w_v, \quad Ju_w = -\lambda_v, \quad Jv_\lambda = -w_u, \quad Jv_w = \lambda_u$$

where

$$(27) \quad J = \frac{\partial(\lambda, w)}{\partial(u, v)} = \frac{1}{A} (fg_\lambda - gf_\lambda).$$

It follows from (26), (24) and (27) that

$$(28) \quad u_w = \frac{g_\lambda}{fg_\lambda - gf_\lambda}, \quad v_w = \frac{-f_\lambda}{fg_\lambda - gf_\lambda}$$

so that

$$u_{ww} = v_{ww} = 0.$$

From equations (25) we note that

$$(29) \quad w_u^2 w_{vv} - 2w_u w_v w_{uv} + w_v^2 w_{uu} = -AJ^2,$$

$$(30) \quad w_{uu} + w_{vv} = -AJ^2 \frac{f_\lambda^2 + g_\lambda^2}{(fg_\lambda - gf_\lambda)^2}.$$

From (29), (30) it follows that

$$(32) \quad (1 + w_u^2)w_{vv} - 2w_u w_v w_{uv} + (1 + w_v^2)w_{uu} = -AJ^2 b^2$$

where

$$(33) \quad b(\lambda) = \left[1 + \frac{f_\lambda^2 + g_\lambda^2}{(fg_\lambda - gf_\lambda)^2} \right]^{\frac{1}{2}}.$$

4. Characteristic Variables. The form of (18) for $\Phi(u, v, t)$ suggests introducing the function $\Psi(u, v, t)$ defined by

$$(34) \quad \Psi = -\Phi - \frac{t}{2}(q^2 - c^2)$$

so that from (10) the acoustic speed will now be given in terms of Ψ_t by the formula

$$(35) \quad a^2 = (\gamma - 1)\Psi_t .$$

Using (32) the equation (18) may now be written in the form

$$(36) \quad -AJ^2a^2b^2 = -\Phi_{tt}(w_{uu}\Phi_{vv} - 2w_{uv}\Phi_{uv} + w_{vv}\Phi_{uu}) \\ + w_{vv}\Psi_{tt}^2 - 2w_{uv}\Psi_{tt}\Psi_{vt} + w_{uu}\Psi_{tt}^2 .$$

Consider now the characteristics of the differential expression in Φ

$$(37) \quad w_{vv}\Phi_{uu} - 2w_{uv}\Phi_{uv} + w_{uu}\Phi_{vv}$$

which represents the only terms in (36) involving the second derivatives of Φ with respect to u, v . The characteristics are the integral curves of

$$w_{vv} dv^2 + 2w_{uv} du dv + w_{uu} du^2 = 0$$

which, by (25), is equivalent to

$$(\lambda_u du + \lambda_v dv)^2 = 0 .$$

Therefore the differential expression (37), which is the coefficient of Φ_{tt} in (36), is parabolic with the function $\lambda(u, v)$ introduced in the integration of (20) as the fundamental variable. This indicates that in the equation (36) the variables (u, v) be replaced by (λ, w) . The following transformations can be written down:

$$(38) \quad \Psi_w = \Psi_u u_w + \Psi_v v_w .$$

In virtue of (25), (26)

$$(39) \quad w_{uu} = -AJ^2v_w^2, \quad w_{uv} = +AJ^2u_wv_w, \quad w_{vv} = -AJ^2u_w^2$$

and

$$u_{ww} = v_{ww} = 0 .$$

The expansion for Ψ_{ww} reduces to

$$\Psi_{ww} = \Psi_{uu}u_w^2 + 2\Psi_{uv}u_wv_w + \Psi_{vv}v_w^2 .$$

Using the equations (39) this may be rewritten

$$(40) \quad -AJ^2\Psi_{ww} = w_{vv}\Psi_{uu} - 2w_{uv}\Psi_{uv} + w_{uu}\Psi_{vv} ,$$

and a similar equation holds for Φ .

Analogously to (38) one may write

$$(41) \quad \Psi_{wt} = \Psi_u u_{wt} + \Psi_v v_{wt} .$$

Using (39), equation (41) implies that

$$(42) \quad -AJ^2\Psi_{w^2}^2 = w_{vv}\Psi_{u^2}^2 - 2w_{uv}\Psi_{ut}\Psi_{vt} + w_{uu}\Psi_{v^2}^2.$$

Therefore when the equation (36) is transformed in terms of the new variables (λ, w, t) using (40), (42) it becomes

$$(43) \quad \Phi_{ww}\Phi_{tt} - \Psi_{w^2}^2 + a^2b^2 = 0.$$

Now differentiating (34) and putting $u_{ww} = v_{ww} = 0$ gives

$$(44) \quad \Phi_{tt} = -\Psi_{tt}, \quad \Phi_{ww} = -\Psi_{ww} - tb^2$$

after employing (28), (33).

Substituting equations (44) in (43) and setting by (35)

$$a^2 = \nu^2\Psi_t, \quad \nu^2 = \gamma - 1$$

gives finally

$$(45) \quad \Psi_{ww}\Psi_{tt} - \Psi_{w^2}^2 + b^2(t\Psi_{tt} + \nu^2\Psi_t) = 0.$$

This is an equation of the Monge-Ampère type for $\Psi(\lambda, w, t)$ in which derivatives with respect to λ do not appear. The variable λ appears only in the coefficient $b = b(\lambda)$.

5. Integration of the Monge-Ampère Equation (45). Letting

$$(46) \quad \xi = \Psi_t, \quad \eta = \Psi_w$$

and proceeding in the usual way to obtain the characteristics of (45), one obtains

$$b^2\nu^2\xi dw^2 - d\xi^2 = 0, \quad d\eta dw + d\xi dt + b^2t dw^2 = 0.$$

Denoting the characteristic variables by $r(\lambda, w, t)$, $s(\lambda, w, t)$ there result the two systems

$$(47) \quad b\nu w_r - \xi^{-\frac{1}{2}}\xi_r = 0, \quad b\nu w_s + \xi^{-\frac{1}{2}}\xi_s = 0,$$

$$(48) \quad \eta_r + b\nu\xi^{\frac{1}{2}}t_r + b^2tw_r = 0, \quad \eta_s - b\nu\xi^{\frac{1}{2}}t_s + b^2tw_s = 0.$$

From (47) the variables (r, s) may be selected in the forms

$$(49) \quad r = b\nu w + 2\xi^{\frac{1}{2}}, \quad s = b\nu w - 2\xi^{\frac{1}{2}}$$

so that equations (48) become

$$(50) \quad (4\nu/b)\eta_r + \nu^2(r-s)t_r + 2t = 0,$$

$$(4\nu/b)\eta_s - \nu^2(r-s)t_s + 2t = 0.$$

It is of interest to note that elimination of η and t respectively leads to the Euler-Poisson equations

$$(51) \quad (r-s)t_{rs} - n(t_r - t_s) = 0,$$

$$(52) \quad (r-s)\eta_{rs} - m(\eta_r - \eta_s) = 0$$

in which

$$2n = \frac{\nu^2 + 2}{\nu^2} = \frac{\gamma + 1}{\gamma - 1}, \quad 2m = 2n - 2 = \frac{2 - \nu^2}{\nu^2} = \frac{3 - \gamma}{\gamma - 1}.$$

For diatomic gases $\gamma = \frac{7}{5}$, $n = 3$, $m = 2$ and the general solutions of (51), (52) may be written down.

6. General Solution for $\Psi(\lambda, w, t)$. The forms (49) of the variables (r, s) in terms of which the solution for $\Psi(\lambda, w, t)$ may be expressed suggests that $\xi = \Psi_t$ be chosen as a new independent variable. This change is best carried out using a partial contact transformation to define a function $\psi(\lambda, w, t)$ as follows:

$$(53) \quad \psi = \Psi - t\Psi_t,$$

in which Ψ is regarded as a function of (λ, w, t) and ψ a function of (λ, w, ξ) . It follows that

$$(54) \quad \psi_\lambda = \Psi_\lambda, \quad \psi_w = \Psi_w, \quad \psi_\xi = -t;$$

therefore (53) is equivalent to

$$(55) \quad \Psi = \psi - \xi\psi_\xi.$$

Employing the relation (34) one finds that

$$\psi = t\Phi_t - \Phi$$

so that, from (3), (5),

$$(56) \quad \psi = t\phi_t + ux + vy + wz - \phi.$$

To construct the solution for ψ let us transform the Monge-Ampère equation (45) directly in terms of ψ by means of the following relations which result from equations (54):

$$(57) \quad \Psi_{tt}\psi_{\xi\xi} = -1, \quad \Psi_{wt}\psi_{\xi\xi} = -\psi_{w\xi}, \quad \Psi_{ww}\psi_{\xi\xi} = \psi_{ww}\psi_{\xi\xi} - \psi_{w\xi}^2.$$

With these transformations (45) reduces to the linear form

$$b^2(\nu^2\xi\psi_{\xi\xi} + \psi_\xi) - \psi_{ww} = 0.$$

Finally in terms of the characteristic variables (r, s) defined by (49) this reduces further to the required Euler-Poisson equation

$$(58) \quad (r - s)\psi_{rs} - m(\psi_r - \psi_s) = 0.$$

Confining attention to the case $m = 2$ the general solution of (58) may be obtained [5] from

$$(59) \quad \psi(r, s, \lambda) = \frac{\partial^2}{\partial r \partial s} \left[\frac{A(r, \lambda) - B(s, \lambda)}{r - s} \right],$$

the functions $A(r, \lambda)$, $B(s, \lambda)$ being arbitrary. In terms of the acoustic speed defined by (35), (46) the characteristic variables (r, s) defined by (49) may be expressed in the forms

$$r = v \left[bw + \frac{2a}{\gamma - 1} \right], \quad s = v \left[bw - \frac{2a}{\gamma - 1} \right],$$

the quantity $b(\lambda)$ being constant along each line $\mathbf{q} = \text{constant}$. It is observed from (51) that $t(r, s, \lambda)$ satisfies the same equation that $t(r, s)$ satisfies in the unsteady rectilinear motion of a gas, but, in the general case, the characteristic variables now have a wider significance than the Riemann invariants

$$r = u + \frac{2a}{\gamma - 1}, \quad s = u - \frac{2a}{\gamma - 1}.$$

Similarly the function $\psi(r, s, \lambda)$ satisfies the same equation (58) as the Riemann dependent function $W(r, s)$. To illustrate the significance of this similarity consider the following particular integral of the system (28):

$$u = w \frac{g_\lambda}{fg_\lambda - gf_\lambda}, \quad v = -w \frac{f_\lambda}{fg_\lambda - gf_\lambda};$$

then (33) gives

$$bw = (u^2 + v^2 + w^2)^{\frac{1}{2}}$$

so that $q = bw$. Now in the integration of the Monge-Ampère equation (45), or the equivalent Euler-Poisson equation (58), the variable λ is temporarily held constant, and the variables r, s for the above integral reduce essentially to the Riemann invariants. These results may be summarised in the following theorems:

1. *In the unsteady flow of the type $\mathbf{q}(\lambda, \mu)$ the curves of constant velocity degenerate into a system of straight lines.*

2. *The equations for the velocity of a double wave admit an integral of the type*

$$w = uf(\lambda) + vg(\lambda) + h(\lambda),$$

$$0 = uf'(\lambda) + vg'(\lambda) + h'(\lambda),$$

the parameter λ being constant along each of a series of ruled surfaces generated by a system of straight lines forming the lines of constant velocity in the physical space.

3. *The variation of the gas speed q and the acoustic speed a over each surface $\lambda = \text{constant}$ is governed by the Riemann theory of unsteady rectilinear gas flow, in which the Riemann function $W(r, s)$ is replaced by the transformed potential*

$$\psi = x\phi_x + y\phi_y + z\phi_z + t\phi_t - \phi.$$

The general solution of the unsteady degenerate flow problem has been expressed by the equations (21), (22) and (59) in terms of the functions of integration $f(\lambda)$, $g(\lambda)$, $h(\lambda)$, $A(r, \lambda)$ and $B(s, \lambda)$. These functions are not, however,

quite arbitrary. They must be selected so that together they will produce a pair of functions $w(u, v)$, $\Phi(u, v, t)$ satisfying the additional Monge-Ampère relation (19). Substitution of the integrals (21), (59) leads to relations connecting the five arbitrary functions. However, these relations are so complex, in the general case, that we shall not attempt to carry the general theory any further but remark at this stage that the integral representations of the type (21), (59) are sufficient to establish certain flow properties in §§7, 8. It is desirable to investigate the possible existence of simpler flow patterns. The case of greatest interest arises when conditions of velocity are constant along each of a series of concurrent straight lines, admitting, however, the possibility that the point of concurrency may not necessarily be fixed. Such a flow would be termed a "centered double wave." It has already been demonstrated that no simple centered wave exists in the unsteady plane flow of a gas [3]. That the same result is also true for double waves of the type considered here is proved in §7. Still more generally it is shown in §8 that the lines of constant velocity may not be generated from a fixed straight line. Finally, in §9 a simplified theory of general spatial simple waves is presented, representing flows of considerable generality.

7. Non-Existence of Centered Double Waves. If the lines of constant velocity are concurrent then there exist functions $x_0(t)$, $y_0(t)$, $z_0(t)$ such that the equations

$$(60) \quad x_0 + z_0 w_u + \Phi_u = 0, \quad y_0 + z_0 w_v + \Phi_v = 0$$

are identically satisfied for all values of (u, v, t) . From these by differentiation there follow the equations

$$(61) \quad z_0 w_{uu} + \Phi_{uu} = 0, \quad z_0 w_{uv} + \Phi_{uv} = 0, \quad z_0 w_{vv} + \Phi_{vv} = 0.$$

The functions B_1 , B_2 , B_3 of §3 are now given by

$$(62) \quad B_1 = (\theta - z_0)w_{uu}, \quad B_2 = (\theta - z_0)w_{uv}, \quad B_3 = (\theta - z_0)w_{vv}.$$

According to (17) then

$$D = B_1 B_3 - B_2^2 = 0;$$

therefore the analysis leading up to the equations (17), (18), (19) is inapplicable in this case since in solving the equations (6) for the derivatives of λ and μ it was assumed that the quantity D was non-vanishing. When the expressions (62) are used one obtains instead

$$(63) \quad \begin{aligned} D_0(\theta - z_0) \operatorname{grad} u &= w_{uv} \mathbf{q}_v - w_{vv} \mathbf{q}_u, \\ D_0(\theta - z_0) \operatorname{grad} v &= w_{uv} \mathbf{q}_u - w_{vv} \mathbf{q}_v, \\ D_0(\theta - z_0) u_t &= w_{uv} \Phi_{vt} - w_{vv} \Phi_{ut}, \\ D_0(\theta - z_0) v_t &= w_{uv} \Phi_{ut} - w_{vv} \Phi_{vt}, \end{aligned}$$

where

$$D_0 = w_{uu}w_{vv} - w_{uv}^2$$

is now assumed non-zero.

When the revised formulas (63) are used the equation (9) is replaced by

$$\begin{aligned} & -a^2[w_{uu}\mathbf{q}_v^2 - 2w_{uv}\mathbf{q}_u \cdot \mathbf{q}_v + w_{vv}\mathbf{q}_u^2] + q^2[q_v^2w_{uu} - 2q_uq_vw_{uv} + q_u^2w_{vv}] \\ & = D_0(\theta - z_0)\Phi_{tt} - \Psi_{ut}(w_{uv}\Phi_{vt} - w_{vv}\Phi_{ut}) - \Psi_{vt}(w_{uv}\Phi_{ut} - w_{uu}\Phi_{vt}). \end{aligned}$$

In this equation it is only necessary to observe that the variable θ , which is independent of u and v , occurs only in the combination $D_0(\theta - z_0)\Phi_{tt}$. As D_0 is non-zero it follows that Φ_{tt} vanishes identically.

From the equations (60) by differentiation it follows that

$$(x_0)_{tt} + (z_0)_{tt}w_u = 0, \quad (y_0)_{tt} + (z_0)_{tt}w_v = 0.$$

Since w_u, w_v do not involve t these relations imply that

$$(x_0)_{tt} = (y_0)_{tt} = (z_0)_{tt} = 0$$

after ignoring trivial cases in which w_u, w_v reduce to constants. Without loss of generality one may then write, in terms of constants a_1, a_2, a_3 ,

$$x_0 = a_1t, \quad y_0 = a_2t, \quad z_0 = a_3t$$

according to which the center (x_0, y_0, z_0) moves with constant speed in a fixed direction. The equations (60) now become

$$\Phi_u + t(a_1 + a_3w_u) = 0, \quad \Phi_v + t(a_2 + a_3w_v) = 0.$$

Taken in conjunction with the field equations

$$\Phi_u + x + zw_u = 0, \quad \Phi_v + y + zw_v = 0$$

it follows that

$$x - a_1t + (z - a_3t)w_u = 0, \quad y - a_2t + (z - a_3t)w_v = 0$$

which define u and v in terms of the quantities $(x - a_1t)/(z - a_3t)$, $(y - a_2t)/(z - a_3t)$. The lines $\mathbf{q} = \text{constant}$ in the physical space are then carried along without rotation in a fixed direction. Therefore, referred relative to the moving center, the flow is essentially not unsteady.

To complete the proof establishing the non-existence of centered waves it is only necessary to dispose of the case for which D_0 vanishes. With the aid of the relations (60) the equations for $\text{grad } u$, $\text{grad } v$ become

$$\begin{aligned} (64) \quad & (\theta - z_0)(w_{uu} \text{grad } u + w_{uv} \text{grad } v) + \mathbf{q}_u = 0, \\ & (\theta - z_0)(w_{uv} \text{grad } u + w_{vv} \text{grad } v) + \mathbf{q}_v = 0. \end{aligned}$$

Since

$$w_{uu}w_{vv} - w_{uv}^2 = 0$$

it follows that the vectors $\mathbf{q}_u$, $\mathbf{q}_v$ must be parallel (or zero) in order to satisfy (64). According to §3, however,

$$\mathbf{q}_u \times \mathbf{q}_v = (-w_u, -w_v, 1)$$

which may never vanish if $\mathbf{q}$ remains a function of (u, v) in terms of which w is to be expressed. The quantity $\mathbf{q}_u \times \mathbf{q}_v$ vanishes, apart from trivial cases, only if $\mathbf{q}$ reduces to a function of just one independent variable, that is, only if the double wave further degenerates into a simple wave. At this stage this possibility is ignored and the theorem is proved. However, in §9 a simplified theory of unsteady waves of the type $\mathbf{q}(\lambda)$ is investigated.

8. Characteristic Lines Radiated from a Straight Line. Generalising the concept of a centered wave it is natural to seek flows generated by a system of characteristic lines each intersecting a given straight line. Attention is confined to the case in which this line is fixed. To construct the flow field in the general case it is necessary to draw the lines

$$(65) \quad x + zw_u + \Phi_u = 0, \quad y + zw_v + \Phi_v = 0.$$

Through each point $(-\Phi_u, -\Phi_v, 0)$ in the xy -plane a line is drawn in the direction $(w_u, w_v, -1)$. Along each such line u, v and $w(u, v)$ are constant. To obtain a flow pattern generated by lines emanating from points of the z -axis one requires that, from (65),

$$zw_u + \Phi_u = 0, \quad zw_v + \Phi_v = 0$$

so that

$$\frac{\partial(\Phi, w)}{\partial(u, v)} = 0,$$

therefore $\Phi = \Phi(w, t)$ and $\Phi_\lambda = 0$. The equations (18), (19) may now be simplified using the expressions

$$(66) \quad \begin{aligned} \Phi_{uu} &= \Phi_{ww}w_u^2 + \Phi_w w_{uu}, & \Phi_{uv} &= \Phi_{ww}w_u w_v + \Phi_w w_{uv}, \\ \Phi_{vv} &= \Phi_{ww}w_v^2 + \Phi_w w_{vv}, \end{aligned}$$

and

$$(67) \quad \Phi_{uu}\Phi_{vv} - \Phi_{uv}^2 = -AJ^2\Phi_w\Phi_{ww}$$

after using (29), (20).

Equation (18) now reduces to

$$(68) \quad \begin{aligned} a^2[(1 + w_u^2)w_{vv} - 2w_u w_v w_{uv} + (1 + w_v^2)w_{uu}] \\ = AJ^2\Phi_{ww}\Phi_{tt} + w_{uu}\Psi_{vt}^2 - 2w_{uv}\Psi_{ut}\Psi_{vt} + w_{vv}\Psi_{ut}^2. \end{aligned}$$

When the formulas (66), (67) are inserted in (19) and (68) is used there remains

$$(69) \quad a^2(w_u^2 + w_v^2)\Phi_{ww} = (w_u\Psi_{vt} - w_v\Psi_{ut})^2\Phi_{ww}.$$

From the inversion relations (26) it follows that

$$w_u \Psi_{ut} - w_t \Psi_{ut} = -J \Psi_{\lambda t}, \quad w_u^2 + w_t^2 = J^2(u_\lambda^2 + v_\lambda^2)$$

so that (69) reduces to

$$(70) \quad [a^2(u_\lambda^2 + v_\lambda^2) - \Psi_{\lambda t}^2] \Phi_{ww} = 0.$$

In addition, $\Psi(\lambda, w, t)$ must also satisfy (45), the transformed version of (68),

$$(71) \quad \Psi_{ww} \Psi_{tt} - \Psi_{wt}^2 + b^2(t \Psi_{tt} + \nu^2 \Psi_t) = 0.$$

If Φ_{ww} vanishes identically then (70) is satisfied and by (44)

$$(72) \quad \Psi_{ww} + b^2 t = 0;$$

then (71) reduces to

$$(73) \quad \Psi_{wt}^2 - b^2 \nu^2 \Psi_t = 0.$$

Equations (72), (73) are incompatible in general. First of all, notice that since $a^2 = \nu^2 \Psi_t$ equation (73) shows that for non-trivial flows Ψ_{wt} may not vanish. Differentiation of (73) with respect to w and removing the non-zero factor Ψ_{wt} produces

$$(74) \quad 2\Psi_{wwt} - b^2 \nu^2 = 0.$$

Similarly by differentiating (72) with respect to t one obtains

$$(75) \quad \Psi_{wwt} + b^2 = 0.$$

The equations (74), (75) can never be simultaneously satisfied. It would be necessary to have $\nu^2 = -2$ so that $\gamma = -1$. It is of interest to note that this is equivalent to the approximation adopted by CHAPLYGIN to simplify the hodograph equations in steady plane flow. If this approximation were accepted one could obtain unsteady waves generated by a degenerate system of characteristic lines. Here, however, the case $\Phi_{ww} = 0$ is ignored and replaced by

$$(76) \quad \nu^2(u_\lambda^2 + v_\lambda^2) \Psi_t = \Psi_{\lambda t}^2$$

which follows from (70) after using (35).

The condition $\Phi_\lambda = 0$ implies by (34) that

$$(77) \quad \Psi_\lambda + t q q_\lambda = 0$$

and

$$(78) \quad \Psi_{\lambda t} + q q_\lambda = 0.$$

Since q is a function of (λ, w) only it follows at once from (78) and (76) that $\Psi_{tt} = 0$ and that (71) becomes

$$(73) \quad \Psi_{wt}^2 - b^2 \nu^2 \Psi_t = 0.$$

The equations for Ψ are now (76), (77) and (73). It will be demonstrated that no solution of this system can be found; therefore any possible curve of inter-

section of the characteristic lines may not degenerate into a fixed straight line. This result, along with the theorem of §7, show that degenerate systems of (u, v) lines exist only if one accepts the hypothetical gas of CHAPLYGIN.

Write the equations (73), (76), (78) in the forms

$$(79) \quad \Psi_{wt}^2 - \nu^2 \mathbf{q}_w^2 \Psi_t = 0,$$

$$(80) \quad \Psi_{\lambda t}^2 - \nu^2 \mathbf{q}_\lambda^2 \Psi_t = 0,$$

$$(81) \quad \Psi_{\lambda t} + \mathbf{q} \cdot \mathbf{q}_\lambda = 0.$$

From (28) it follows that

$$(82) \quad \mathbf{q}_{ww} = 0.$$

Introduce the function

$$k(\lambda, w) = 2\Psi_t^{\frac{1}{2}};$$

then (79), (80), (81) become

$$(83) \quad k_w^2 = \nu^2 \mathbf{q}_w^2,$$

$$(84) \quad k_\lambda^2 = \nu^2 \mathbf{q}_\lambda^2,$$

$$(85) \quad k k_\lambda + 2\mathbf{q} \cdot \mathbf{q}_\lambda = 0.$$

Differentiating (83) with respect to λ produces

$$(86) \quad k_w k_{\lambda w} = \nu^2 \mathbf{q}_w \cdot \mathbf{q}_{\lambda w}.$$

Differentiating (85) twice with respect to w yields

$$(87) \quad k_w k_{\lambda w} = -2\mathbf{q}_w \cdot \mathbf{q}_{\lambda w},$$

after using $k_{ww} = 0$ which follows at once from (82), (83). Comparing (86), (87) it is seen that

$$(88) \quad k_{\lambda w} = \mathbf{q}_w \cdot \mathbf{q}_{\lambda w} = 0,$$

again ignoring the possibilities $k_w = 0$, $\nu^2 = -2$. From (84), (88) it follows that

$$\mathbf{q}_\lambda \cdot \mathbf{q}_{\lambda w} = 0,$$

from which by further differentiation one obtains $\mathbf{q}_{\lambda w}^2 = 0$. Therefore in addition to (82) one has that

$$(89) \quad \mathbf{q}_{\lambda w} = 0$$

so that $\mathbf{q}_w$ is a constant vector.

Differentiating (85) with respect to w and using (88), (89) gives

$$k_w k_\lambda + 2\mathbf{q}_w \cdot \mathbf{q}_\lambda = 0$$

which may be converted into

$$(90) \quad \pm \nu^2 |\mathbf{q}_w| \cdot |\mathbf{q}_\lambda| + 2\mathbf{q}_w \cdot \mathbf{q}_\lambda = 0.$$

Since

$$\mathbf{q}_\lambda = (u_\lambda, v_\lambda, 0), \quad \mathbf{q}_w = (u_w, v_w, 1)$$

(90) may be written

$$(91) \quad \pm \nu^2 (1 + u_w^2 + v_w^2)^{\frac{1}{2}} (u_\lambda^2 + v_\lambda^2)^{\frac{1}{2}} + 2(u_\lambda u_w + v_\lambda v_w) = 0.$$

A second equation relating $u(\lambda, w)$, $v(\lambda, w)$ is obtained by differentiating equations (84), (85) with respect to λ and eliminating the function $k(\lambda, w)$ giving

$$(\nu^2 + 2)(\mathbf{q}_\lambda^2)^2 + 2(\mathbf{q} \cdot \mathbf{q}_{\lambda\lambda})\mathbf{q}_\lambda^2 - 2(\mathbf{q} \cdot \mathbf{q}_\lambda)(\mathbf{q}_\lambda \cdot \mathbf{q}_{\lambda\lambda}) = 0$$

which gives finally

$$(92) \quad 2(u_\lambda v_{\lambda\lambda} - v_\lambda u_{\lambda\lambda})(u v_\lambda - v u_\lambda) = (\nu^2 + 2)(u_\lambda^2 + v_\lambda^2)^2.$$

The equations (91), (92) are inconsistent in general. Since u_w, v_w are constant then, assuming for example that v_λ does not vanish identically, (91) implies that the quantity u_λ/v_λ is also constant so that $u_\lambda v_{\lambda\lambda} - v_\lambda u_{\lambda\lambda} = 0$. Equation (92) then implies the contradiction $u_\lambda = v_\lambda = 0$ unless $\nu^2 = -2$. Alternatively, if $v_\lambda = 0$ then from (92) it still follows that $u_\lambda = 0$ so that the only solution of these equations is the trivial solution $u_\lambda = v_\lambda = 0$. In this case $\mathbf{q} = \mathbf{q}(w)$; the flow represents a degenerate simple wave and is considered in the next section.

9. Simplified Theory of Unsteady Plane Waves. At this point we shall not attempt to carry the general theory any further but shall concentrate on flows of the type $\mathbf{q}(\lambda)$. Such flows have occurred in §§7, 8 as degenerate instances of the double wave $\mathbf{q}(\lambda, \mu)$. The object now is to extend to three dimensions the theory of unsteady simple waves in plane flow developed in [3]. It will be shown that for such flows the surfaces of constant velocity comprise a system of planes. The equation for the velocity obtained in the general case in §2 now reduces to, since $\mathbf{q}$ and Φ now depend on λ ,

$$(93) \quad (a^2 \mathbf{q}_\lambda - q q_\lambda \mathbf{q}) \cdot \text{grad } \lambda = \Phi_{,\lambda} + (\Phi_{,\lambda} + 2q q_\lambda) \lambda_t$$

whilst the simplified forms of the equations (6) are

$$(94) \quad (\Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda}) \text{grad } \lambda + \mathbf{q}_\lambda = 0,$$

$$(95) \quad (\Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda}) \lambda_t + \Phi_{,\lambda} = 0.$$

Elimination of $\text{grad } \lambda, \lambda_t$ between (93), (94), (95) produces

$$(96) \quad a^2 \mathbf{q}_\lambda^2 = (\Phi_{,\lambda} + q q_\lambda)^2 - \Phi_{,\lambda} (\Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda}).$$

The system (4) is now replaced by the single equation

$$(97) \quad \Phi_\lambda + \mathbf{r} \cdot \mathbf{q}_\lambda = 0,$$

Φ now reducing to a function of (λ, t) only. Equation (97) represents a one parameter family of planes along each of which λ and $\mathbf{q}(\lambda)$ are constant. Therefore

the surfaces of constant velocity degenerate into a system of planes. These planes depend on the time since Φ_λ involves t . From (96), since a^2 , $\mathbf{q}$, Φ depend only on λ and t it is necessary to demand that either $\Phi_{,t}$ vanishes identically or that a relation of the following type exists:

$$F(\lambda, t) + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda} = 0,$$

from which by differentiation is obtained

$$(98) \quad (F_\lambda + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda\lambda}) \text{grad } \lambda + \mathbf{q}_{\lambda\lambda} = 0.$$

Comparing (94), (98) it follows that

$$(99) \quad \mathbf{q}_\lambda \times \mathbf{q}_{\lambda\lambda} = 0.$$

From (99) the following integral is readily deduced:

$$(100) \quad \mathbf{q}(\lambda) = \mathbf{b}f(\lambda) + \mathbf{c},$$

$\mathbf{b}$, $\mathbf{c}$ being constant vectors and $f(\lambda)$ arbitrary. The formula (100), however, implies that $\mathbf{q}$ has no component in the fixed direction $\mathbf{b} \times \mathbf{c}$ so that the flow is plane. Returning to the general case it follows that

$$(101) \quad \Phi_{,t} = 0$$

and that, from (96),

$$(102) \quad a^2 \mathbf{q}_\lambda^2 = (\Phi_{,\lambda} + q q_\lambda)^2.$$

The general integral of (101) is

$$(103) \quad \Phi(\lambda, t) = t\psi(\lambda) + K(\lambda),$$

ψ , K being arbitrary. Then (102) gives

$$(104) \quad a^2 \mathbf{q}_\lambda^2 = (\psi_\lambda + q q_\lambda)^2$$

where, from (10),

$$(105) \quad 2a^2 = v^2(c^2 - q^2 - 2\psi).$$

Taking K to be zero in (103) simplifies (97) to

$$(106) \quad t\psi_\lambda + \mathbf{r} \cdot \mathbf{q}_\lambda = 0$$

which defines λ as a function of $(x/t, y/t, z/t)$. Therefore the planes of constant velocity are propagated at constant speed in a fixed direction. Such waves have been termed "plane progressive waves."

From (104), (105) one obtains

$$(107) \quad u_\lambda^2 + v_\lambda^2 + w_\lambda^2 = 25a_\lambda^2$$

after using the value $v^2 = \frac{8}{3}$ corresponding to $\gamma = \frac{7}{5}$. The equation of the wave planes are now given by (106) with

$$-\psi_\lambda = uu_\lambda + vv_\lambda + ww_\lambda + 5aa_\lambda$$

which follows from (105).

The equation (107) relates the four functions u, v, w, a ; therefore three may be prescribed arbitrarily so that a great number of different flows may be obtained by the method.

10. Progressive Waves. We return briefly to the alternative possibility $\Phi_{tt} = 0$ which is the case put to one side in §2. In this case the relation (17) is satisfied automatically and, in general, there follows no integral for $w(u, v)$ of the type provided by the equations (21), (22). Instead there remain the two Monge-Ampère equations (18), (19) simplified by the removal of the terms involving Φ_{tt} . In addition it may be pointed out that the transformation from $\Psi(\lambda, w, t)$ to $\psi(\lambda, w, \xi)$ carried out in §6 is inapplicable in this case since $\xi = \Psi_t$, by (44), reduces to a function of (λ, w) only. According to (35) the acoustic speed also remains, in this case, independent of t . Therefore along any line for which $\mathbf{q}$ remains constant the acoustic speed also remains constant for all time. The flow then reduces precisely to a double wave, in the notation of GIESE.

One may write

$$\Phi = t\Gamma(u, v) + \Lambda(u, v)$$

and the field equations (4) become

$$t\Gamma_u + \Lambda_u + \mathbf{r} \cdot \mathbf{q}_u = 0, \quad t\Gamma_v + \Lambda_v + \mathbf{r} \cdot \mathbf{q}_v = 0.$$

In particular, when Λ is taken to be constant these become

$$t\Gamma_u + x + zw_u = 0, \quad t\Gamma_v + y + zw_v = 0$$

which imply that (u, v, w) reduce to functions only of the quantities $x/t, y/t, z/t$. Such flows are commonly referred to as “progressive waves” since the lines of constant speed advance linearly with the time. Therefore by confining attention to the case

$$\Phi = t\Gamma(u, v)$$

progressive waves may be constructed by the method.

REFERENCES

- [1] J. H. GIESE, Compressible Flows with Degenerate Hodographs, *Quart. Appl. Math.* **9** (1951), pp. 237–246.
- [2] J. H. GIESE, *Degenerate Unsteady Compressible Flows*, Aberdeen Proving Ground Ballistic Research Laboratories Report 894 (1954).
- [3] D. NAYLOR, Unsteady Simple Waves in Gases. *J. Rat. Mech. Anal.* **5** (1956), pp. 687–714.
- [4] C. E. WEATHERBURN, *Differential Geometry*, Vol. 1, Cambridge University Press, 1947, p. 76.
- [5] G. DARBOUX, *Leçons sur la théorie générale des surfaces*, Pt. 2, Gauthier-Villars, Paris, pp. 54–70.

University of Toronto