Kinematics of Homogeneous Turbulence

GARRETT BIRKHOFF &

J. KAMPÉ de FÉRIET*

INTRODUCTION

The idea of a statistical theory of turbulence goes back as far as J. BOUSSINESQ (1872) and O. REYNOLDS (1885); many of the ideas, now almost universally accepted, were introduced by Sir GOEFFREY TAYLOR between 1920 and 1935.† But it was really the creation, by the work of N. WIENER, P. LEVY and A. KOLMOGOROFF, of the modern theory of random functions which made possible a rigorous mathematical approach to the problem; the components of the velocity in a turbulent flow have to be considered as *random functions* of $\mathbf{x}$ and t .

In this paper, we consider homogeneous turbulent velocity fields at a particular time t , limiting ourselves to the kinematics of the problem. Hence we are concerned with random vector fields (R.V.F.) defined in infinite 3-dimensional Euclidean space R^3 —though for the sake of generality we consider R.V.F. with q components in p -dimensional space R^p .

There are several ways to define R.V.F. Since 1939 (see [8]), one of us has systematically considered an R.V.F. to be given by a vector function $\mathbf{u}(\mathbf{x}, \omega)$ defined and measurable on the product space $R^n \times \Omega$, where Ω is an unspecified *abstract* probability space. By choosing ω in Ω , one gets a *sample* velocity field. In [7], a systematic exposition of the kinematics of turbulence was given from this point of view, which has various mathematical advantages. It has, however, the disadvantage that the construction of such a function $\mathbf{u}(\mathbf{x}, \omega)$ from observational data is not described.

Therefore, in this paper, a different procedure is followed. We consider first a *sample probability space* Λ (see [11, p. 155]), which consists of all velocity fields $\mathbf{u}(\mathbf{x})$ which are measurable and have finite energy on compact sets. This class of fields seems sufficiently general. We define Λ as a topological linear space, and assume that the measure is “regular”, in the sense of being determined by the measures of Borel sets in Λ . In principle, one could consider the frequency

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†For a brief historical sketch, see [7, pp. 568–571].

with which $\alpha_{ik} < \int_{D_i} u_k(\mathbf{x}) dm(\mathbf{x}) < \beta_{ik}$ as a multivariate distribution determinable experimentally in the usual way as a statistical frequency. From such frequencies μ , the measure on Λ could be (approximately) determined experimentally, to any accuracy.

This was the point of view implicitly adopted by one of us in [2].* In Part A of the present paper, we show that it is effectively equivalent to that followed in [7]. Define a regular measure on Λ as "admissible" if it has finite kinetic energy expectation $\bar{E}_D$ on any compact set D of physical space R^p ; we show how to construct $\mathbf{u}(\mathbf{x}, \alpha)$ on the product space $R^p \times I$, where I is the unit interval, from any such admissible measure. We prove that any regular measure is strictly separable and metrically separable; also, we establish the connections between admissible measures and "measurably random vector fields" (i.e., $u_i(\mathbf{x}, \omega)$ measurable with respect to the product measure $m \times \mu$). We prove that the sets $(\mathcal{C})$, $(\mathcal{C}^k)$ and $(\mathcal{C}^\infty)$ of continuous vector fields, k -ply continuously differentiable vector fields, and vector fields having continuous derivatives of all orders are all measurable for any regular probability measure on Λ .

Part B is devoted to covariances and correlations. First we establish the fundamental properties of the covariance matrix $\Gamma_{i,k}(\mathbf{x}, \mathbf{y})$ of an admissible random velocity field; a prominent role is played by an integral equation of Hilbert-Schmidt type; our main result is that any matrix satisfying a set of conditions (explicitly stated) on every compact D is the covariance matrix of a unique normal admissible R.V.F. The consequences of continuity in the mean are established; they allow one to supplement the Hilbert-Schmidt technique by Mercer's Theorem and to prove pointwise convergence on compact D . The notions of homogeneity and isotropy (introduced by G. I. TAYLOR and TH. v. KARMAN) are examined. We prove that, for any homogeneous admissible R.V.F., the correlation matrix $R_{i,k}(\mathbf{x} - \mathbf{y}) = \Gamma_{i,k}(\mathbf{x}, \mathbf{y})$ is uniformly continuous. Finally, we show that the usual identities $\sum \partial R_{ik} / \partial h_i = \sum \partial R_{ik} / \partial h_k = 0$, and the new equation $\sum \partial^2 R_{ik} / \partial h_i \partial h_k(\mathbf{0}) = 0$, are *sufficient* as well as necessary for incompressibility.

In Part C, we introduce *spectral matrix measures* $S_{i,k}(K)$; the $S_{i,k}(K)$ are complex-valued Lebesgue-Stieltjes measures bounded on R^p and Hermitian symmetric for each measurable set K in wave vector space. We say that a complex homogeneous R.V.F. is associated with the spectral matrix measure $S_{i,k}(K)$ if its covariance $R_{i,k}(\mathbf{h})$ is the latter's Fourier-Stieltjes transform. We prove that the class of continuous positive definite $R_{i,k}(\mathbf{h})$ is identical with that of Fourier-Stieltjes transforms of spectral matrix measures; the correlation matrix of each admissible homogeneous R.V.F. is associated with one and only one spectral matrix measure and conversely. We examine with special attention the case of *real* vector fields, establishing a one-one correspondence between the class of normal admissible real homogeneous R.V.F. and the class of spectral matrix

*It corresponds closely to what Doob [3, p. 67] calls a process "of function space type", for which the sample function and the element of Ω coincide.

measures satisfying the symmetry condition $S_{j,k}(-K) = S_{j,k}(K)$. The particular case of isotropic, incompressible vector fields is examined; we conclude that the class of admissible normal R.V.F. having the properties usually assumed in the kinematic theory of homogeneous isotropic turbulence corresponds one-one to the class of energy distributions $\mathcal{E}(\kappa)$ in wave number space.

In Part D, to follow soon, it will be proved that any normal homogeneous random velocity field whose spectral matrix measure is absolutely continuous is *metrically transitive*.

A. RANDOM VECTOR FIELDS

1. Basic concepts. We shall consider below the set Ω of all vector fields

$$(1) \quad \mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), \dots, u_q(\mathbf{x})), \quad \mathbf{x} = (x_1, \dots, x_p),$$

with q components, in the p -dimensional Euclidean space R^p . In Ω , we shall identify vector fields which differ on a set of measure zero (are equal a.e.). One can also use the notation $\mathbf{u}(\mathbf{x}, \omega)$, where each $\omega \in \Omega$ refers to a *sample* vector field.

A *random vector field* (R.V.F) may be defined generally as a *probability measure* on Ω . That is, it is a triple $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is the σ -field of "measurable" sets, and $\mu(S)$ is a σ -additive, non-negative real function defined on $\mathcal{F}$, with $\mu(\Omega) = 1$. Since any such choice makes Ω into a probability measure "space", one can also regard $(\Omega, \mathcal{F}, \mu)$ as defined by the set of q real-valued functions $u_i(\mathbf{x}, \omega)$ defined on $R^p \times \Omega$. (For each ω , we allow ambiguity in $u_i(\mathbf{x}, \omega)$ on a set of Lebesgue measure zero in R^p . The special case $p = q = 1$ is often called a stochastic "process.")

The proper choice of $\mathcal{F}$ is far from clear *a priori*, and various suggestions have been made—the purpose of which is to ensure that the sets in $\mathcal{F}$ correspond to physically measurable quantities [2, p. 20, Postulate II].

An influential school, following KHINCHINE, defines $\mathcal{F}$ through the requirement that it must contain all "window sets" $a_{ij} < u_i(\mathbf{x}^j, \omega) < b_{ij}$ (cf. [3, p. 46]). Following WIENER [14], one of us has written a series of papers (including [7] and [8]) based on the different assumption that $\mathbf{u}(\mathbf{x}, \omega)$ was measurable on the *product space* $R^p \times \Omega$. BATCHELOR [1, p. 15, (2.1.1)], though avoiding the technical language of measure theory, requires in effect that "any" function

$$F(u_1(\mathbf{x}^1, \omega), \dots, u_q(\mathbf{x}^r, \omega))$$

should be integrable.

The main purpose of Part A is to show that the preceding requirements (properly interpreted) are *all* roughly equivalent to CARATHEODORY's concept of a "regular" measure, on the subspace consisting of Borel functions, suitably topologized. Moreover, "regularity" specializes in separable Banach spaces to the defining property of L -measure, as defined by Mlle. MOURIER [12], following FRÉCHET [4]. It is also closely related to the concept of a "perfect"

measure, as defined by KOLMOGOROFF & GNEDENKO.* Finally (see §7) it agrees with VON NEUMANN's general ideas about measurable quantities in classical physics.

2. Regular measures. We shall assume that μ is a "regular" probability measure on a suitable topological linear space Λ , to be defined in §4. To define "regularity", we shall need various standard measure-theoretic concepts.**

By the (Borel) *extension* μ of a given finitely additive set-function λ , is meant simply its extension by σ -additivity to a σ -field of sets. Such an extension is possible if and only if λ is "compatible". Any σ -additive set-function μ has a (Lebesgue) *completion* μ^* , defined by making $\mu^*(T) = \mu(S)$ if $(S \cap T') \cup (S' \cap T) \leq N$, where $\mu(N) = 0$.

Definition. By a *regular* measure on a topological space Ω , is meant one which is the Lebesgue completion of a measure defined on the field $B(\Omega)$ of all Borel sets† of Ω .

Lemma 1. If Ω is a metric space, the preceding definition of regularity is equivalent to that of Caratheodory.††

We omit the proof of this result, which is primarily of historical interest. Much more useful to us will be the following elementary principle.

Theorem 1. Let $(\Omega, \mathfrak{F}, \mu)$ and $(\Omega, \mathfrak{G}, \nu)$ be regular measures on any topological space Ω , and let $\mu(S) = \nu(S)$ for any generating family of Borel sets (e.g., any sub-basis of closed or open sets). Then $\mathfrak{F} = \mathfrak{G}$, and $\mu(S) = \nu(S)$ for all $S \in \mathfrak{F}$.

Proof. By σ -additivity, $\mu(S) = \nu(S)$ on all Borel sets; hence μ and ν have the same completion of their restrictions to Borel sets. Hence $\mathfrak{F} = \mathfrak{G}$, and the final conclusion is also obvious.

3. Banach spaces. Technically useful to us will also be the relation between "regularity", as defined above, and the Fréchet-Mourier concept of an L -measure, which we now give.

Definition. A measure μ on a Banach space $\mathfrak{B}$ is an L -measure, if and only if all linear functionals on $\mathfrak{B}$ are μ -measurable.

**Limit distributions for sums of independent random variables*, Cambridge, 1954, p. 18. See also D. BLACKWELL, *Third Berkeley Symposium on Statistics and Probability* (1955), Vol. II, pp. 1-6.

**We shall use, in general, the terminology of P. HALMOS, *Measure Theory*. The facts about measure theory used below are mostly proved there or in G. BIRKHOFF, *Lattice Theory*, Ch. XI, §8, where "compatibility" is defined.

†See [10, §26] for a relevant analysis of this classic concept; also, G. MACKEY, *Trans. Am. Math. Soc.* 85 (1957), pp. 134-165. We have had many stimulating discussions of Part A with Prof. MACKEY.

††C. CARATHEODORY, *Vorlesungen über reelle Funktionen*, Ch. V, esp. §§251, 271. The concept of a "Radon measure" in a locally compact space (N. BOURBAKI, *Integration*, Ch. III, esp. pp. 43, 50) is almost equivalent.

Lemma 2. *A measure on a Banach space is an L -measure, if and only if it is an extension of a “regular” measure for the weak topology.*

The proof is trivial. Since any extension of an L -measure is an L -measure, L -measures need not be regular, however.

The following result is known (cf. [12, p. 168]).

Lemma 3. *The Borel sets of any separable Banach space are the same, whether the weak or the metric topology is used.*

Corollary 1. *A measure on a separable Banach space is “regular” in the metric topology, if and only if it is “regular” in the weak topology.*

Corollary 2. *A measure on a separable Banach space is an L -measure, if and only if it is an extension of a “regular” measure in the strong topology.*

We shall now apply the above results to some special Banach spaces,* closely related to the topological linear spaces Λ (§4) and Γ (§8) which arise most naturally in connection with turbulence. First, we consider the space $C(D)$ of continuous functions on a compact domain D in R^n .

Lemma 4. *If D is compact, a probability measure on $C(D)$ is “regular” if and only if it is the (Lebesgue) completion of the (Borel) extension of a compatible measure defined on window sets.*

Proof. By a well known theorem of F. RIESZ, the linear functionals on $C(D)$ are the ordinary Stieltjes integrals

$$(2) \quad \int_D \sum u_i(\mathbf{x}) d\alpha_i(\mathbf{x}) = (\mathbf{u}, \alpha), \quad \int_D |d\alpha_i(\mathbf{x})| < +\infty.$$

For a suitable point-concentrated set function α_{ik} , the condition $(\mathbf{u}, \alpha) \leq b$ is, however, equivalent to $u_i(\mathbf{x}^i) \leq b_{ii}$. Hence the (weak or metric) Borel sets on $C(D)$ are generated by the “window sets”

$$(3) \quad a_{ij} \leq u_i(\mathbf{x}^j) \leq b_{ij} \quad (i = 1, \dots, q; j = 1, \dots, p; \mathbf{x}^j \in R^n).$$

The conclusion of Lemma 4 is now evident.

Next, we consider the space $L_2(D)$ of all functions quadratically integrable on a bounded domain D of R^n . Let $\mathbf{e}_1(\mathbf{x})$, $\mathbf{e}_2(\mathbf{x})$, $\mathbf{e}_3(\mathbf{x})$, $\dots$ be any orthonormal basis for $[L_2(D)]^q = [L_2(R^n)]^q$. Then every $\mathbf{u}(\mathbf{x}) \in [L_2(D)]^q$ can be expanded,

$$\mathbf{u}(\mathbf{x}) = \sum c_k \mathbf{e}_k(\mathbf{x}), \quad c_k = (\mathbf{u}, \mathbf{e}_k),$$

defining the usual isomorphism $[L_2(D)]^q \leftrightarrow l_2(J^+)$, where J^+ is the set of positive integers, and $l_2(J^+)$ is the set of all sequences $(c_1, c_2, c_3, \dots)$ such that $\sum c_n^2 < +\infty$.

*Many are considered in [12]; others by one of us in *C. R. Paris* 244 (1957), pp. 1119, 1850. See also [2]. We use below the notation of S. BANACH, *Théorie des opérations linéaires*, esp. p. 59. Here Riesz' Theorem is also proved.

Lemma 5. *The Borel sets in $L_2(D)$ are generated by the pseudo-rectangles $a_k \leq (\mathbf{e}_k, \mathbf{f}) \leq b_k$, for any (countable) orthonormal basis in $L_2(D)$.*

Proof. By [10, §26], it suffices to show that the closed spheres are so generated. But the closed sphere with center $\mathbf{f} = (f_1, f_2, f_3, \dots)$ and radius r in $L_2(J^+)$ is the intersection of the cylinder sets $S_n(\mathbf{f}, r)$ defined by $\sum_{k=1}^n |c_k - f_k|^2 \leq r$, and hence is in the σ -field generated by the pseudo-rectangles.

Corollary 1. *A measure in $L_2(D)$ is regular if and only if it is the completed Borel extension of a (σ -finite) additive set-function on the pseudo-rectangles $a_k \leq (\mathbf{e}_k, \mathbf{f}) \leq b_k$.*

4. The space Λ . In our opinion, the class of vector fields of greatest interest for turbulence consists of those defined by Borel functions which are *square-integrable on compact sets*. The linear “space” Λ of these vector fields can be topologized, by the definition

$$(4) \quad \mathbf{u}_n(\mathbf{x}) \rightarrow \mathbf{u}(\mathbf{x}) \quad \text{means} \quad \int_D |\mathbf{u}_n(\mathbf{x}) - \mathbf{u}(\mathbf{x})|^2 dm(\mathbf{x}) \rightarrow 0,$$

for any compact domain D . (Note that any compact domain, as a Borel set, is measurable; hence the integral on the right of (4) is defined for all $\mathbf{u}_n, \mathbf{u} \in \Omega$. Note also that (4) defines Λ as a *complete* topological linear space, i.e., every Cauchy sequence in Λ converges to a limit.)

Actually, there is one such space Λ_{pq} for each choice of positive integers p and q , but by (5) below these are all related by linear homeomorphisms, and so we shall usually ignore p and q in the sequel. In other words, the case $p = q = 1$ of a “stochastic process” is typical.

Alternatively, one can define Λ as the unrestricted direct topological product of countably many Hilbert spaces

$$(5) \quad \prod_1^\infty [L_2(I^n)]^q,$$

where the $I^n = [I(u)]^n$ are the unit cubes $k_i \leq x_i < k_i + 1$, for the different vectors $\mathbf{k} = (k_1, \dots, k_n)$ of integers. Or, finally, one can let D_n be the cube $|x_i| \leq 2^n$, and define Λ as the limit space defined by the inverse sequence of natural homomorphisms (actually, orthogonal projections)

$$(6) \quad [L_2(D_1)]^q \leftarrow [L_2(D_2)]^q \leftarrow [L_2(D_3)]^q \leftarrow \dots \leftarrow \Lambda.$$

We omit proving the equivalence of these definitions; formula (6) simply expresses the fact that the open sets in Λ are generated by the inverse images of the open sets in the $L_2(D_n)$; i.e., the latter form a neighborhood-basis in Λ .

Finally, since any sequence of nested compact sets $D_1 \leq D_2 \leq D_3 \leq \dots$ satisfying $\bigcup_1^\infty D_n = R^n$ could have been used in (6) instead of the cubes $|x_i| \leq 2^n$, we see that the definition of Λ is *invariant* under all *volume-preserving homeomorphisms* of R^n .

5. Borel sets in Λ ; separability. It is our claim that the Borel sets in Λ represent *exactly* properties of physical interest—just as the Borel sets in phase-space are commonly believed to represent measurable quantities in classical physics.* For a general bounded measurable subset D in R^n , the “energy in D ”

$$(7) \quad \int_D |\mathbf{u}(\mathbf{x}, \omega)|^2 dm(\mathbf{x}) = E_D(\omega)$$

is such a quantity of physical interest. We have

Lemma 1. *For any E , the set $S_D(E)$ of ω such that $E_D(\omega) \leq E$ is a Borel set in Λ .*

Proof. We use an analog of (5) to represent Λ as the direct topological union of $[L_2(D_i)]^a$, with $D_1 = D$. Then we observe that all closed spheres are Borel sets in $L_2(D)$ —see Lemma 5, §3, or [12, p. 168].

Corollary. *For any regular measure on Λ , the energy $E_D(\omega)$ on any compact $D \subseteq R^n$ is a non-negative measurable function, and so the energy expectation $\bar{E}(D) = \int_0^\infty E d\mu(S_D(E))$ is a well-defined non-negative real number or $+\infty$.*

Lemma 2. *For any measurable function $\mathbf{e}(\mathbf{x})$ with “compact support” (i.e., vanishing identically outside some compact D), the inner product $(\mathbf{e}, \mathbf{u})$ is a Borel function on Λ , and hence measurable under any regular measure.*

The proof is similar; again see Lemma 5 of §3. Other related results, of a deeper nature, will be proved in §8.

In turbulence, the energy expectation on compact sets must be finite, for physical reasons. Hence we make the following definition.

Definition. By an *admissible* measure, we mean a regular probability measure on Λ , for which the energy expectation on every compact set D is finite.

In this connection, we note that fluid motions whose difference involves zero energy are identified in Λ .

The word “separable” has been given at least three meanings in the literature on measure theory. Two of these are as follows.

Definition. A measure μ is called *strictly* separable when the Boolean σ -algebra of measurable sets, modulo null sets, is generated by a countable subset. It is called *metrically* separable when the metric space defined by the distance function

$$(8) \quad |S - T| = \mu(S \cup T) - \mu(S \cap T)$$

contains a countable dense subset.

Lemma 3. *Any regular measure on Λ is strictly separable, and hence metrically separable.*

*See J. VON NEUMANN, *Annals of Math.* **33** (1932), pp. 595–598.

Proof. A countable neighborhood-basis for Λ is provided by the open spheres $S_{r,n}$ in the $L_2(D_n)$ with rational radii r and centers constituting any countable dense set of $\mathbf{u}^{in}$ in the $L_2(D_n)$. Since the $L_2(D_n)$ are separable, such a countable neighborhood-basis exists. Finally, all Borel sets are generated by any neighborhood-basis, in any topological space.

L -measures, of course, need not be strictly separable.

6. Measurable case. Extending the definition of §1, one can define a measurably random vector field (or stochastic process [3, p. 60]) as one in which the function $\mathbf{u}(\mathbf{x}, \omega)$ is measurable on the product space* $R^p \times \Omega$. Random vector fields have been treated using this assumption by one of us, in a series of publications.** We shall now relate this definition to that of an admissible measure.

For such a measurably R.V.F., the energy expectation on any compact set D is naturally defined as

$$(9) \quad \overline{E(D)} = \iint_{\Omega \times D} |\mathbf{u}(\mathbf{x}, \omega)|^2 dm(\mathbf{x}) d\mu,$$

where $dm(\mathbf{x})$ denotes ordinary Borel measure on R^p . We assume $\overline{E(D)} < +\infty$. Since $\mathbf{u}(\mathbf{x}, \omega)$ is $(m \times \mu)$ -measurable on $R^p \times D$, one has first, by the Tonelli Theorem, $\mathbf{u}(\mathbf{x}, \omega) \in L_2(D \times \Lambda)$, and then, by the Fubini Theorem, $\mathbf{u}(\mathbf{x}, \omega) \in L_2(D)$ for almost all $\omega \in \Lambda$.

Lemma 4. Any measurable R.V.F. which satisfies (9) on every bounded set D , defines an admissible measure on Λ .

Proof. Let $\mathcal{E}$ be the subset of ω such that $\mathbf{u}(\mathbf{x}, \omega) \in L_2(D)$. By hypothesis and the Fubini Theorem applied to $|\mathbf{u}(\mathbf{x}, \omega)|^2$, $\mu(\Omega - \mathcal{E}) = 0$. Moreover, if the $\mathbf{e}_k(\mathbf{x})$ are an orthonormal basis for $L_2(D)$, then the inner products

$$(10) \quad c_k(\omega) = \int_D \mathbf{e}_k(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}, \omega) dm(\mathbf{x}) = (\mathbf{e}_k, \mathbf{u})_D$$

are defined and finite for all $\omega \in \mathcal{E}$, by the Schwarz Inequality. By the Fubini Theorem, the pseudo-rectangles of Lemma 5, §3, are also measurable in Ω . Hence every Borel set in $L_2(D)$ is measurable on the probability space Ω . But these sets generate the Borel sets of Λ , as shown in §4, completing the proof.

In §9, we shall prove a converse to Lemma 4, but this is much harder.

7. Essentially continuous functions. A fundamental result of N. WIENER† is the fact that particle-paths in (ideal) Brownian motion are continuous with

*The measurable sets being defined as in P. HALMOS, *Measure Theory*, p. 137.

**Beginning with [8]; see esp. [7], and *Sem. Darboux Institut Henri Poincaré*, 1956, pp. 1–19.

†*J. Math. Phys. MIT* 2 (1923), pp. 131–174. For DOOB's ideas, see J. L. DOOB, *Trans. Am. Math. Soc.* 42 (1937), pp. 107–140, and [3]. An excellent discussion is given in J. L. DOOB & W. AMBROSE, *Annals of Math.* 41 (1940), pp. 737–745.

probability one. However, the “measurability” of the class of continuous functions is not easy to correlate with measures defined by window sets. One solution to this problem is given by Doob’s concept of “separability”, a concept which is only remotely related to strict and metric separability as defined in §5.

We shall now show that CARATHEODORY’s concept of a “regular” measure, as applied to the space Λ , gives a simple alternative solution to the problem. Namely, the class of *essentially* continuous functions (*i.e.*, those differing from continuous functions on a set of measure zero in R^p) is measurable, under *any* regular measure on Λ . This is an obvious corollary of the following result.

Theorem 2. *The set $\mathcal{C}$ of essentially continuous functions is a Borel set in Λ .*

Proof. We use the inner product notation

$$(11) \quad (\mathbf{a}, \mathbf{u})_D = \sum_i \int_D a_i(\mathbf{x}) u_i(\mathbf{x}) \, dm(\mathbf{x})$$

for any bounded domain D (or, equivalently, over R^p , if $\mathbf{a}$ has compact support D). This is always a continuous linear functional on $[L_2(D)]^q$; hence, by (6), it is one on Λ . Hence, if B is any Borel set of real numbers, the set of “sample” vector fields $\mathbf{u}(\mathbf{x}; \omega)$ such that

$$(12) \quad (\mathbf{a}, \mathbf{u})_D \in B \quad \text{defines a Borel set in } \Lambda.$$

For the set (12) is the inverse image of a Borel set under a continuous transformation.

In particular, suppose $\mathbf{a}^i(\mathbf{x}) = \delta_{ii}$, the Kronecker delta-function (*i.e.*, $a_i^i(\mathbf{x}) = 1$, while $a_i^j(\mathbf{x}) = 0$ if $i \neq j$). Then, for any compact domain D in R^p with positive measure $m(D)$, clearly $(\mathbf{a}^i, \mathbf{u})_D / m(D)$ denotes the *average* $A(u_i, D)$ of $u_i(x)$ on D . Hence, by (12), if D and D^* are any two such domains the set $S_i(D, D^*, 1/s)$ of $\mathbf{u} \in \Lambda$ such that

$$(13) \quad |A(u_i, D) - A(u_i, D^*)| = \frac{1}{m(D)} [(\mathbf{a}^i, \mathbf{u})_D - (\mathbf{a}^i, \mathbf{u})_{D^*}] < \frac{1}{s}$$

is a Borel set in Λ .

Clearly $\mathbf{u} \in \mathcal{C}$ if and only if $\mathbf{u}(\mathbf{x})$ is essentially *uniformly* continuous on each compact cube D_n , defined by $|x_i| \leq 2^n$ for all i . Let us call T_r the countable set of pairs of compact cubes $D, D^* \subset D_n$ with rational vertices, such that the diameter of $D \cup D^*$ in R^p is less than $1/r$. Then for each D_n the set C^n of all functions essentially uniformly continuous in D_n is definable as the Borel set

$$\bigwedge_{i=1}^{s-1} \{ \bigvee_{r=1}^{r-1} [\bigwedge_{T_r} S_i(D, D^*, 1/s)] \}, \quad m(D) = m(D^*).$$

Hence $C = \bigwedge C_n$ is also a Borel set.

A similar proof applies to the set $\mathcal{C}^{(k)}$ of k -ply continuously differentiable vector fields on R^p , for any positive integer k . Thus

Theorem 3. *For any positive integer k , the set $\mathcal{C}^{(k)}$ of k -ply continuously differ-*

entiable vector fields is measurable, under any regular probability measure on $\mathcal{C}$. Hence so is the set $\mathcal{C}^{(\infty)}$ of vector fields having continuous derivatives of all orders.*

8. The space Γ . We now define a *continuous* R. V. F. as a regular probability measure on Λ such that $\mu(\mathcal{C}) = 1$. Since any continuous function is quadratically integrable on compact sets, such a measure is always “admissible”, in the sense of §5. Moreover, in Λ , each point which represents an essentially continuous function, represents one and only one *continuous* function. This isomorphism maps any continuous R.V.F. onto a probability measure on the set Γ of continuous vector fields.

Indeed, it may be argued that viscosity always makes $\mathbf{u}(\mathbf{x})$ continuous, and that therefore Γ is more fundamental than Λ . However, our approach has the advantage of permitting *discontinuous* fluid motions (shock waves and slip-streams), as well as other discontinuous R.V.F. It also accords with general ideas about measurable quantities in classical physics, as being defined by Borel sets in phase-space (see §5). Our view is that the case $\mu(\mathcal{C}) = 1$ is simply a very important special case.

In this special case, one can develop a consistent analog of the basic general theory of §5, by making Γ into an appropriate *topological linear space*. In this space, we define convergence to mean uniform convergence on compact sets.† Equivalently, we can define it by the inverse sequence of continuous linear transformations, analogous to those of (6),

$$(15) \quad [C(D_1)]^a \leftarrow [C(D_2)]^a \leftarrow [C(D_3)]^a \leftarrow \cdots \leftarrow \Gamma.$$

The Borel sets of Γ are generated by the inverse images of the open spheres in the $[C(D_k)]^a$.

By Lemma 4 of §3, the “regular” measures on Γ are thus the complete extensions of the “compatible” (see §2) measures on window sets as ordinarily defined. Thus, for continuous processes, the Borel sets in Γ are the measurable sets of KHINCHINE.** We now prove that they are also the Borel sets in Λ , when $\mu[\mathcal{C}] = 1$. Stated informally, the class of continuous processes is the class of regular probability measures on Γ . More precisely,

Theorem 4. *The regular measures on Λ with $\mu(\mathcal{C}) = 1$ define, through the natural isomorphism $\mathcal{C} \rightleftharpoons \Gamma$, the regular measures on Γ .*

Proof. The Khinchine window set (3) is the intersection of $K_{i,j}$ in Λ , each defined by a single inequality $a_{i,j} \leq u_i(\mathbf{x}^j) \leq b_{i,j}$ for a single $\mathbf{x}^j \in R^p$. But clearly, in $\mathcal{C}$, $a \leq u_i(\mathbf{x})$ refers to the set of $\mathbf{u} \in \Lambda$ satisfying $\limsup A(u_i, I_m) \geq a$, where

*In [1, Ch. III], the assumption $\mu[\mathcal{C}^{(\infty)}] = 1$ is apparently used tacitly. In [7, p. 603], the weaker assumption $\mu[\mathcal{C}^{(4)}] = 1$ is made.

†This is the most convenient topology for continuous functions. See R. F. ARENS, *Annals of Math.* 47 (1946), pp. 480–495; also R. H. FOX, *Bull. Am. Math. Soc.* 51 (1945), pp. 429–432.

**A. KHINCHINE, *Math. Annalen* 109 (1934), pp. 604–615. Since intervals generate Borel sets in any σ -field, this is equivalent to Doob’s definition [3, p. 47].

I_m denotes the cube of side 2^{-m} centered at $\mathbf{x}$. Hence, if $S(i, m, s)$ denotes the set of $\mathbf{u}(\mathbf{x})$ satisfying $A(u_i, I_m) \geq a - 2^{-s}$, then $S(i, m, s)$ is a Borel set in Λ by (12), and so

$$(16) \quad K_{ii} = \bigvee_i \left\{ \bigwedge_m \left\{ \bigwedge_{m>M} S(i, m, s) \right\} \right\}$$

is a Borel set in Λ . Hence, if $(\Lambda, \mathfrak{F}, \mu)$ is any continuous process, the Borel sets of Γ all lie in $\mathfrak{F}$ (cf. the application of Lemma 4 of §3, three paragraphs above).

Conversely, let $(\Gamma, \mathfrak{F}, \nu)$ be any regular measure on Γ . Let $(\Lambda, \mathfrak{G}, \nu^*)$ be the "equivalent" measure on Λ , defined by making $\nu^*(S) = \nu(S \cap \mathcal{C})$, through the isomorphism $\Gamma \rightleftharpoons \mathcal{C}$. Trivially, this is σ -additive, non-negative and satisfies $\nu^*(\mathcal{C}) = 1$. To complete the proof of Theorem 4, it suffices to show that every Borel set T in Λ *does* correspond under $T \rightarrow T \cap \mathcal{C}$ to a Borel set in Γ . By Lemma 5 of §3, and (6), it suffices to show that, for some orthonormal basis of $\mathbf{e}_k$ in any $[L_2(D_n)]^q$, the set of $S(k, \alpha)$ defined by the inequality $(\mathbf{e}_k, \mathbf{u}) < \alpha$ is a Borel set in Γ . Bases of *continuous* orthonormal functions can, however, be found (e.g., by multiple Fourier series), and, for any continuous $\mathbf{e}_k$ and $\mathbf{u}$, $(\mathbf{e}_k, \mathbf{u}) < \alpha$ is equivalent to a statement about the limit of a sequence of finite Riemann sums. Since any inequality involving finite Riemann sums defines a set in the σ -field generated by window sets, it is clear that $(\mathbf{e}_k, \mathbf{u}) < \alpha$ *does* define a Borel set in Γ , completing the proof.

Corollary. For probability measures with $\mu(\mathcal{C}) = 1$, regularity in Λ is equivalent to regularity in Γ .

9. Measurable case, II. One of us has previously assumed [7, pp. 596, 602] that the R.V.F. which arose in turbulence were *measurably* random (with $\mu[\mathcal{C}^{(4)}] = 1$), and had finite energy expectation $\overline{E(D)}$ on any compact set D . By Lemma 4 of §6, they therefore define *admissible* measures on Λ . We shall now prove that, conversely, any admissible measure on Λ defines a measurably R.V.F. This result resembles Thm. 2.5 of [3, p. 60], but makes weaker hypotheses. It also settles some questions* raised in [7, pp. 572–3].

The correspondence between such measurably R.V.F. $\mathbf{u}(\mathbf{x}, \omega)$ and admissible measures $(\Omega, \mathfrak{F}, \mu)$ is at best many-one. This is evident since, if $\mathbf{u}(\mathbf{x}, \omega)$ is measurable, and $\omega' = f(\omega)$ is any one-one measure-preserving Borel transformation of Λ , then $\mathbf{u}(\mathbf{x}, \omega)$ and $\mathbf{v}(\mathbf{x}, \omega) = \mathbf{u}(\mathbf{x}, f(\omega))$ define "equivalent" measures on Λ , in the sense that $\mu(S) = \nu(S)$ for all $S \in \mathfrak{F}$. (Cf. Thm. 1 of §2.)

Our main result (Thm. 5 below) asserts that such a many-one correspondence does, in fact, hold; the range of the correspondence defined by Lemma 4 of §6 includes *all* admissible measures. To formulate and prove this result, it will be convenient, however, to take a special orthonormal basis of p -dimensional *Haar functions* in R^p ; these we now define.

*Thus the "swollen monster" of class Ω of all real vector fields is effectively deflated by the simple requirement $\mu[\Lambda] = 1$.

If $\mathbf{n} = (n_1, \dots, n_p)$ and $\mathbf{k} = (k_1, \dots, k_p)$ are any vectors having non-negative integers, resp. integers, as components, then we can define

$$(17) \quad H_{\mathbf{kn}}(\mathbf{x}) = \begin{cases} 0 & \text{unless } k_i \leq x_i < k_i + 1 \\ h_{n_1}(x_1 - k_1) \cdots h_{n_p}(x_p - k_p) & \text{if } k_i \leq x_i < k_i + 1. \end{cases}$$

Here $h_n(x)$ is the ordinary one-dimensional Haar function* defined on $0 \leq x < 1$. The $H_{\mathbf{kn}}(\mathbf{x})$ are orthonormal *step*-functions, with the fundamental property that the characteristic function of any cube having faces $x_i =$ dyadic fractions, is a *finite* linear combination of the $H_{\mathbf{kn}}(\mathbf{x})$ with constant coefficients.

Theorem 5. *If μ is any admissible probability measure on Λ , then there exists a Borel function $c(\alpha)$ from the unit interval I , $0 \leq \alpha \leq 1$, to Λ such that*

$$(18) \quad u_i(\mathbf{x}, \alpha) = \sum_{\mathbf{k}, \mathbf{n}} c_{\mathbf{k}, \mathbf{n}, i}(\alpha) H_{\mathbf{k}, \mathbf{n}}(\mathbf{x})$$

is a Borel function of $R^p \times I$ to R^q . Moreover, the measure μ^* defined by $\mathbf{u}(\mathbf{x}, \alpha)$, through Lemma 4 of §6, is equivalent to μ .

This is our main result; it can be proved by adapting a construction of HALMOS & VON NEUMANN** to the present case.

10. Details of proof. We first consider the case of a compact cube D in R^p . We prove the

Lemma. *If μ is any admissible probability measure on Λ , and D is any compact cube in R^p , then there exists a Borel function $\gamma(\alpha)$ from I to $l_2(J^+)$, such that*

$$(19) \quad \mathbf{u}(\mathbf{x}, \alpha) = \sum \gamma_k(\alpha) \mathbf{H}_k(\mathbf{x})$$

is a Borel function from $D \times I$ to R^q . This defines μ on $L_2(D)$ up to equivalence, through the inverse mapping from pseudo-rectangles $a_k \leq (\mathbf{H}_k, \mathbf{u}) \leq b_k$ to Borel sets in I .

Proof. The p -dimensional scalar Haar functions $\mathbf{H}_{on}(x)$ on the unit cube $0 \leq x_i < 1$ are countable; so are the *vector* Haar functions having all q components zero save one; hence we can arrange them in an ordinary sequence. By a change of linear scale in R^p , and renormalization, we get a basis of (orthonormal) Haar functions on D . Hence we can expand each $\mathbf{u} \in \Lambda$ on D as a series

$$(20) \quad \mathbf{u}(\mathbf{x}) = \sum c_k(\omega) \mathbf{H}_k(\mathbf{x}) \quad \text{on } D, \quad c_k = (\mathbf{u}, \mathbf{H}_k)_D.$$

In (20), ω refers to the "sample" $\mathbf{u} \in \Lambda$; by Lemma 5 of §3, the $c_k(\omega)$ are Borel functions on $\mathfrak{S} = L_2(D)$ and so, by (6), on Λ . Moreover, $\sum c_k^2(\omega) < +\infty$ for all $\mathbf{u} \in \Lambda$.

*See S. KACZMARZ & H. STEINHAUS, *Theorie der Orthogonalreihen*, Warsaw, 1935, pp. 120, 125.

***Annals of Math.* 43 (1942), pp. 332–350, esp. Theorem 2. GEORGE MACKEY has remarked privately that the lemma of §10 holds for general complete separable metric spaces D , using results of KURATOWSKI [10].

We now construct an almost measure-preserving Borel mapping $c(\alpha, \mu)$ from the unit interval I onto the Hilbert space $L_2(C) = \mathfrak{S}$ of coefficients $c_k(\omega)$; by (20), this will be onto $L_2(D)$ as well. To construct $c(\alpha, \mu)$, we make a nested sequence of partitions π_s of $\mathfrak{S}$ into a finite number $N(s)$ of coordinate rectangles $E_{s,i}$, and associate with each such $E_{s,i}$ an open interval $\sigma(E_{s,i})$ of length $\mu(E_{s,i})$ on I . Except for the countable points of subdivision, each $\alpha \in I$ will be contained in (usually the limit of) a unique nested sequence of $\sigma(E_{s,i})$. By the construction, the corresponding $E_{s,i}(\alpha)$ will be a *convergent* nested sequence in $\mathfrak{S}$, whose limit, as $s \rightarrow \infty$, we take for $c(\alpha, \mu)$. The details follow.

Let $k(s)$ be the s^{th} term in the sequence

$$(21) \quad 1; 1, 2; 1, 2, 3; 1, 2, 3, 4; \dots$$

We suppose π_{s-1} to have been constructed, and consider an individual $E = E_{s-1,i}$. By induction on s , we may assume that $c_{k(s)}$ has either a fixed value γ_0 on E , or is defined by an inequality $a < c_{k(s)} < b$, the possibilities $a = -\infty$, $b = +\infty$ being allowed. If $c_{k(s)} = \gamma_0$, we do not subdivide further, but let E be a coordinate rectangle for s . Otherwise, $c_{k(s)}$ is defined by an inequality $a < c_{k(s)} < b$. If a and b are finite, we subdivide E into subrectangles $a < c_{k(s)} < \frac{1}{2}(a+b)$, $\frac{1}{2}(a+b) < c_{k(s)} < b$, and $c_{k(s)} = \frac{1}{2}(a+b)$. If $a = -\infty$, $b = +\infty$, we subdivide into $-\infty < c_{k(s)} < 0$, $c_{k(s)} = 0$, and $0 < c_{k(s)} < +\infty$. If $a = -\infty$, but b is finite, we subdivide into $-\infty < c_{k(s)} < b-1$, $c_{k(s)} = b-1$, and $b-1 < c_{k(s)} < b$; and similarly if a is finite but $b = +\infty$. In all cases, we subdivide $\sigma(E)$ into corresponding open subintervals $\sigma(E')$, $\sigma(E'')$ and $\sigma(E''')$ of lengths $\mu(E')$, $\mu(E'')$ and $\mu(E''')$, respectively. Because of the possibility $\mu(E') = 0$, etc., a set of μ -measure zero in $\mathfrak{S}$ may be mapped onto an empty set.

By construction, since $\|c\| < \infty$ for all $c \in \mathfrak{S}$, each coordinate of *almost* all $c \in \mathfrak{S}$ will define a nested sequence of rectangles $\{E_{s,i(s)}\}$ mapped onto a nested sequence of *open* intervals in I . Conversely, almost all $\alpha \in I$ will be limits of nested sequences of $\sigma(E_{s,i(s)})$, on which each c_k converges to a *finite* value with probability one. This defines a *Borel function* $c(\alpha, \mu)$, under which the inverse image of any $E_{s,i(s)}$ is precisely $\sigma(E_{s,i(s)})$. This defines a basis of Borel sets in $\mathfrak{S}$, by Lemma 5 of §3; since each $\sigma(E_{s,i})$ has length $\mu(E_{s,i})$, the Borel mapping is measure-preserving; this proves the lemma (*i.e.*, Theorem 5 for D).

To extend this result to Λ , it suffices to consider *all* Haar functions on R^p , generalizing suitably the order (21). The analog of (19) will then define $\mathbf{u}(\mathbf{x}, \alpha)$ on *all* R^p . On any compact cube D , the preceding argument will apply; $\mathbf{u}(\mathbf{x}, \alpha) \in L_2(D)$ with probability one on D . Hence $\mathbf{u}(\mathbf{x}, \alpha) \in \Lambda$ with probability one, giving the Borel function whose existence is asserted in Thm. 5. We omit the discussion of measure-conservation.

11. Essentially compact measures. For Part B, we shall need the concept of an “essentially compact” measure.* We define a probability μ on a topological

*This is related to the “semi-compactness” of HOPF [6, §16]. S. ULAM has obtained results very close to ours.

space H to be *essentially compact* if and only if H contains compact subsets T with arbitrarily small measure deficiency $1 - \mu(T)$.

Lemma. *A probability μ on a complete metric space H is essentially compact if and only if, for every $\epsilon > 0$, H contains a subset T_ϵ with measure deficiency $1 - \mu(T_\epsilon) < \epsilon$, having a finite ϵ -dense subset.*

Proof. The necessity is obvious, since any compact metric space contains, for every $\epsilon > 0$, a finite ϵ -dense subset. We now prove sufficiency. Without loss of generality, we can assume the T_ϵ of the lemma to be closed. For any $\eta > 0$, form the intersection T of $T_{\eta/2}$, $T_{\eta/4}$, $T_{\eta/8}$, $\dots$. This has measure deficiency η ; yet, for any $\epsilon > 0$, T contains a finite ϵ -dense subset. Hence, since T is closed, it is sequentially compact,* completing the proof.

Theorem 6. *Any "regular" measure μ , on the Hilbert space $H = [L_2(D)]^q$ of vector fields $\mathbf{u}(\mathbf{x})$ having finite energy on D , is essentially compact.*

Proof. Fix an orthogonal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots$ in H . Let Σ_n denote the finite-dimensional subspace of H spanned by $\mathbf{e}_1, \dots, \mathbf{e}_n$; let $\Sigma(R)$ be the closed sphere $\|\mathbf{u}\| \leq R$; let $\Sigma_n(R) = \Sigma_n \cap \Sigma(R)$; let $T_n(\delta)$ be the (closed) set of $\mathbf{u} = \sum h_i \mathbf{e}_i$, $\mathbf{u} \in H$ such that $\sum_{i=1}^{\infty} h_i^2 \leq \delta^2$. Clearly $\bigcup_n \Sigma(R) = H$; moreover since $\sum_{i=1}^{\infty} h_i^2(\mathbf{u}) \downarrow 0$ for all $\mathbf{u} \in H$, $\bigcup_n T_n(\delta) = H$.

Hence, for any $\epsilon > 0$, we can choose n so large that $\mu(T_n(\frac{1}{2}\epsilon)) > \mu(H) - \frac{1}{2}\epsilon$. We can also choose R so large that $\mu(\Sigma(R)) > \mu(H) - \frac{1}{2}\epsilon$. Therefore, if $T_\epsilon = T_n(\frac{1}{2}\epsilon) \cap \Sigma(R)$, $\mu(T_\epsilon) > \mu(H) - \epsilon$. Moreover, if $\mathbf{u} \in T_n(\delta)$ and $\mathbf{u}_n$ denotes $\sum_{i=1}^n h_i \mathbf{e}_i$, $\mathbf{u}_n \in \Sigma_n$, then clearly $\|\mathbf{u} - \mathbf{u}_n\| \leq \delta$. Hence $\|\mathbf{u} - \mathbf{u}_n\| \leq \frac{1}{2}\epsilon$ for any $\mathbf{u} \in T_\epsilon$; but also $\mathbf{u}_n \in \Sigma_n(R)$ for any $\mathbf{u} \in T_\epsilon$, and $\Sigma_n(R)$ contains a finite $(\frac{1}{2}\epsilon)$ -dense subset. This subset will therefore be ϵ -dense on T_ϵ . By the preceding lemma, this completes the proof.

12. Hypotheses of Batchelor. As BATCHELOR [1, Ch. II] did not formulate his hypotheses in precise measure-theoretic terms (*cf.* [1, p. x]), it is not easy to show the relation between his ideas and ours. His basic assumption seems to be the *integrability* of "any" function $F(u_{11}, \dots, u_{ra})$ on R^{r^q} , where $u_{ik} = u_k(\mathbf{x}^i)$, the choice of $\mathbf{x}^1, \dots, \mathbf{x}^r$ being arbitrary [1, p. 15, (12.1.1)].

If we replace the concept of "any" function by that of *Borel* function on R^{r^q} ("definable" function, in the sense of Borel), and integrability by measurability,** we can show that our framework provides an adequate measure-theoretic foundation for BATCHELOR's ideas also.

For the class of Borel functions is generated by that of continuous functions, under rational operations and passage to the limit. For any *continuous* $F(u_{11}, \dots, u_{ra})$, each set $\alpha < F(u_{11}, \dots, u_{ra}) < \beta$ is however in the σ -field

*F. HAUSDORFF, *Mengenlehre*, 2nd ed., p. 108.

**The assumptions are equivalent for *bounded* functions; the integrability of all unbounded Borel functions would exclude normal R. V. F. (see Part C).

generated by the sets $\alpha_{ik} < u_k(\mathbf{x}^i) < \beta_{ik}$, and the latter are μ -measurable under any regular measure μ , by Lemma 4 of §3. Since the class $\mathfrak{M}$ of measurable sets is a σ -field, it follows that

Theorem 7. Any Borel function $F(u_{11}, \dots, u_{rq})$, where $u_{ik} = u_k(\mathbf{x}^i)$, is a measurable function on Ω for any regular measure μ .

For any regular measure μ , the partial derivatives of the $\mathbf{u}(\mathbf{x}^i)$, when they exist (cf. Thm. 3), are also measurable functions—and so is any Borel function of these partial derivatives. These remarks support some of the tacit assumptions of [1, p. 22], by giving them plausible yet rigorous mathematical interpretations.

BACHELOR's hypotheses as to *integrability* are, however, questionable as to substance. Thus, it is now generally conceded that the discussion of "Loitsiansky's invariant" in [1, p. 92] breaks down, because the measurable functions involved are not integrable ("the integrals do not converge").*

The implications of various other statements of [1] should probably also be re-examined, in the light of the general theory of measure and integration. The statements of [1, p. 21] suggests that measures are not only required to be *admissible* ([1, (2.3.1)] with $m = 2$), but more. For the limit to exist in [1, (2.1.2)], V cannot be arbitrary, but must be restricted (e.g., to a family of spheres or cubes). Finally, the existence of the left side of [1, (2.5.1)] seems improbable. However, we do not attempt a systematic study of these questions here.**

B. COVARIANCE AND CORRELATION

1. Covariance matrix. From now on, we will consider only admissible R. V. F. ($\Lambda, \mathfrak{F}, \mu$) in the sense of Part A, §5. By §9 of Part A, any such R.V.F. can be represented by a measurable Borel function $\mathbf{u}(\mathbf{x}, \alpha)$ from $R^p \times I$ to R^q , where I is the unit interval $0 \leq \alpha \leq 1$. Such a representation defines a measure-preserving correspondence between Λ and I , and hence allows one to think of $d\mu$ as $d\alpha$.

For any compact set D in R^p , we will then have

$$\bar{E}_D = \sum_i \int_I \left[\int_D u_i^2(\mathbf{x}, \alpha) dm(\mathbf{x}) \right] d\mu(\alpha) < +\infty.$$

We will call $\bar{E}_D$ the mean value or expectation of the kinetic energy on D , disregarding the factor $\frac{1}{2}\rho$, which could always be made equal to one by a suitable choice of mass unit.

Hence, by the Tonelli and Fubini Theorems, as in §6 of Part A, we have

$$(a) \quad \mathbf{u}(\mathbf{x}, \alpha) \in [L_2(D \times \Lambda)]^q$$

*See [2, p. 44], and I. PROUDMAN & W. H. REID, *Proc. Roy. Soc. A*247 (1954), esp. p. 183.

**Some of them are equivalent to hypotheses about the "energy spectrum", for large or small wave-numbers.

$$(b) \quad \bar{E}(\mathbf{x}) = \sum_i \left[\int_{\Lambda} u_i^2(\mathbf{x}, \alpha) d\mu(\alpha) \right] < +\infty, \quad \text{a.a. } \mathbf{x} \in D,$$

$$(b') \quad E_D(\alpha) = \sum_i \int_D u_i^2(\mathbf{x}, \alpha) dm(\mathbf{x}) < +\infty, \quad \text{a.a. } \alpha \in \Lambda,$$

$$(c) \quad \bar{E}_D = \int_D \bar{E}(\mathbf{x}) dm(\mathbf{x}) = \int_{\Lambda} E_D(\alpha) d\mu(\alpha).$$

Since $\mu(\Lambda) = 1$, (b) implies the existence of

$$(1) \quad v_i(\mathbf{x}) = \int_{\Lambda} u_i(\mathbf{x}, \alpha) d\mu(\alpha)$$

for almost all (a.a.) $\mathbf{x} \in D$, hence for almost all $\mathbf{x} \in R^p$.

Clearly, $\mathbf{v}(\mathbf{x})$ defines the *mean velocity* of the given R.V.F. It is important for turbulence in pipes and channels. In such cases, the fluctuating or *turbulent* component of the velocity is defined as

$$(2) \quad u'_i(\mathbf{x}, \alpha) = u_i(\mathbf{x}, \alpha) - v_i(\mathbf{x}),$$

and the *relative turbulence* level in D as the ratio

$$(2') \quad \frac{\sum_i \left[\int_D u_i'^2(\mathbf{x}, \alpha) dm(\mathbf{x}) d\mu(\alpha) \right]}{\int_D \left[\sum_i u_i^2(\mathbf{x}) dm(\mathbf{x}) \right]}.$$

In such cases, $\mathbf{v}(\mathbf{x})$ is very important.

However, in *homogeneous* turbulence, we can make $\mathbf{v}(\mathbf{x}) = 0$. In this case, the *covariance matrix*

$$(3) \quad \Gamma_{ik}(\mathbf{x}, \mathbf{y}) = \int_{\Lambda} u_i(\mathbf{x}, \alpha) u_k(\mathbf{y}, \alpha) d\mu$$

is far more important. By (a) and (b) above, this is defined for almost all $(\mathbf{x}, \mathbf{y})$ in $D \times D$. Moreover, by the Schwarz inequality and (b),

$$(4) \quad \sum_{i,k} \Gamma_{ik}^2(\mathbf{x}, \mathbf{y}) \leq \bar{E}(\mathbf{x}) \bar{E}(\mathbf{y})$$

whence, due to (c),

$$(4') \quad \Gamma_{ik}(\mathbf{x}, \mathbf{y}) \in [L_2(D \times D)]^{2q}.$$

Elementary computations also prove that

$$(5) \quad \Gamma_{ik}(\mathbf{x}, \mathbf{y}) = \Gamma_{ki}(\mathbf{y}, \mathbf{x}) \quad (\text{symmetry}),$$

$$(6) \quad \sum_{i,k} \iint_{D \times D} \Gamma_{ik}(\mathbf{x}, \mathbf{y}) h_i(\mathbf{x}) h_k(\mathbf{y}) dm(\mathbf{x}) dm(\mathbf{y}) \geq 0$$

for any $\mathbf{h}(\mathbf{x}) \in [L_2(D)]^q$ (positive type).

In summary, we have proved

Lemma 1. *The covariance matrix (3) of any admissible R.V.F. satisfies (4), (4'), (5) and (6).*

2. Eigenfunctions on D . On any compact domain D , one may consider the vector integral equation

$$(7) \quad \int_D [\sum_k \Gamma_{jk}(\mathbf{x}, \mathbf{y}) \phi_k(\mathbf{y}) dm(\mathbf{y})] = \lambda \phi_j(\mathbf{x}), \quad j = 1, \dots, q.$$

This is a system of integral equations of Hilbert-Schmidt type,* by Lemma 1. Hence, by the classical theory of symmetric Hilbert-Schmidt kernels of non-negative type, we know that

Lemma 2. *On any compact domain D , the covariance matrix (3) of any admissible R.V.F. determines a countable orthonormal basis of vector eigenfunctions $\phi_n(\mathbf{x})$, with non-negative eigenvalues λ_n , such that*

$$(8) \quad \sum_{i,k} \iint_{D \times D} \Gamma_{ik}(\mathbf{x}, \mathbf{y}) \phi_{ni}(\mathbf{x}) \phi_{nk}(\mathbf{y}) dm(\mathbf{x}) dm(\mathbf{y}) = \lambda_n,$$

and

$$(8') \quad \sum_1^\infty \lambda_n^2 \leq \sum_{i,k} \left\{ \iint_{D \times D} \Gamma_{ik}(\mathbf{x}, \mathbf{y})^2 dm(\mathbf{x}) dm(\mathbf{y}) \right\} < +\infty.$$

(See also (4)–(4').) In fact, we can strengthen the inequality (8') to the sharper relation

$$(8'') \quad \sum_1^\infty \lambda_n = \bar{E}_D.$$

To prove (8''), we consider the sequence of random variables defined by

$$(9) \quad a_n(\alpha) = \int_D \sum_i u_i(\mathbf{x}, \alpha) \phi_{ni}(\mathbf{x}) dm(\mathbf{x}).$$

Averaging the formula

$$a_n(\alpha)^2 = \sum_{i,k} \int_{D \times D} u_i(\mathbf{x}, \alpha) u_k(\mathbf{y}, \alpha) \phi_{ni}(\mathbf{x}) \phi_{nk}(\mathbf{y}) dm(\mathbf{x}) dm(\mathbf{y})$$

and using (8), we have

$$(10) \quad \overline{a_n(\alpha)^2} = \int_A a_n^2(\alpha) d\mu(\alpha) = \lambda_n.$$

*A. C. Zaanen, *Linear analysis*, Noordhoff, 1954, Ch. XIV, §§3–4.

Now, from (10), by the Parseval-Bessel equality

$$(10') \quad \sum_1^{+\infty} a_n(\alpha)^2 = \sum_i \int_D u_i(\mathbf{x}, \alpha)^2 dm(\mathbf{x}) = E_D(\alpha).$$

Integrating over Λ (*i.e.*, averaging), and applying the dominated convergence theorem of Lebesgue to the left hand member, we have, by (10), (10') and (b'),

$$(11) \quad \sum_1^{\infty} \lambda_n = \sum_1^{+\infty} \overline{a_n(\alpha)^2} = \int_{\Lambda} E(\alpha) d\mu(\alpha) = \bar{E}_D$$

which establishes (8*).

Moreover, due to (8) and (8'), and the convergence of the series (10), one has also

$$(12) \quad \Gamma_{ik}(\mathbf{x}, \mathbf{y}) = \sum_n \lambda_n \phi_{ni}(\mathbf{x}) \phi_{nk}(\mathbf{y}),$$

in the sense of metric convergence on $[L_2(D \times D)]^q$. This completes the proof of the following result.

Lemma 3. *The covariance matrix (3) of any admissible R.V.F. can be written in the form (12), where the vector eigenfunctions $\phi_n(\mathbf{x})$ are orthonormal, and the λ_n are non-negative and satisfy (8*).*

Physical significance. The vector eigenfunctions $\phi_n(\mathbf{x})$ presumably depend on the domain D —hence their “physical” significance is questionable, in general. However, in the case of triply periodic solutions of the Navier-Stokes equations, such as have been considered by TAYLOR, GREEN, and DARRIEUS,* $\Gamma_{ik}(\mathbf{x}, \mathbf{y})$ and the $\phi_n(\mathbf{x})$ are also periodic, and a fundamental cell is the natural choice for D . Hence the claims that there are a natural “statistical equilibrium” and an associated universal theory of “decay” imply that there is a natural class of vector eigenfunctions (unique except for different eigenfunctions having the same eigenvalue). One may conjecture that these are trigonometric polynomials associated with the fundamental periods, in this case.

A more difficult case, to which similar remarks apply, is furnished by turbulence having a fundamental longitudinal period in a straight pipe, with constant mean pressure gradient.

3. Admissible covariance matrices. We shall now show that the preceding conditions characterize completely the covariance matrices of admissible R.V.F. on compact sets. Our construction makes use of a diagonalization process which has been used elsewhere for related purposes.**

*G. I. TAYLOR & A. E. GREEN, *Proc. Roy. Soc. A* **138** (1937), pp. 499–521; G. DARRIEUS, *Proc. 5th Int. Congr. Appl. Mech.*, Cambridge (1938), pp. 422–427.

For the homogeneous normal case, with $p = q = 1$, this construction was used by M. KAC & A. J. SIEGERT, *Annals of Math. Statistics* **18 (1947), pp. 438–442, and for $p = q = 3$, by one of us in [2, p. 27]; see also [3, pp. 72–75]. For continuous covariance, the process was used

Theorem 1. A matrix function $\Gamma_{ik}(\mathbf{x}, \mathbf{y})$ defined on $D \times D$ is the covariance matrix of an admissible R.V.F. on D , if and only if it is the sum of a series (12), where the $\phi_n(\mathbf{x})$ are orthonormal, the λ_n non-negative, $\sum \lambda_n < +\infty$, and convergence is metric ("strong") in $[L_2(D \times D)]^q$.

Proof. Let $A_1(\alpha), \dots, A_n(\alpha), \dots$ be any sequence of independent random variables such that

$$(13) \quad \overline{A_n} = 0, \quad \overline{A_n^2} = \lambda_n,$$

where the λ_n are the eigenvalues of the integral equation (7); for example, the A_n might be normally distributed as in §5. This is always possible.* Consider the R.V.F. defined by

$$(14) \quad \mathbf{u}(\mathbf{x}, \alpha) = \sum_1^{+\infty} A_n(\alpha) \phi_n(\mathbf{x}), \quad \alpha \in I.$$

Since $\sum_1^{+\infty} \lambda_n = \bar{E}_D$ is finite, the series $\sum_1^{+\infty} A_n(\alpha)^2$ converges with probability one,** and thus the series (14) converges (strongly) in the Hilbert space $[L_2(D)]^2 = \mathfrak{H}$, with probability one in the probability space $(I, \mathfrak{B}, m)$. One must remark that using criterion (A), the independence of the random variables is not needed to prove that $\mathbf{u}(\mathbf{x}, \alpha) \in [L^2(D)]^q$. But the assumption of independence is needed to prove that the covariance matrix of the R.V.F. defined by (14) is effectively equal to the given matrix $\Gamma_{i,k}(\mathbf{x}, \mathbf{y})$.

Let us first observe that, due to the orthogonality of the eigenvectors $\phi_n(\mathbf{x})$,

$$\int_{D \times I} A_n^2(\alpha) \phi_n^2(\mathbf{x}) d\mu(\alpha) dm(\mathbf{x}) = \lambda_n.$$

Thus the series

$$\sum_1^{\infty} \int_{D \times I^{\infty}} A_n(\alpha)^2 \phi_n^2(\mathbf{x}) d\mu(\alpha) dm(\mathbf{x})$$

converges, and this implies, by the Fatou Lemma,

$$\mathbf{u}(\mathbf{x}, \alpha) \in L^2(D \times I^{\infty}).$$

(appealing to Mercer's Theorem) by one of us in the *Seminaire Darmois, Inst. Henri Poincaré*, Paris, 1956, pp. 1–19; see also *C. R. Paris* 244 (1957), pp. 1850–1853. H. CRAMÉR, *Annals of Math.* 41 (1940), pp. 215–230, used a quite different construction to get a similar result, in the case of continuous processes, with $p = 1$.

*See [5, p. 211]; the crucial point is the construction of a (product) measure μ for the A_n on the probability space of α as in Thm. B of [5, p. 157]. See also [2, §8], where a similar construction was used. If the A_n are normally distributed, then $\overline{A_n^4} = 3(\overline{A_n^2})^2$, and Kolmogoroff's criterion (B) below is also applicable.

**This is a consequence of the following criterion [11, p. 173, 10(b)]:

(A) If $X_1, \dots, X_n, \dots$ is a sequence of random variables (independent or not), then

$$\sum_1^{+\infty} \overline{|X_n|} < +\infty \Rightarrow \text{Prob} \left[\sum_1^{+\infty} |X_n| < +\infty \right] = 1.$$

whence, using the Tonelli Theorem, $u(\mathbf{x}, \alpha) \in L^2(I^\infty)$ for almost all $\mathbf{x}$ in D . Thus the covariance matrix

$$\int_{I^\infty} u_i(\mathbf{x}, \alpha) u_k(\mathbf{y}, \alpha) d\mu(\alpha)$$

is defined for almost all $(\mathbf{x}, \mathbf{y})$ in $D \times D$. But, due to their independence, the $A_n(\alpha)$ are orthogonal in I :

$$\int_I A_m(\alpha) A_n(\alpha) d\mu(\alpha) = \delta_{m,n} \lambda_n.$$

Thus, by the Parseval Formula, one has, for the two functions $u_i(\mathbf{x}, \alpha)$, $u_k(\mathbf{y}, \alpha) \in L^2(I^\infty)$

$$\int_{I^\infty} u_i(\mathbf{x}, \alpha) u_k(\mathbf{y}, \alpha) d\mu(\alpha) = \sum_1^\infty \lambda_n \phi_{n,i}(\mathbf{x}) \phi_{n,k}(\mathbf{y}),$$

which is precisely $\Gamma_{i,k}(\mathbf{x}, \mathbf{y})$ due to (12).

Many other equivalent formulations are possible. A particularly simple one is given by

Corollary. *For $\Gamma_{i,k}(\mathbf{x}, \mathbf{y})$ to be the covariance function of an “admissible” R.V.F. on D , it is necessary and sufficient that there exist a symmetric matrix $K_{ik}(\mathbf{x}, \mathbf{y}) \in [L_2(D \times D)]^{2^q}$ such that*

$$(16) \quad \Gamma_{ik}(\mathbf{x}, \mathbf{y}) = \sum_r \int_D K_{ir}(\mathbf{x}, \mathbf{z}) K_{rk}(\mathbf{z}, \mathbf{y}) d\mu(\mathbf{z}).$$

Proof. Due to the convergence of the series (10),

$$(17) \quad K_{ik}(\mathbf{x}, \mathbf{y}) = \sum_n \sqrt{\lambda_n} \phi_{ni}(\mathbf{x}) \phi_{nk}(\mathbf{y})$$

is in $[L_2(D \times D)]^q$. (The series converges in mean square.) The verification that $\Gamma_{ik}(\mathbf{x}, \mathbf{y})$ satisfies (16) for this $K_{ik}(\mathbf{x}, \mathbf{y})$, also in the sense of mean square convergence, is trivial. Conversely, for any symmetric square-integrable $K_{ik}(\mathbf{x}, \mathbf{y})$, it may be verified that (16) defines a $\Gamma_{ik}(\mathbf{x}, \mathbf{y})$ having the desired properties.

4. Continuity in the mean. It is easy to construct R. V. F. whose covariance matrix is discontinuous. Nevertheless, the case of *continuous* $\Gamma_{ik}(\mathbf{x}, \mathbf{y})$ is so important as to deserve special mention; in this case, the corresponding R.V.F. is said to be “continuous in the mean”.* For a R.V.F. to be continuous in the mean on D , it is sufficient that the diagonal entries $\Gamma_{ii}(\mathbf{x}, \mathbf{y})$ of the covariance matrix be continuous at every point $(\mathbf{a}, \mathbf{a})$ of the diagonal of $D \times D$. This results

*See M. LOEVE [11, pp. 464–89], who has considered especially the case $p = q = 1$. For the general case, see J. KAMPÉ DE FÉRIET, *Séminaire Darmon* 1956, esp. pp. 6, 15, where further results may be found.

from the inequality

$$\begin{aligned} |\Gamma_{ik}(\mathbf{x}, \mathbf{y}) - \Gamma_{ik}(\mathbf{a}, \mathbf{b})| &\leq A + B, \\ A &= [\Gamma_{ij}(\mathbf{a}, \mathbf{a})]^\frac{1}{2} [\Gamma_{kk}(\mathbf{x}, \mathbf{x}) - 2\Gamma_{kk}(\mathbf{x}, \mathbf{b}) + \Gamma_{kk}(\mathbf{b}, \mathbf{b})]^\frac{1}{2}, \\ B &= [\Gamma_{kk}(\mathbf{b}, \mathbf{b})]^\frac{1}{2} [\Gamma_{ii}(\mathbf{y}, \mathbf{y}) - 2\Gamma_{ii}(\mathbf{y}, \mathbf{a}) + \Gamma_{ii}(\mathbf{a}, \mathbf{a})]^\frac{1}{2}. \end{aligned}$$

Moreover, if D is compact, the continuity is uniform.

Continuity in the mean of R.V.F. does *not* imply the continuity of sample vector fields; there exist R.V.F. with every sample discontinuous at every $\mathbf{x} \in D$, which are nevertheless “continuous in the mean” (have continuous covariance).

For R.V.F. which are continuous in the mean, condition (6) is equivalent to the simpler and frequently used* condition

$$(18) \quad \sum_{r,s} \sum_{i,k} \Gamma_{ik}(\mathbf{x}^{(r)}, \mathbf{x}^{(s)}) \xi_i^{(r)} \xi_k^{(s)} \geq 0$$

for any finite set of points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ of D and corresponding complex vectors $\mathbf{q}^{(1)}, \dots, \mathbf{q}^{(m)}$. Here all sums are from 1 to m .

For R.V.F. which are continuous in the mean, moreover, the use of the Hilbert-Schmidt theory in proving Theorem 1 can be replaced by Mercer’s Theorem. As one of us has proved in the reference cited above, there will follow

Theorem 1’. *For R.V.F. which are continuous in the mean, the series (12) converges pointwise and uniformly on any compact $D \times D$. Moreover, the series (14) converges pointwise in $D \times I$, except possibly on a set Δ of $(m \times \mu)$ -measure zero.*

We content ourselves with proving the second assertion here; the first is a part of Mercer’s Theorem as usually stated—e.g., by ZAAANEN, *op. cit.* in §2.

Proof. Let us apply the Kolmogoroff criterion** to the independent random variables $X_n = A_n(\alpha)\phi_{n,i}(\mathbf{x})$; for a given $\mathbf{x}$ one has

$$\overline{X_n} = 0, \quad \overline{X_n^2} = \lambda_n \phi_{n,i}^2(\mathbf{x}).$$

In the case $p = q = 1$; see the preceding footnote. Usually, the Γ_{ik} are allowed to be complex Hermitian. In this case, the $h_k^{(s)}$ in (18) must be taken as the complex conjugate $h_k^{(s)}$.

**This criterion [A. N. KOLMOGOROFF, *Math. Annalen* 99 (1928), p. 309] reads:

(B) *If $X_1, \dots, X_n, \dots$ is a sequence of independent random variables such that $X_n = 0$ (for all n), then*

$$\sum_1^{+\infty} \overline{X_n^2} < +\infty \Rightarrow \text{Prob} \left[\sum_1^{+\infty} X_n \text{ converges} \right] = 1.$$

To prove the convergence of $\sum A_n^2$ with probability one we could also have used criterion (B) in place of criterion (A) (whose power we did not use in full, because, as we have noticed, we have here, for the end of our proof, to suppose the A_n independent). But in that case we should have had to introduce hypotheses about $\sum \tilde{A}_n^4$ which we have avoided, using criterion (A) above.

Thus the series

$$\sum_1^{+\infty} \overline{X_n^2} = \sum_1^{+\infty} \lambda_n \phi_{n,i}^2(\mathbf{x}) = \Gamma_{i,i}(\mathbf{x})$$

converges pointwise for all $\mathbf{x} \in D$ (by Mercer's Theorem). Thus, by Kolmogoroff's criterion (B), the set Δ on which (14) diverges has μ -measure 0 for all x . But Δ is evidently $(m \times \mu)$ -measurable, because each term of the series (14) is $(m \times \mu)$ -measurable. Thus by a corollary of the Tonelli Theorem, $(m \times \mu)(\Delta) = 0$.

5. Normal random vector fields. Normal (alias Laplacian or Gaussian) R. V. F. have a special physical interest for turbulence, because velocity fields in homogeneous turbulence are approximately distributed according to the normal law [1, p. 169]. In §§5–6, they will have another, purely mathematical, interest for us; they will permit us to extend the results of §3 to infinite space, through the judicious use of uniqueness theorems.

Normal R.V.F. can be defined in various ways. By Lemma 5 of Part A, §3, any regular measure is determined by its behavior on the pseudo-rectangles of some orthonormal basis in $[L_2(D)]^q$. This suggests the following definition.

Definition. A regular R.V.F. is *normal* if and only if, for every compact D and some orthonormal basis $\{\phi_n\}$ in $[L_2(D)]^q$, all the multivariate distributions $(c_1, \dots, c_n)$ defined by

$$(19) \quad c_k = (\mathbf{u}, \phi_k) = \int_D \sum_i u_i(\mathbf{x}, \alpha) \phi_{ki}(\mathbf{x}) d\mu(\alpha)$$

are normal as multivariate distributions.

Equivalently, one may require that every linear functional on $[L_2(D)]^q$ be normally distributed. This is the definition of FRÉCHET [4, p. 7]; the equivalence follows directly from the following proposition of PAUL LÉVY:*

(C) *If, for a given finite set of random variables $A_1, \dots, A_m$, every linear combination $B = b_1 A_1 + \dots + b_m A_m$ is normally distributed, then $A_1, \dots, A_m$ have a multivariate normal distribution.*

Referring to (14), we see that the R.V.F. constructed in Thm. 1 is *normal* if and only if the $A_n(\alpha)$ are *normally* distributed, for each $\lambda_n > 0$. But this is always possible in (13), and in only one way.

If the admissible R.V.F. is continuous in the mean, then the series (14) is convergent in D , by Thm. 1'. Hence, for almost all $\mathbf{x}$, $u_i(\mathbf{x}, \alpha)$ is itself normal, being the sum of a series of independent normal random variables. More generally, by LÉVY's proposition (C) above, we have the

Lemma. *If $\mathbf{u}(\mathbf{x}, \alpha)$ is continuous in the mean, then the nq components $u_i(\mathbf{x}^{(r)}, \alpha)$*

*P. LÉVY, personal communication.

are normally distributed, for all $(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}) \in D^n$ excepting a set of measure zero in D^n .

A stronger uniqueness theorem can be proved, as follows. Let $\mathbf{u}(\mathbf{x}, \alpha)$ be any admissible R.V.F. on D . Then each $\mathbf{u}(\mathbf{x}, \alpha)$, being in $[L_2(D)]^q$, has a unique expansion $\sum A_n(\alpha) \phi_n(\mathbf{x})$, where the $\phi_n(\mathbf{x})$ are the vector eigenfunctions of $\Gamma_{ik}(\mathbf{x}, \mathbf{y})$. Moreover, the $A_n(\alpha)$ are normally distributed and satisfy (13), which determines them uniquely. Hence, by Lemma 5 of Part A, §3, the R.V.F. is uniquely determined (see Thm. 1 of Part A). This proves

Theorem 2. *There is one and only one normal R.V.F. with mean $\mathbf{u}(\mathbf{x}) = \mathbf{0}$ on a compact domain D , whose covariance matrix is a given square-integrable symmetric matrix of positive type, with eigenvalues satisfying (8*).*

Corollary. *Given an admissible R.V.F. on a compact domain D , there always exists an admissible normal R.V.F. having the same covariance matrix (and mean).*

We omit the details. The corresponding generalization of [3, p. 74] to continuous R.V.F. is false (see §9). It would be interesting to know whether or not a corresponding result holds for R.V.F. (or stochastic processes) which are “separable” in the sense of Doob [3, p. 51].

6. Extension to infinite space. The extension of the preceding results from compact domains D to infinite space is not trivial (cf. [2, §11]); it uses *uniqueness* very essentially.

To show this, we consider a nested sequence $D_1 \subset D_2 \subset D_3 \subset \dots$ of compact cubes, whose union is R^p . On each D_m , there is, by Theorem 2, a unique *normal* random vector field $\mathbf{u}^{(m)}(\mathbf{x}, \omega_m)$ having the covariance $\Gamma_{ik}(\mathbf{x}, \mathbf{y})$ on $D_m \times D_m$. Because the $\Gamma_{ik}(\mathbf{x}, \mathbf{y})$ agree on D_m , for any $\mathbf{u}^{(n)}(\mathbf{x}, \omega_n)$ with $n > m$, the measures agree so far as measurable sets in D_m are concerned. That is, the natural many-one correspondence $\mathbf{u}^{(n)}(\mathbf{x}, \omega_n) \rightarrow \mathbf{u}^{(m)}(\mathbf{x}, \omega_m)$ is measure preserving, from the ω_n to the ω_m .

Referring to formula (6) of Part A, we see further that there is an inverse sequence of natural many-one correspondences, defined in Λ ,

$$(20) \quad \omega_1 \leftarrow \omega_2 \leftarrow \dots \leftarrow \omega_n \leftarrow \dots \leftarrow \omega$$

such that the “inverse sequence” (19) defines Λ . Since the measures defined on Λ by (19) are all consistent with each other, they obviously define a *finitely additive* measure on Λ . This proves the following result.

Lemma 4. *Any covariance matrix satisfying (4')–(6) on all compact D is the covariance matrix of a normal R.V.F. corresponding to a finitely additive set-function μ on Λ , countably additive on compact subsets of R^p .*

We shall now prove that this set-function is *compatible*, so that it can be extended (Borel) to a measure on Λ and completed (Lebesgue), as in §2 of Part A. This Borel extension is “regular”, so that this proof will imply

Theorem 3. Any covariance matrix satisfying (4')–(6) on all compact D is the covariance of a (unique) normal admissible R.V.F.

Our (crucial) proof of compatibility (see §2) is analogous to KOLMOGOROFF's proof* of the consistency of the product measure on $\bar{R}^\infty$, relying on a diagonal process and compactness. We could not find a proof based on (20) alone.

Proof of compatibility. Using a representation of the form (14), we can generate the measure μ_n on each D_n referred to above, as the outer measure** μ_n^* associated with the μ of Lemma 4, restricted to a (countable) basis of open cylinders $b_i < a_i < c_i$ [$i = 1, \dots, s$]. The Caratheodory-Lebesgue outer measure defined through countable coverings by such open cylinders will give the measure on D_n , and the cylinders in question are open in $[L_2(D_n)]^q$. We now define the outer measure μ^* in Λ generated by the open cylinders of all D_n , $n = 1, 2, 3, \dots$. To prove compatibility, it suffices to show that $\sum \mu(S_i) < 1$ is incompatible with $\bigvee_{i=1}^\infty S_i \supset \Lambda$; this we now do.

Since μ is essentially compact (Part A, §11), we can find $S_i^* \supset S_i$ with compact complements T_i , such that $\sum \mu(S_i^*) < 1$ (and $\bigvee_i S_i^* \supset \Lambda$, trivially); to ensure this, we need only make the $\mu(S_i^*) - \mu(S_i)$ sufficiently small. Consider $S_i \cup \dots \cup S_m = U_m$; this is an open cylinder in some $D_{n(m)}$ with $n(m) > n$, and has a non-void compact complement $T_1 \cap \dots \cap T_m$ in $[L_2(D_{n(m)})]^q$, since $\mu(S_1 \cup \dots \cup S_m) < 1$.

Choose any $\mathbf{u}_m \in T_1 \cap \dots \cap T_m$, and consider the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots$ of vector fields, extended arbitrarily to vector fields on R^p (i.e., elements of Λ). We can extract a subsequence of $\{\mathbf{u}_m\}$ convergent in $D_{n(1)}$, then a subsubsequence convergent in $D_{n(2)}$, $\dots$, all because the $T_1 \cap \dots \cap T_m$ are (sequentially) compact. Finally, applying Cantor's Diagonal Process, we get a subsequence of vector fields $\mathbf{v}_r(\mathbf{x}) = \mathbf{u}_{m(r)}(\mathbf{x})$ on R^p , which converges on every D_n . The limit $\mathbf{v}_\infty(\mathbf{x})$ is in Λ , but is in no $S_1 \cup \dots \cup S_m$, completing the proof.

Corollary. A matrix function $\Gamma_{ik}(\mathbf{x}, \mathbf{y})$ defined on $R^p \times R^p$ is the covariance matrix of an admissible R.V.F. if and only if it is admissible,† symmetric, and of positive type.

7. Homogeneity and isotropy. Clearly, for every translation $\mathbf{x} \rightarrow \mathbf{x} + \xi$ or rotation $\mathbf{x} \rightarrow \mathbf{x}K$ (K an orthogonal matrix), the transformations

$$(21) \quad T_\xi : \mathbf{u}(\mathbf{x}) \rightarrow \mathbf{u}(\mathbf{x} + \xi) \quad \text{and} \quad R_K : \mathbf{u}(\mathbf{x}) \rightarrow \mathbf{u}(\mathbf{x}K)$$

are linear transformations and homeomorphisms of the topological linear space Λ of Part A, §4. Hence, if $(\Lambda, \mathfrak{F}, \mu)$ is any admissible R.V.F., there is one and

*A. N. KOLMOGOROFF, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, p. 27. It is also analogous to the proof of ANDERSEN & JESSEN [5, p. 157].

**In the sense of CARATHEODORY. See, for example, G. BIRKHOFF, *Lattice theory*, rev. ed., Chap. XI, §8.

†I.e. as before, in $[L_2(D \times D)]^q$ on every compact D .

only one function $\alpha' = \phi(\alpha, \xi)$ on Λ to Λ such that $\mathbf{u}(\mathbf{x}, \alpha') = \mathbf{u}(\mathbf{x} + \xi, \alpha)$; moreover,

$$(22a) \quad \phi(\phi(\alpha, \xi), \xi') = \phi(\alpha, \xi + \xi').$$

Similarly, there is just one function $\alpha' = \psi(\alpha, K)$ such that $\mathbf{u}(\mathbf{x}, \alpha') = [\mathbf{u}(\mathbf{x}K, \alpha)]K$ and

$$(22b) \quad \psi(\psi(\alpha, K), K') = \psi(\alpha, KK').$$

Definition. A *homogeneous* random vector field is an R.V.F. such that every $\phi(\alpha, \xi)$ leaves the measure μ on Λ invariant. A homogeneous R.V.F. is called *isotropic* when μ is also invariant under all rotations $\psi(\alpha, K)$ and reflections.*

The discussion of Part A shows that the preceding definition of homogeneity is equivalent to the usual condition [7, p. 587] that $\mathbf{u}(\mathbf{x} + \xi; \omega) = \mathbf{u}(\mathbf{x}; \omega T\xi)$ for a suitable measure-preserving (point- or set-) transformation T_ξ of the measure space $(\Lambda, \mathfrak{F}, \mu)$.

8. Correlation matrix. If $(\Omega, \mathfrak{F}, \mu)$ is homogeneous, then clearly

$$(23) \quad \Gamma_{jk}(\mathbf{x}, \mathbf{y}) = R_{jk}(\mathbf{x} - \mathbf{y})$$

for the *correlation matrix* $R_{jk}(\mathbf{h}) = \Gamma_{jk}(\mathbf{h}, \mathbf{0})$. Conversely, when (23) holds, $(\Lambda, \mathfrak{F}, \mu)$ is said to have “second order” or “wide sense” homogeneity.† In the case of strict or wide sense homogeneity, we can prove

Theorem 4. For any admissible homogeneous R.V.F., the correlation matrix $R_{jk}(\mathbf{h})$ is uniformly continuous.**

Proof. For any compact D of positive measure, the set $T(\epsilon, \eta)$ of $\theta \in \Lambda$ such that

$$(24) \quad \int_D |\mathbf{u}(\mathbf{x} + \mathbf{h}) - \mathbf{u}(\mathbf{x})|^2 dm(\mathbf{x}) < \epsilon, \quad \text{for all } |\mathbf{h}| \leq \eta,$$

is a Borel set in Λ . This may be shown by the methods of Part A. That is, the function

$$(25) \quad S(\eta, \theta) = \sup_{|\mathbf{h}| \leq \eta} \left\{ \int_D |\mathbf{u}(\mathbf{x} + \mathbf{h}) - \mathbf{u}(\mathbf{x})|^2 dm(\mathbf{x}) \right\}$$

is μ -measurable, for any admissible measure μ on Λ . An easy application of the

*See H. P. ROBERTSON, *Proc. Camb. Phil. Soc.* **36** (1940), pp. 209–223, for the significance of assuming invariance under other groups.

†See [3, p. 95] or [11, p. 482]. Note that $R_{jk}(\mathbf{h})$ can be *measured* experimentally, using a hot wire anemometer.

**In the case $q = 1$, M. M. CRUM has proved, in the paper mentioned in Part C, §2, that the correlation $R(h)$ of a stationary random function $u(x, \omega)$, measurable with respect to $m \times \mu$, is always continuous.

Schwarz inequality gives also

$$\int_D S(\eta, \theta) d\mu \leq 4\bar{E}_D ;$$

moreover, $S(\eta, \theta)$ decreases monotonely with η . Hence the set of $S(|\mathbf{h}|, \theta)$ with $|\mathbf{h}| < \eta$ is dominated by the μ -integrable function $S(\eta, \theta)$. On the other hand, for any fixed $\theta \in \Lambda$, $S(\eta, \theta)$ tends to zero by the Lebesgue Density Theorem, applied to $\mathbf{u}(\mathbf{x}) \in L_2(D)$. Hence, by Lebesgue's Dominated Convergence Theorem, we have

$$(26) \quad \lim_{\mathbf{h} \rightarrow 0} S(|\mathbf{h}|, \theta) d\mu(\theta) = 0.$$

By the Schwarz Inequality again, the quantity

$$\begin{aligned} L &= m(D) |R_{ik}(\mathbf{h} + \mathbf{h}') - R_{ik}(\mathbf{h}')| \\ &= \left| \iint_{D \times \Omega} u_i(\mathbf{x} - \mathbf{h}') [u_k(\mathbf{x} + \mathbf{h}) - u_k(\mathbf{x})] dm(\mathbf{x}) d\mu \right| \\ &\leq \left\{ \iint_{D \times \Omega} |\mathbf{u}(\mathbf{x} - \mathbf{h}')|^2 dm(\mathbf{x}) d\mu \right\}^{\frac{1}{2}} \cdot \left| \int_{\Omega} S(|\mathbf{h}|, \theta) d\mu \right|^{\frac{1}{2}} \\ &= \left\{ m(D) \bar{E}_D \int_{\Omega} S(|\mathbf{h}|, \theta) d\mu \right\}^{\frac{1}{2}}. \end{aligned}$$

Simplifying, we get the basic inequality

$$(27) \quad |R_{ik}(\mathbf{h} + \mathbf{h}') - R_{ik}(\mathbf{h}')| \leq \left\{ \frac{\bar{E}_D}{m(D)} \int_{\Omega} S(|\mathbf{h}|, \theta) d\mu \right\}^{\frac{1}{2}}.$$

Letting $\mathbf{h} \rightarrow \mathbf{0}$ in (27), and applying (26), we get the conclusion that the $R_{ik}(\mathbf{h})$ are uniformly continuous.

Corollary 1. Any admissible homogeneous R.V.F. is "continuous in the mean".

Therefore the results of §4 are applicable to any homogeneous R.V.F.

Theorem 5. A given matrix $R_{ik}(\mathbf{h})$ is the correlation matrix of an admissible homogeneous R.V.F. if and only if it is continuous, symmetric, and positive definite (see (18)).

Proof. That the conditions are necessary is a corollary of Theorem 4 and the discussion of (18) in §4. But to prove sufficiency, we need to complete Theorems 1-1'.

By these theorems, given a continuous and positive definite matrix function $R_{ik}(\mathbf{x} - \mathbf{y}) = \Gamma_{i..k}(\mathbf{x}, \mathbf{y})$, we can always define an admissible R.V.F. by (14) having this matrix as covariance matrix. In fact, by Theorem 2, there is a unique normal R.V.F. with this $R_{ik}(\mathbf{x} - \mathbf{y})$ as covariance matrix, all on any compact D . Finally, the R.V.F. defined in this way for different compact D are all com-

patible, in the sense of §6. Thus, by Theorem 3, they are all compatible with the same R.V.F. on R^p . *Q.E.D.*

9. A counterexample. Though homogeneous R. V. F. are always continuous in the mean, almost all individual sample vector fields can be discontinuous everywhere, even in the normal case.* This fact is very important for us, because it shows that the space Γ of continuous vector fields is inadequate for our purposes.

Theorem 6. *There exist admissible homogeneous normal R.V.F., which are unbounded with probability one, and hence everywhere discontinuous.*

Proof. We take the case $p = q = 1$ of random functions. Let

$$(28) \quad u(x) = \sum_1^{\infty} A_k \cos [(k!)x + \alpha_k],$$

where the A_k have independent normal distributions with mean zero, and the α_k are independently distributed on the circle $-\pi < \alpha_k \leq \pi$ with the usual "random phase" distribution $d\mu_k(\alpha_k) = d\alpha_k/2\pi$. As before, such series will define an admissible normal random function provided $\sum \overline{A_k^2} < +\infty$.

Now, given a sample function $u(x) = u(x, A, \alpha)$ and an interval $c - \epsilon < x < c + \epsilon$, first choose k so large that $\epsilon < 1/2k$. Then choose x_k as the real number minimizing $|x_k - c|$, among those satisfying

$$(29) \quad (k!)x_k + \alpha_k = \frac{1}{2}\pi(1 - \operatorname{sgn} A_k) \pmod{2\pi}$$

—i.e., among those making $u(x_k) = |A_k|$. (If there are two such, then choose the smaller.)

Recursively, we can then define x_{h+1} as the real number nearest to x_h such that

$$(29') \quad (h+1)!x_{h+1} + \alpha_{h+1} \equiv \frac{1}{2}\pi(1 - \operatorname{sgn} A_{h+1}) \pmod{2\pi};$$

if there are two such, we choose the smaller. Clearly, $|x_{h+1} - x_h| \leq \pi/(h+1)!$. If x_∞ denotes the limit of the convergent sequence so defined, evidently

$$(h!) |x_h - x_\infty| < \frac{1}{h+1} + \frac{1}{(h+1)(h+2)} + \cdots < \frac{1}{h}.$$

It follows that, for all $h = k, k+1, k+2, \dots$,

$$(30) \quad A_k \cos [(k!)x_\infty + \alpha_k] \geq \frac{1}{2} |A_k|.$$

Hence, substituting back into (27), we get

$$u(x_\infty) > \frac{1}{2} \sum_2^{\infty} |A_k|.$$

*For a similar example which is not normal, see [8, p. 9].

By choosing $\overline{A_k^2} = k^{-4/3}$ and thus $|\overline{A_k}| = \sqrt{\frac{1}{2}\pi} k^{-2/3}$ we will have at the same time

$$\sum \overline{A_k^2} < +\infty \quad \text{and} \quad \sum |\overline{A_k}| = +\infty,$$

thus assuring the existence of the random function, but the divergence with probability one of $\sum |A_k|$; which completes the proof.

Since (28) defines an ordinary Fourier series with gaps, the preceding argument can easily be completed by observing that if $u(x)$ were continuous, then $u(x_\infty)$ would be the first Cesaro mean of its finite partial sums, whereas these obviously diverge to $+\infty$.

The preceding example has a discrete spectrum; we do not know whether similar counterexamples exist having a continuous spectrum.

10. Incompressibility and irrotationality. Though homogeneity cannot be tested by considering the covariance property (*i.e.*, “wide sense” homogeneity does not imply “strict” homogeneity*), both incompressibility and irrotationality can be so tested. The proof will only be sketched below, and then only for the case that the $\mathbf{u}(\mathbf{x}, \alpha)$ have two continuous derivatives** with probability one. In principle, the idea applies to all homogeneous *linear* conditions; it consists in interpreting the condition that the square of the expectation of a quantity vanish as a condition on the covariance.

Theorem 7. *If $\mu(\mathcal{C}^{(2)}) = 1$, then almost all samples of a homogeneous R.V.F. with $p = q$ are **incompressible**, if and only if*

$$(31) \quad \sum_{i,j} \frac{\partial^2 R_{ij}}{\partial h_i \partial h_j}(\mathbf{0}) = 0.$$

Proof. A condition for incompressibility is

$$(32) \quad \int_{\Lambda} |\text{Div } \mathbf{u}|^2 d\mu(\alpha) = 0 \quad \text{for all } \mathbf{x} \in R^p.$$

We consider the approximations by difference quotients

$$\begin{aligned} \int_{\Lambda} d\mu(\alpha) \left\{ \delta^{-2} \sum_{i,j} [u_i(\mathbf{x} + \delta \mathbf{e}_i) - u_i(\mathbf{x})][u_j(\mathbf{x} + \delta \mathbf{e}_j) - u_j(\mathbf{x})] \right\} \\ = \delta^{-2} \sum_{i,j} R_{ii}(\delta \mathbf{e}_i - \delta \mathbf{e}_j) - R_{ij}(\delta \mathbf{e}_i) - R_{ij}(-\delta \mathbf{e}_j) + R_{ij}(\mathbf{0}), \end{aligned}$$

where $\mathbf{e}_i$ is the i^{th} unit vector. From Lebesgue's Dominated Convergence Theorem, it seems apparent that this will tend to (32) as $\delta \downarrow 0$. On the other hand, the limit may be evaluated directly, giving

*We have not studied the corresponding question for isotropy, in a homogeneous R. V. F.

It seems likely that the conclusion could be extended to admissible R. V. F. by considerations similar to those of ZAREMBA & H. WEYL *Duke Math. J.* **7 (1940), pp. 411-444.

$$(32^*) \quad \int_{\Lambda} |\text{Div } \mathbf{u}|^2 d\mu(\alpha) = \sum_{i,j} \frac{\partial^2 R_{ij}}{\partial h_i \partial h_j}(\mathbf{0}).$$

This completes the proof.

By Theorem 7, (31) implies incompressibility. But it is known ([1, pp. 26–27] or [7, p. 604]) that incompressibility implies

$$(33) \quad \sum \frac{\partial R_{jk}}{\partial h_j}(\mathbf{h}) = \sum \frac{\partial R_{jk}}{\partial h_k}(\mathbf{h}) = 0 \quad \text{for all } \mathbf{h}.$$

Finally, (33) obviously implies (31), which proves the

Corollary. *Condition (33) is necessary and sufficient for a homogeneous R.V.F., with $p = q$, to be incompressible.*

Theorem 8. *If $\mu(\mathcal{C}^{(2)}) = 1$, then almost all samples of a homogeneous R.V.F. are irrotational if and only if*

$$(34) \quad 2 \frac{\partial^2 R_{ij}}{\partial h_i \partial h_j}(\mathbf{0}) = \frac{\partial^2 R_{ii}}{\partial x_j^2}(\mathbf{0}) + \frac{\partial^2 R_{jj}}{\partial x_i^2}(\mathbf{0}) \quad \text{for all } i \neq j.$$

Remark. The interesting application is to acoustic *noise* fields. We cannot use the condition of [1, p. 39, (3.2.2)], which assumes incompressibility. In fact, irrotational *and* incompressible vector fields have unbounded velocities, and are mathematically pathological. However, we note that [1, p. 39, (3.2.2)] would make $-\sum \nabla^2 R_{ii}(\mathbf{0}) = 0$ if $\int |\mathbf{v}(\mathbf{x}, \alpha)|^2 d\mu(\alpha)$ were zero, and hence, $\nabla^2 R_{ii}(\mathbf{0}) = 0$.

Proof. The condition of zero vorticity is equivalent to the condition that, for (almost) all $i, j, \mathbf{x}$,

$$(35) \quad \int_{\Lambda} \left[\frac{\partial u_i}{\partial x_j}(\mathbf{x}) - \frac{\partial u_j}{\partial x_i}(\mathbf{x}) \right]^2 d\mu(\alpha) = 0.$$

We again consider the approximation by difference quotients to (35),

$$\begin{aligned} & \frac{1}{4\delta^2} \int d\mu(\alpha) \{u_i(\mathbf{x} + \delta \mathbf{e}_i) - u_i(\mathbf{x} - \delta \mathbf{e}_i) - u_j(\mathbf{x} + \delta \mathbf{e}_i) + u_j(\mathbf{x} - \delta \mathbf{e}_i)\}^2 \\ &= \frac{1}{2\delta^2} \{R_{ii}(\mathbf{0}) - R_{ii}(2\delta \mathbf{e}_i) + R_{ii}(\mathbf{0}) - R_{ii}(2\delta \mathbf{e}_i) - R_{ij}(\delta \mathbf{e}_j - \delta \mathbf{e}_i) \\ & \quad + R_{ij}(\delta \mathbf{e}_j + \delta \mathbf{e}_i) + R_{ji}(\delta \mathbf{e}_i + \delta \mathbf{e}_j) - R_{ji}(\delta \mathbf{e}_i - \delta \mathbf{e}_j)\}. \end{aligned}$$

Expanding in Taylor series, and using the relation $R_{ij}(\mathbf{h}) = R_{ij}(-\mathbf{h})$, the above quotient tends in the limit as $\delta \downarrow 0$ to the difference between the two sides of (34).

C. ENERGY SPECTRUM

1. Spectral matrix measure. The spectral analysis of R. V. F. is commonly and most conveniently expressed using *complex*-valued functions $u_i(\mathbf{x})$. Hence,

in Part C, though continuing to consider only real position vectors $\mathbf{x} = (x_1, \dots, x_p)$ and real wave vectors $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_p)$, we shall treat complex velocity vectors $\mathbf{u} = (u_1, \dots, u_q)$. For such complex homogeneous R.V.F., the correlation matrix is defined by

$$(1) \quad R_{ik}(\mathbf{h}) = \int u_i(\mathbf{x}, \omega) u_k^*(\mathbf{x} + \mathbf{h}, \omega) d\mu(\omega)$$

where a star denotes complex conjugation. Obviously

$$(2) \quad R_{ik}(\mathbf{h}) = R_{ki}^*(-\mathbf{h}) \quad (\text{Hermitian symmetry}),$$

while a brief consideration shows that*

$$(3) \quad \sum_{i,k} \sum_{r,s} R_{ik}(\mathbf{h}^{(r)} - \mathbf{h}^{(s)}) \xi_i^{(r)} \xi_k^{*(s)} \geq 0 \quad (\text{positive definite})$$

for any finite set of real p -vectors $\mathbf{h}^{(1)}, \dots, \mathbf{h}^{(m)}$ and corresponding complex p -vectors $\xi^{(1)}, \dots, \xi^{(m)}$.

Theorem 1. *A matrix function $R_{ik}(\mathbf{h})$ is the correlation matrix of an admissible homogeneous (complex) R.V.F. if and only if it is continuous and positive definite.*

Proof. It is known (MATHIAS, *op. cit.*) that (3) implies (2); the rest of the proof is contained in Theorem 5 of Part B. (As with real R.V.F., “admissibility” means that the “energy expectation”

$$(4) \quad \bar{E}_D = \sum_i \int_D \Gamma_{ii}(\mathbf{x}, \mathbf{x}) dm(\mathbf{x}) < +\infty$$

is finite on any compact D —i.e., in the homogeneous case, that $\sum R_{ii}(\mathbf{0}) = \bar{E}_D/m(D) < +\infty$.)

Example. The concept of spectral measure matrix is suggested by a consideration of the special case

$$(5) \quad \mathbf{u}(\mathbf{x}) = \mathbf{c}(\omega) \exp(i\boldsymbol{\kappa} \cdot \mathbf{x}) \quad \text{with fixed } \boldsymbol{\kappa}.$$

In this class of examples, clearly

$$(6) \quad R_{ik}(\mathbf{h}) = \overline{c_i c_k^*} \exp(i\boldsymbol{\kappa} \cdot \mathbf{h}) = \Gamma_{ik}(\mathbf{x} - \mathbf{y}),$$

for $\mathbf{h} = \mathbf{x} - \mathbf{y}$; the covariance is always stationary. Hence, if all $\bar{c}_i = 0$, the (obviously) *monochromatic* R.V.F. (5) has wide sense homogeneity. However, it is no longer true for such R.V.F. that normality and wide sense homogeneity imply strict sense homogeneity. To make this hold, it seems most natural to

*[11, p. 207]; the original discussion is due to M. MATHIAS *Math. Zeits.* 16 (1923), pp. 101–125. Obviously, (3) is simply (18) of Part B.

require *complex-isotropy*: invariance of the measure under the complex rotations $u \rightarrow e^{i\alpha}u$. We omit a detailed discussion of this concept *

More generally, let the probability sample space consist of complex monochromatic vector fields

$$(5') \quad \mathbf{u}(\mathbf{x}, \omega) = \mathbf{c}(\omega) \exp(i\mathbf{\kappa}(\omega) \cdot \mathbf{x})$$

with variable wave vectors $\mathbf{\kappa}(\omega)$. (I.e., let the $\mathbf{c} \exp(i\mathbf{\kappa} \cdot \mathbf{x})$ have probability one in Λ .) Then, similarly,

$$(6') \quad R_{ik}(\mathbf{h}) = \int \exp(i\mathbf{\kappa}(\omega) \cdot \mathbf{h}) c_i(\omega) c_k^*(\omega) d\mu.$$

We now consider the Hermitian symmetric matrix-valued function defined on Borel subsets K of R^p by

$$S_{ik}(K) = \int_{\mathbf{\kappa}(\omega) \in K} c_i(\omega) c_k^*(\omega) d\mu(\omega).$$

For any admissible R.V.F. $\mathbf{u}(\mathbf{x}, \omega)$, each $S_{ik}(K)$ is bounded and σ -additive, and the $S_{ii}(K)$ are measure functions. Clearly, $\sum S_{ii}(K)$ represents the spatial *energy density* of those samples whose wave vectors are in K . Moreover, by (6'), we have

$$(7) \quad R_{ik}(\mathbf{h}) = \int_{R^p} \exp(i\mathbf{\kappa} \cdot \mathbf{h}) dS_{ik}(\mathbf{\kappa}),$$

in the sense of *Lebesgue-Stieltjes* integration [11, pp. 98–99].

The simple examples just described suffice to motivate the following definitions—deeper interpretations will be given in §4, and in §5 of Part D.

Definitions. A (complex) homogeneous R.V.F. will be said to be *associated* with the complex† *spectral matrix* measure $S_{ik}(K)$ if and only if its correlation matrix satisfies (7). A *Hermitian matrix measure* is a bounded,** σ -additive Hermitian matrix set-function $S_{ik}(K)$, defined and non-negative definite for all Borel sets K in R^p .

As usual, the Hermitian matrix $S_{ik}(K)$ will be called non-negative definite if and only if

$$(8) \quad \sum \xi_i S_{ik}(K) \xi_k^* \geq 0$$

for all complex vectors $\xi = (\xi_1, \dots, \xi_p)$.

*It is discussed by DOOB in [3, pp. 72–73] and by LOEVE in [11, pp. 462–465]. Complex-isotropy is one of the two meanings given to “strong normality” by LOEVE; it also corresponds to the concept of “random phase”, discussed by G. BIRKHOFF & J. KOTIK, *Proc. Gravity Wave Symposium Nat. Bu. Standards*, Washington, D. C., 1952.

†[5, p. 115], P. HALMOS considers only *signed* measures. For an exposition of $S' + iS''$ where S' and S'' are signed measures, one can refer to M. H. STONE, *Linear transformations in Hilbert space*, 1932, pp. 198–208. The $S_{ij}(K)$ are (positive) measure functions.

**For additive set-functions, being bounded is equivalent to being of bounded variation. See for example G. BIRKHOFF, *Lattice theory*, rev. ed., p. 245.

The following result is an almost immediate corollary of the general theory of Lebesgue-Stieltjes integrals, for bounded continuous functions like $\exp(i\kappa \cdot \mathbf{h})$.

Lemma 1. *For any Hermitian matrix measure $S_{ik}(K)$, the Lebesgue-Stieltjes integral (7) defines a continuous, positive definite correlation matrix.*

The only point to check is (3), the positive definiteness of $R_{ik}(\mathbf{h})$. This can be verified for approximating sums and passage to the limit; see [11, p. 208]. Combining Lemma 1 and Theorem 1, we have the

Corollary. *Any positive Hermitian matrix measure $S_{jk}(K)$ is associated with the correlation matrix of an admissible homogeneous R.V.F.*

2. Bochner's Theorem. In a classic paper, BOCHNER* proved a converse of Lemma 1 of §1, for the case $q = 1$ of scalar $u(x)$. We quote his main result, as follows.

Bochner's Theorem. *A scalar function $R(h)$ can be represented as a Lebesgue-Stieltjes integral*

$$(9) \quad R(h) = \int_{R^p} \exp(i\kappa \cdot h) dS(\kappa),$$

with respect to a bounded (real) measure function [11, p. 98] $S(K)$, if and only if it is continuous and positive definite.

This result has been sharpened by CRUM,** who proved that a measurable function $R(h)$ is pointwise positive definite if and only if $R(h) = R_e(h) + R_o(h)$, where $R_e(h)$ is continuous and positive definite, while $R_o(h)$ vanishes almost everywhere and is pointwise positive definite.

BOCHNER himself (*op. cit.*, p. 402) proved that $R(h)$ uniquely determined $S(K)$ in (9); the proof goes back to an inversion formula discovered by P. LÉVY in the case $p = 1$, and later generalized by E. K. HAVILAND to the general p -dimensional case.*** This justifies calling $S(K)$ the spectral distribution associated with the correlation function $R(h)$ —or of any homogeneous random function with correlation $R(h)$.

Bochner's Theorem can be applied directly to the energy correlation (or trace)

$$(10) \quad \rho(\mathbf{h}) = \sum_{i=1}^q R_{ii}(\mathbf{h})$$

of any homogeneous R.V.F. More generally, one easily verifies the following

*S. Bochner, *Math. Annalen* **108** (1933), pp. 378–410, esp. p. 406.

M. M. CRUM, *Proc. London Math. Soc.* **6 (1956), pp. 548–608. A measurable function will be called pointwise positive definite if and only if it satisfies (3). This obviously implies (6) of Part B, but is not implied by it for general discontinuous functions.

***For proofs and precise references, see H. CRAMÉR, *Random variables and probability distributions*, Cambridge Univ. Press, 1937, pp. 28–29 and 104.

Lemma 2. *If the matrix $R_{ik}(\mathbf{h})$ is continuous and positive definite, then, for every $\mathbf{n} = (\eta_1, \dots, \eta_a)$,*

$$(11) \quad \rho(\mathbf{h}, \mathbf{n}) = \sum_{i,k} R_{ik}(\mathbf{h}) \eta_i \eta_k^*$$

is a continuous and positive definite function of $\mathbf{h}$.

The proof of (2) and (3) for $\rho(\mathbf{h}, \mathbf{n})$ follows from (2) and (3) for $R_{ik}(\mathbf{h})$ by simple algebraic computations. Thus $\sum \rho(\mathbf{h}^{(r)} - \mathbf{h}^{(s)}, \mathbf{n}) \xi^{(r)} \xi^{*(s)} \geq 0$ by (3), with the special choice $\xi^{(r)} = \xi^{(r)} \mathbf{n}$.

Corollary 1. *For every $\mathbf{n} = (\eta_1, \dots, \eta_a)$,*

$$(12) \quad \sum_{i,k} R_{ik}(\mathbf{h}) \eta_i \eta_k^* = \int \exp(i\boldsymbol{\kappa} \cdot \mathbf{h}) d\sigma(\boldsymbol{\kappa}, \mathbf{n}),$$

for a (unique) suitable Lebesgue-Stieltjes measure $d\sigma(\boldsymbol{\kappa}, \mathbf{n})$.

Corollary 2. *The energy correlation (10) of any admissible homogeneous R.V.F. can be expressed as the Fourier-Lebesgue-Stieltjes integral*

$$(13) \quad \rho(\mathbf{h}) = \int \exp(i\boldsymbol{\kappa} \cdot \mathbf{h}) d\sigma(\boldsymbol{\kappa})$$

for a unique energy spectrum distribution $\sigma(K)$.

In (13), evidently $\sigma(R^n) = \bar{E}$, the energy density distribution. Hence $\sigma(K)$ may be thought of as the energy corresponding to wave vectors in K ; this is essentially the result of [1, p. 31].

3. Existence of spectral matrix measure. Using Lemma 2, it is not hard to extend Bochner's Theorem so as to prove the existence of a unique Hermitian matrix measure $S_{jk}(K)$, "associated" with each continuous, positive definite matrix point-function $R_{jk}(\mathbf{h})$. The problem is to apply Lemma 2 and Bochner's Theorem to individual (scalar) matrix entries.

Setting $\eta_i = \eta_k = 1$ in Lemma 2, and all other components zero, we get

$$(14) \quad \rho(\mathbf{h}, \mathbf{n}) = R_{ij}(\mathbf{h}) + R_{jk}(\mathbf{h}) + R_{ki}(\mathbf{h}) + R_{kk}(\mathbf{h}).$$

If we set $\eta'_i = 1$ and $\eta'_k = i = \sqrt{-1}$, we get similarly

$$(14') \quad \rho(\mathbf{h}, \mathbf{n}') = R_{ij}(\mathbf{h}) - iR_{jk}(\mathbf{h}) + iR_{ki}(\mathbf{h}) + R_{kk}(\mathbf{h}).$$

By Lemma 2, $\rho(\mathbf{h}, \mathbf{n})$ and $\rho(\mathbf{h}, \mathbf{n}')$ are Fourier-Lebesgue-Stieltjes transforms of suitable real measure functions $\sigma(K)$ and $\sigma'(K)$; by Bochner's Theorem itself,* $R_{ij}(\mathbf{h})$ and $R_{kk}(\mathbf{h})$ are both transforms of scalar measures $S_{ij}(K)$ and $S_{kk}(K)$. Combining with (14)–(14'), we get

*Applied to the scalar fields $u_j(x)$ and $u_k(x)$. Alternatively, one can appeal to Lemma 2, with $\eta_j = 1$ and all other components zero.

$$(15) \quad R_{ik}(\mathbf{h}) = \int_{R^p} \exp(i\mathbf{\kappa} \cdot \mathbf{h}) dS_{ik}(\mathbf{\kappa}),$$

where

$$(15') \quad S_{ik}(K) = \frac{1}{2}[\sigma(K) + i\sigma'(K) - (1+i)S_{ii}(K) - (1+i)S_{kk}(K)]$$

is a bounded σ -additive Hermitian matrix set-function. One easily verifies that, for any Borel set K , $S_{ki}(K) = S_{ik}^*(K)$, and that, for any complex ξ , $\sum \xi_i S_{ik}(K) \xi_k^* \geq 0$. This completes the proof of

Theorem 2. *The class of continuous, positive definite matrix functions $R_{ik}(\mathbf{h})$ is identical with that of Fourier-Lebesgue-Stieltjes transforms (15) of Hermitian matrix measures.*

As a corollary of Theorem 2 and §1, we have the following fundamental result.

Theorem 2'. *The correlation matrix of each admissible homogeneous R.V.F. is associated with one and only one Hermitian matrix measure $S_{ik}(K)$, and conversely, any Hermitian matrix measure is associated with an admissible homogeneous R.V.F.*

We have seen, in §2, that the trace $\sum S_{ii}(K) = \varepsilon(K)$ of $\|S_{ik}(K)\|$ represents the energy associated with wave vectors $\mathbf{\kappa} \in K$, at least in a simple case. The rest of Part C will be devoted to analyzing this idea in more detail.

4. Real case. In the case of *real* homogeneous R. F. V., evidently the correlation matrix is also real. An easy exercise in Fourier transform theory gives the following, essentially known,* corresponding restriction on the spectral matrix measure.

Lemma 3. *In (7), $R_{ik}(\mathbf{h})$ is real if and only if the spectral matrix measure satisfies*

$$(16) \quad S_{ik}(-K) = S_{ik}^*(K),$$

for every Borel set K in R^p .

Explanation. By $-K$ we mean the image of K under reflection in the origin, $\mathbf{\kappa} \rightarrow -\mathbf{\kappa}$. Since the $S_{ik}(K)$ are Hermitian, we have the

Corollary. *The spectral matrix measure of an admissible homogeneous R.V.F. is real, if and only if*

$$(16') \quad S_{ik}(-K) = S_{ik}(K).$$

In the real case, it is appropriate to replace (7) by sine and cosine transforms,

*For the case $p = 1$, $q = 2$, see H. CRAMÉR, *Annals of Math.* 41 (1940), pp. 226-228.

writing

$$(17) \quad R_{ik}(\mathbf{h}) = \int_{\frac{1}{2}R^p} \cos \boldsymbol{\kappa} \cdot \mathbf{h} \, dA_{ik}(\boldsymbol{\kappa}) + \sin \boldsymbol{\kappa} \cdot \mathbf{h} \, dB_{ik}(\boldsymbol{\kappa}),$$

where $\frac{1}{2}R^p$ is the set on which $\kappa_1 + \dots + \kappa_p > 0$ plus half of its boundary, while $A_{ik}(K)$ and $B_{ik}(K)$ obviously satisfy

$$(17') \quad 2S_{ik}(K) = A_{ik}(K) - iB_{ik}(K).$$

Since $S_{ik}(K)$ is Hermitian, the following basic result is immediate.

Theorem 3. *The covariance matrix of any admissible homogeneous real R.V.F. is the Lebesgue-Stieltjes cosine transform of a positive definite symmetric matrix measure function $A_{ik}(K)$, plus the sine transform of a skew-symmetric matrix measure function $B_{ik}(K)$.*

The characterization of the precise class of $B_{ik}(K)$ compatible with a given $A_{ik}(K)$ seems to be fairly involved, and we shall not enter into this question here. (CRAMÉR, *op. cit. supra*, has treated the case $p = 1$, $q = 2$.) Instead, we shall consider an instructive example.

Example. Let the probability sample space consist (much as in the Example of §1) of simple harmonic functions, of the form

$$(18) \quad \mathbf{u}(\mathbf{x}, \omega) = \mathbf{a} \cos(\boldsymbol{\kappa} \cdot \mathbf{x} + \alpha) + \mathbf{b} \sin(\boldsymbol{\kappa} \cdot \mathbf{x} + \alpha),$$

where $\mathbf{a} = \mathbf{a}(\omega)$, $\mathbf{b} = \mathbf{b}(\omega)$, $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\omega)$, and the α are uniformly distributed on $0 \leq \alpha < 2\pi$, to ensure homogeneity.

Since all samples are in Λ , any such R.V.F. for which $\mathbf{a}$, $\mathbf{b}$, and $\boldsymbol{\kappa}$ are all Borel functions is necessarily admissible. Moreover, for each fixed $\boldsymbol{\kappa}$, an easy computation gives

$$(18') \quad 2R_{ik}(\mathbf{h}) = (\overline{a_i a_k} + \overline{b_i b_k}) \cos \boldsymbol{\kappa} \cdot \mathbf{h} + (\overline{a_i b_k} - \overline{b_i a_k}) \sin \boldsymbol{\kappa} \cdot \mathbf{h}.$$

That is, in (18)–(18'), if bars denote averages with respect to $d\mu(\omega)$, for fixed $\boldsymbol{\kappa}$,

$$dA_{ik}(\boldsymbol{\kappa}) = (\overline{a_i a_k} + \overline{b_i b_k}), \quad dB_{ik}(\boldsymbol{\kappa}) = (-\overline{a_i b_k} + \overline{b_i a_k}).$$

By a simple translation of the phase-parameter α , we can make $\mathbf{a} \cdot \mathbf{b} = 0$ and (writing $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$) $a \geq b$. By a rigid rotation of axes, we can thus reduce (19) to the special form

$$(19) \quad \begin{aligned} 2R_{11}(\mathbf{h}) &= a^2 \cos \boldsymbol{\kappa} \cdot \mathbf{h}, & 2R_{22}(\mathbf{h}) &= b^2 \cos \boldsymbol{\kappa} \cdot \mathbf{h}, \\ 2R_{12}(\mathbf{h}) &= -2R_{21}(\mathbf{h}) = ab \sin \boldsymbol{\kappa} \cdot \mathbf{h}. \end{aligned}$$

Formulas (19) are closely related to those of CRAMÉR, *op. cit.*, p. 228.

Various probability averages in (a, b) -space give rise to the same covariance matrix (19); hence there is no obvious general intuitive interpretation of the spectral matrix only of its *trace*, which is a cosine transform defined by (13).

In the case $p = q = 3$ of greatest interest for us, B_{ik} is a spin-matrix, by a

classic theorem of EULER. We can then resolve $\mathbf{u}(\mathbf{x}, \omega)$ into components parallel and perpendicular to the spin-vector; the latter *rotates* about the spin vector as $\mathbf{x}$ moves parallel to $\boldsymbol{\kappa}$.

5. Normal case. In the case of (homogeneous, real) *normal* R. V. F., the existence theorem of §3 can be complemented by a “strict sense” uniqueness theorem on the probability measure. In fact, by Theorem 3 of Part B, the “wide sense” uniqueness of $R_{ik}(\mathbf{h})$ —or, equivalently, of $S_{ik}(K)$ —implies such a strict sense uniqueness theorem. That is, we can assert

Theorem 4. *Formulas (1) and (15) establish a one-one correspondence between the class of normal admissible (real, homogeneous) R.V.F., and the class of positive Hermitian measures (“spectral measures”) which satisfy (16).*

In Theorem 4, the relation (16') could obviously be substituted for (16). However, to obtain a uniqueness theorem for the case of complex R.V.F., one must assume complex-isotropy as defined in §2.

6. Isotropic case. In the case of homogeneous *isotropic* R. V. F., with $p = q$ (cf. Part B, §7), the trace function

$$(20) \quad \rho(\mathbf{h}) = \sum R_{ii}(\mathbf{h})$$

is a function of $h = |\mathbf{h}|$ alone. One can express $\rho(h)$ as a Lebesgue-Stieltjes Hankel transform of a scalar *energy distribution function*, a (scalar) *energy spectral measure* $\Phi(K)$, defined on Borel subsets on the positive real axis of wave numbers κ . Namely, we can apply the following theorem of SCHOENBERG.*

Schoenberg's Theorem. *A single-valued function $\rho(h) = \rho(|\mathbf{h}|)$ is continuous and positive definite on R^p if and only if*

$$(21) \quad \rho(h) = \Gamma(\tfrac{1}{2}p) \int_0^{+\infty} \left(\frac{2}{kh}\right)^{\frac{1}{2}(p-2)} J_{\frac{1}{2}(p-2)}(kh) d\Phi(k), \quad h = |\mathbf{h}|,$$

for some bounded non-decreasing function $\Phi(k)$.

Corollary 1. *The class of trace functions of admissible, real, homogeneous isotropic random velocity fields, in p dimensions, is the class of Lebesgue-Stieltjes Hankel transforms (21) of “energy distribution” functions $\Phi(k)$.*

In the special case $p = 3$, this simplifies to

Corollary 2. *The class of trace functions (20) of admissible three-dimensional homogeneous isotropic random velocity-fields is the class of sine transforms*

$$(22) \quad \rho(h) = \int_0^{+\infty} \frac{\sin kh}{kh} d\Phi(k),$$

where $\Phi(k)$ is non-decreasing and bounded on R^p .

*I. J. SCHOENBERG, *Annals of Math.* 39 (1938), pp. 811–841. By positive definite, we mean that (3) holds.

Parity. In the case of reflection symmetry (alias “parity”), by definition $\mathbf{u}(-\mathbf{x}, \theta) = \mathbf{u}(\mathbf{x}, \theta')$ for some measure-preserving transformation $\theta \rightarrow \theta'$ of Λ . Hence $R_{ik}(-\mathbf{h}) = R_{ik}(\mathbf{h})$. Combining with (2), we get $R_{ik}(\mathbf{h}) = R_{ik}(\mathbf{h})$ if the R.V.F. are real. This proves

Theorem 5. *In the case of parity, $R_{ik}(\mathbf{h})$ is a symmetric matrix, and so $B_{ik}(\mathbf{\kappa}) = 0$ and the correlation matrix is a pure cosine transform.*

7. Incompressible case. In Part B, §10, each of two conditions (31) and (33) was shown to be necessary and sufficient for the incompressibility of (sufficiently differentiable) R.V.F. However, differentiating (7) with respect to the h_i under the integral sign and using (33), we get

$$(23) \quad 0 = \sum_i \frac{\partial^2 R_{ik}}{\partial h_i}(\mathbf{h}) = \int \sin(\mathbf{\kappa} \cdot \mathbf{h}) \sum \kappa_i dA_{ik}(\mathbf{\kappa}) - \int \cos(\mathbf{\kappa} \cdot \mathbf{h}) \sum \kappa_i dB_i(\mathbf{\kappa}).$$

By the usual uniqueness theorem for Lebesgue-Stieltjes transforms,* this implies the following result, which is well known [7, p. 604] under somewhat less precise assumptions.

Theorem 6. *If differentiable, a homogeneous admissible R.V.F. is incompressible if and only if its spectral measure is confined to the set where*

$$(24) \quad \sum \kappa_i A_{ik} = \sum \kappa_i B_{ik} = \sum A_{ik} \kappa_k = \sum B_{ik} \kappa_k = 0.$$

This is equivalent, for the Example of §4, by Theorem 7 of Part B, to the conditions that

$$(25) \quad \mathbf{a} \cdot \mathbf{\kappa} = \mathbf{b} \cdot \mathbf{\kappa} = 0 \quad \text{in (18).}$$

It is interesting to specialize these conditions to the case $p = q = 3$ of real turbulence; we can build up general flows, by linear combination and rotation and translation of axes, from the following two cases:

$$(26a) \quad \mathbf{u}(\mathbf{x}) = a(\cos kz, 0, 0) \quad (\text{Boussinesq flows}^{**})$$

and

$$(26b) \quad \mathbf{u}(\mathbf{x}) = a(\cos kz, \sin kz, 0),$$

which we shall call *circularly polarized*.

For the homogeneous R.V.F. defined by the Boussinesq flow (26a) and its translates, $R_{11}(\mathbf{h}) = (\frac{1}{2}a^2) \cos kh_3$, and all other components of the correlation

*We assume here that $|\mathbf{\kappa}| dA_{ik}(\mathbf{\kappa})$ and $|\mathbf{\kappa}| dB_{ik}(\mathbf{\kappa})$ are both associated with bounded measure functions. This is associated with the differentiability restrictions discussed in Part B, §10.

**So named in [2], where it was shown that homogeneous and isotropic incompressible normal R. V. F. could be built up from such “Boussinesq flows” by “Gaussian convolution”.

matrix are zero. For the circularly polarized homogeneous R.V.F. defined by (26b) and its translates, $R_{ik}(\mathbf{h})$ is defined by the matrix of Fig. 1.

$$\frac{a^2}{2} \begin{bmatrix} \cos kh_3 & \sin kh_3 & 0 \\ -\sin kh_3 & \cos kh_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Fig. 1

For the *homogeneous, isotropic* R.V.F. defined by the Boussinesq flow (26a), and its transforms under the Euclidean group, $R_{ik}(\mathbf{h})$ is symmetric and given by the usual Karman-Howarth formulas; these will be discussed in §9. This is true whether reflection-symmetry is postulated or not.

For that defined by the circularly polarized flow (26b) and its transforms under rigid motions (with determinant $+1$), on the other hand, $R_{ik}(\mathbf{h})$ is *not* symmetric. This is easily shown using the following lemma.

Lemma 4. *Under orthogonal coordinate transformations T of homogeneous R.V.F. with $p = q$, $R_{ik}(\mathbf{h}T)$, $A_{ik}(\mathbf{h}T)$, and $B_{ik}(\mathbf{h}T)$ are given by the usual transformation rules $R \rightarrow T^{-1}RT$, $A \rightarrow T^{-1}AT$, and $B \rightarrow T^{-1}BT$.*

Proof. By direct calculation of $(\mathbf{u}T)_i(\mathbf{u}T)_k$, the formula may be verified for $R_{ik}(\mathbf{h}T)$. Since $\mathbf{h} \cdot \boldsymbol{\kappa} = \mathbf{h}T \cdot \boldsymbol{\kappa}T$, the corresponding formula follows for $A_{ik}(\boldsymbol{\kappa}T)$ and $B_{ik}(\boldsymbol{\kappa}T)$ from (17).

Applying Lemma 4 to the matrix of Fig. 1, we confirm the obvious conclusion that $B_{ik}(K)$ is invariant under rigid motions leaving the κ_3 -axis fixed (*i.e.*, leaving $\boldsymbol{\kappa}$ invariant). However, if *reflection* symmetry is assumed, then the off-diagonal terms cancel, and we have the same correlation and spectral matrices as for Boussinesq flows. This is also obvious from Theorem 5 and known results.

8. Absolutely continuous spectrum. Adopting standard usage, we shall make the following definition.

Definition. An admissible homogeneous R.V.F. will be said to have an *absolutely continuous spectrum* if and only if the total energy in a region K of wave-vector space tends uniformly to zero with the measure $m(\boldsymbol{\kappa})$ of K .

By the Lebesgue-Radon-Nikodym Theorem, this is equivalent to the possibility of writing

$$(27) \quad \rho(\mathbf{h}) = \int_{R^p} \exp(i\boldsymbol{\kappa} \cdot \mathbf{h}) E(\boldsymbol{\kappa}) dm(\boldsymbol{\kappa}),$$

for a suitable *energy density* $E(\boldsymbol{\kappa})$ per unit volume in wave-number space. In particular, setting $\mathbf{h} = \mathbf{0}$, we have

$$(27^*) \quad \bar{E}_D = \int_{R^p \times D} E(\boldsymbol{\kappa}) dm(\boldsymbol{\kappa}) dm(\mathbf{x}) = m(D) \int_{R^p} E(\boldsymbol{\kappa}) dm(\boldsymbol{\kappa}).$$

The physical significance of these formulas is obvious from the discussion of §1.

From the non-negative definiteness of $S_{jk}(K)$, it follows that absolute continuity of its trace $\sigma(K) = \sum S_{jj}(K)$ implies the absolute continuity of each (j, k) -entry. In fact, (8) implies that

$$|S_{jk}(K)| \leq [S_{jj}(K)]^{\frac{1}{2}} [S_{kk}(K)]^{\frac{1}{2}} \leq \sigma(K),$$

and thus if $\sigma(K) \downarrow 0$ as $m(K) \downarrow 0$, then $S_{jk}(K) \downarrow 0$ as $m(K) \rightarrow 0$. Therefore we can state the following result.

Theorem 7. *For any (admissible, homogeneous) R.V.F. with absolutely continuous spectrum, we can write*

$$(28) \quad R_{jk}(\mathbf{h}) = \int_{R_p} \exp(i\mathbf{\kappa} \cdot \mathbf{h}) \Phi_{jk}(\mathbf{\kappa}) \, dm(\mathbf{\kappa}),$$

for a suitable spectral density matrix $\Phi_{jk}(\mathbf{\kappa})$.

This result was stated in [7, p. 604] under less precise hypotheses; the class of spectral density matrices of complex R.V.F. with absolutely continuous spectra was also characterized there. One of us* has proved that the spectrum is absolutely continuous if and only if

$$\lim_{|\mathbf{h}| \rightarrow 0} (2H)^{-p} \int_{-H \leq h_i \leq H} \sum_{j,k} [R_{jk}(\mathbf{h})]^2 \, dm(\mathbf{h}) = 0.$$

In this result, the integrand could be taken as $\rho(\mathbf{h})$ instead of $\sum R_{jk}(\mathbf{h})^2$.

Corollary. *In the absolutely continuous, homogeneous isotropic case, with $p = q = 3$ and $k = |\mathbf{\kappa}|$,*

$$(29) \quad \bar{E} = \int_0^\infty E(k) \, dk, \quad \text{where} \quad E(k) = 4\pi k^2 E(\mathbf{\kappa}).$$

BATCHELOR [1, p. 25] simply assumes absolute continuity without comment; thus he quotes incompletely the Theorem of CRAMÉR and KHINCHINE.** We shall discuss the physical significance of this assumption in §9.

9. Homogeneous isotropic turbulence. The preceding results can be applied directly to the case $p = q = 3$ (with reflection symmetry), so as to shed light on the theory of homogeneous isotropic turbulence.

First, a word as to the assumption of an "absolutely continuous spectrum". The usual schemes for producing homogeneous turbulence involve passing air

*J. KAMPÉ DE FÉRIET, *Nav. Ord. Lab. Rep.* 1136 (1949), p. 11. In the case $p = 1$, the proposition goes back to P. LÉVY, *Calcul des probabilités*, Paris, 1925, p. 171.

**In the same vein, we feel impelled to point out that BATCHELOR's formula (2.5.3) does not have any recognized mathematical meaning for individual velocity fields; thus the discussion on pp. 31-33 of [1] is also hard to interpret rigorously, however attractive it may be intuitively.

through a periodic grid; strictly speaking, the turbulence produced thereby might well have a finite energy component having wave-vectors commensurable with this period.* (We recall that between a “discrete spectrum” and an “absolutely continuous spectrum” lies the possibility of having a finite energy concentrated in a set of measure zero, but with zero energy associated with each individual wave vector. Readers who are unfamiliar with this fact should read carefully standard statements of the Lebesgue-Radon-Nikodym Decomposition Theorem ([5, p. 134] or [11, p. 130]).

However, we do not wish to go into these refinements here; instead, we wish to concentrate on stating a precise general theorem about the class of correlation matrices associated with homogeneous, isotropic (with statistical reflection symmetry), real, incompressible random velocity fields in ordinary three dimensional space. This will serve to make precise the main result of [2].

Theorem 8. *The class of admissible normal R.V.F. having the properties usually assumed in the kinematic theory of homogeneous, isotropic turbulence, corresponds one-one to the class of energy distributions $\mathcal{E}(k)$ on wave-number space. For any such energy distribution, the longitudinal and transverse correlation functions are given, as in [2, (23)], by*

$$(30) \quad f(h) = \int_0^\infty \frac{3}{h^2 k^2} \left[\frac{\sin hk}{hk} - \cos hk \right] d\mathcal{E}(k),$$

$$(31) \quad g(h) = \frac{1}{E} \int_0^\infty \frac{1}{2h^2 k^2} \left[(1 + hk) \cos kh - \frac{\sin hk}{hk} \right] d\mathcal{E}(k).$$

Formula (30) is essentially [1, (3.4.16)], with $u^2 = E = \rho(\mathbf{0})$. To derive (31) from (30), rewrite [1, (3.4.6)] as $2rg = (r^2 f)'$. In our notation, $r = h$, and $f(r) = R_{11}(h)/E$.

For each such $\mathcal{E}(\kappa)$, the rotations, translations and reflections of R^p ($p = q = 3$) generate an interesting *representation* of the *Euclidean group* on the space Λ of $\mathbf{u}(\mathbf{x}, \alpha)$. It would be interesting to work out the structure of these *group representations*, which are examples involving natural but unfamiliar topological linear spaces. (If Λ is treated as a measure space only, we are led to an interesting representation of the Euclidean group on Hilbert space; that of the translation-subgroup is discussed in [7, §15].)

PARTIAL GLOSSARY OF NOTATION

$m(S)$	Borel measure of set S
p	dimension of domain of vector field
q	dimension of range of vector field
$C(D)$	Banach space of functions continuous on D
D	compact set in R^p

*Some experimental evidence supporting this idea has been presented by H. L. GRANT & I. C. T. NISBET, *J. Fluid Mech.* 2 (1957), pp. 263–272.

I	unit interval
$L_2(D)$	Hilbert space of functions quadratically (Lebesgue) integrable on D
R	infinite interval — $-\infty < x < \infty$
α	element of I
θ	element of Λ
μ	probability measure
Γ	space of (essentially) continuous vector fields
Λ	space of Borel vector fields, quadratically integrable on any compact set
Ω	(abstract) probability space

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Harvard University
Cambridge, Massachusetts