Linear Growth, Birth and Death Processes

SAMUEL KARLIN & JAMES MCGREGOR

Communicated by W. FELLER

A birth and death process is a stationary Markoff process $x(t)$ whose state space is the non-negative integers and whose transition probability matrix

$$P_{ij}(t) = \Pr \{x(t) = j | x(0) = i\}$$

satisfies the conditions (as $t \rightarrow 0$)

$$(1) \quad P_{ij}(t) = \begin{cases} \lambda_i t + o(t) & \text{if } j = i + 1 \\ \mu_i t + o(t) & \text{if } j = i - 1 \\ 1 - (\lambda_i + \mu_i)t + o(t) & \text{if } j = i \end{cases}$$

where $\lambda_i > 0$ for $i \geq 0$, $\mu_i > 0$ for $i \geq 1$ and $\mu_0 \geq 0$. The process is called a linear growth process if $\lambda_n = \lambda n + a$ and $\mu_n = \mu n + b$ with $\lambda > 0$ and $\mu > 0$. Such processes occur naturally in the study of biological reproduction and population growth. If the state of the system n describes the size of the population, then the average instantaneous rate of growth is $\lambda n + a$. Similarly, the probability of the state of the process decreasing by one after the lapse of a small period of time is $(\mu n + b)t + o(t)$. The factor λn represents the growth of the population due to the current size of the population. The second factor a may be interpreted as the increase of the state of the system due to an external source. The two components μn and b which compose the death rate possess an analogous meaning.

These birth and death processes represent the most elementary formulations of probabilistic models describing biological growth. Various aspects of these processes have received considerable attention in the recent literature [4] (see page 407). Nevertheless, a thorough understanding of the nature of these simple models is still lacking. For a complete set of references and a lucid survey of the subject matter of linear growth processes, the reader should consult [11] and the extensive bibliography quoted therein.

Our approach to the study of linear growth models derives basically from our knowledge of the elaborate structure of birth and death processes as developed

Prepared under contract Nonr-225(28) (NR-047-019) for Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.

in [6] and [7]. The analysis relies decisively on the integral representation formulas and recognition of the orthogonal polynomial systems associated with linear growth processes. The orthogonal polynomials which are relevant for linear growth models are related to the classical Meixner polynomials and to the Laguerre polynomials. By virtue of the vast detailed information available concerning these polynomial systems, very precise statements can be made regarding these types of linear growth processes.

From the same point of view we studied extensively queueing processes for several servers with Poisson input and exponential service rate. For details we refer the reader to [8].

The infinitesimal matrix of the general birth and death process is of the form

$$(2) \quad A = \begin{pmatrix} -(\lambda_0 + \mu_0) & \lambda_0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix determines a system of polynomials by means of the recurrence relations

$$(3) \quad \begin{cases} Q_0(x) \equiv 1, \\ -xQ_0(x) = -(\lambda_0 + \mu_0)Q_0(x) + \lambda_0Q_1(x), \\ -xQ_n(x) = \mu_nQ_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_nQ_{n+1}(x). \end{cases}$$

It is shown in [6] that there is a positive regular measure ψ on $0 \leq x < \infty$ for which the orthogonality relations

$$\int_0^\infty Q_i(x)Q_j(x) d\psi(x) = \frac{\delta_{ij}}{\pi_j}, \quad i, j = 0, 1, \dots,$$

where

$$\pi_0 = 1, \quad \pi_n = \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n},$$

are valid. In the case of a linear growth process, the measure ψ is unique [6] and, moreover, the transition probability matrix $P(t) = (P_{ij}(t))$ of the process is uniquely determined by A . It has the representation

$$(4) \quad P_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) d\psi(x).$$

This is an extremely useful formula for the transition probability function, and our first task will be to compute the polynomials $\{Q_n(x)\}$ and the spectral measure ψ belonging to the various linear growth processes. Often we shall refer to the state of the process as the position of a diffusing particle subject to the transition laws defining the birth and death process.

Because of their probabilistic interpretations a distinction must be made between the two types of birth and death processes according as $\mu_0 = 0$ or $\mu_0 > 0$. In the former case the state zero is a reflecting barrier in the sense that whenever the particle reaches zero, a transition must occur in finite time which moves the particle back to state 1. But if μ_0 is positive such a transition occurs only with probability $\lambda_0/(\lambda_0 + \mu_0)$. One of the important structural questions regarding such processes is to determine whether recurrence to a prescribed state of the system is a certain event. Since all states communicate it is sufficient to resolve this problem for the state zero. An appeal to the general theory of birth and death processes shows that the states are recurrent if and only if $\sum_k 1/\lambda_k \pi_k$ diverges [7]. In that circumstance the recurrence time distribution has a finite mean value (the process is ergodic) if and only if $\sum_k \pi_k < \infty$. Moreover, $\pi_k/\sum_{k=0}^{\infty} \pi_k$ may then be interpreted as the limiting stationary probability that the state of the system is k .

When the process is non-recurrent or transient, then we distinguish two kinds of transient behavior which measure the rate at which the particle drifts to infinity.

We say that the process is weakly transient if the process is non-recurrent and $\sum_{i=0}^{\infty} P_{ij}(t) \equiv 1$ for all finite t and any i . The occurrence of this phenomenon is equivalent to [7]

$$\sum_{n=0}^{\infty} \pi_n \sum_{k=n}^{\infty} \frac{1}{\lambda_k \pi_k} = \infty.$$

The process is called strongly transient if, for every $t > 0$ and all i , $\sum_{i=0}^{\infty} P_{ij}(t) < 1$. These two kinds of transient behavior describe all the possibilities of non-recurrence [7].

If $\mu_0 > 0$ then the process possesses an ignored absorbing state at -1 , a state in which the system remains forever once it arrives there. When the system is in the zero state and a transition occurs, the system moves to state 1 with probability $\lambda_0/(\lambda_0 + \mu_0)$ and is absorbed with probability $\mu_0/(\lambda_0 + \mu_0)$. Starting in a given state i , the problem can be posed as to whether the particle will be absorbed with certainty into the -1 state. The condition that this happens, as one would expect, is again equivalent to the divergence of the series $\sum_{k=0}^{\infty} 1/\lambda_k \pi_k$. The distribution for the time of absorption for any prescribed initial state can be likewise computed (see [7]).

In the same way the answers to many of the natural probabilistic queries may be obtained by applying the general structure theorems of birth and death processes [7]. Still more precise information about special process can be gained by expressing the orthogonal polynomials and the spectral measure in workable form.

In section one we summarize some of the pertinent formulas connected with Meixner and Laguerre polynomials. Of particular use are the recurrence laws satisfied by these polynomial systems and their generating functions.

In the following section we determine the linear growth models of which

the Meixner polynomials constitute the corresponding polynomial systems. There are four linear growth processes connected with Meixner polynomials. Two of these processes possess an ignored -1 state and the other two have zero as a reflecting barrier. These four processes are obtained by different types of normalization procedures applied to the classical Meixner polynomials. An enumeration of several of the quantities of interest for these processes is given in Table 1.

Recurrence and absorption questions are discussed at the close of Section 2. Section 3 treats all linear processes connected with Laguerre polynomials. One special example was investigated earlier by REUTER & LEDERMAN [12]. In Section 4 we utilize the generating function formulas displayed in Section 1 to derive explicit formulas for the transition probabilities. In Section 5 we restrict attention to special linear growth processes involving an absorbing state (-1) . For these models we investigate the physically significant random variable which counts the number of transitions that transpire before absorption. The explicit determination of the distribution of this random variable derives from properties of systems of ultraspherical polynomials.

The concept of the "related process" where the zero state is converted into an absorbing state is introduced in Section 6. The formalism of the "related process" is exploited so as to characterize the form of the spectral measure corresponding to linear growth processes distinct from those expressed in terms of the Meixner and Laguerre polynomials.

Other quantities of interest related to linear growth processes, such as coincidence probabilities and occupation time probabilities, will be published separately. The final section discusses the orthogonal polynomial system and the spectral measure of the classical Ehrenfest model.

1. Summary of formulas for Meixner polynomials and Laguerre systems [3], [13]. The classical hypergeometric function is given by

$$(5) \quad F(a, b, c, z) = \sum \frac{(a)_n (b)_n z^n}{n! (c)_n}, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Define

$$(6) \quad \phi_n(x) = \phi_n(x; \beta, \gamma) = F\left(-n, -x, \beta, 1 - \frac{1}{\gamma}\right)$$

for $\beta > 0$ and $0 < \gamma < 1$ and set $\phi_{-1} \equiv 0$. Then $\phi_n(x)$ constitute a system of polynomials and

$$m_n(x; \beta, \gamma) = (\beta)_n \phi_n(x; \beta, \gamma)$$

are the Meixner polynomials [3] which are orthogonal with respect to a jump function with jumps at $x = 0, 1, 2, \dots$. Specifically

$$(7) \quad \sum_{x=0}^{\infty} \phi_m(x) \phi_n(x) \rho_x = \delta_{m,n} \frac{n!}{(\beta)_n \gamma^n},$$

where

$$(8) \quad \rho_x = (1 - \gamma)^\beta \frac{(\beta)_x}{x!} \gamma^x.$$

From a contiguity relation satisfied by the hypergeometric function (see [3]) it follows that

$$(9) \quad -x \frac{(1 - \gamma)}{\gamma} \phi_n(x) = \frac{n}{\gamma} \phi_{n-1} - \left(n + \frac{n}{\gamma} + \beta \right) \phi_n + (n + \beta) \phi_{n+1}.$$

An important generating function based on the polynomials ϕ_n is

$$(10) \quad \sum_{n=0}^{\infty} \phi_n(x) \frac{(\beta)_n}{n!} s^n = \left(1 - \frac{s}{\gamma} \right)^x (1 - s)^{-x-\beta}$$

which converges for $|s| < \gamma$, at least.

This last identity may be derived directly from the definition of ϕ_n . Alternatively the reader is referred to [3].

Finally we summarize several formulas concerned with the Laguerre system of polynomials.

Let $L_n^\alpha(x)$ denote the n^{th} Laguerre polynomial normalized so that $L_n^\alpha(0) = \binom{n + \alpha}{n}$ and set

$$(11) \quad \psi_n(x) = \frac{1}{\binom{n + \alpha}{n}} L_n^\alpha\left(\frac{x}{\kappa}\right)$$

so that $\psi_n(0) \equiv 1$. These polynomials satisfy the recurrence relations

$$(12) \quad -x\psi_n(x) = n\kappa\psi_{n-1} - (2n\kappa + (\alpha + 1)\kappa)\psi_n + (n\kappa + (1 + \alpha)\kappa)\psi_{n+1}$$

with $\psi_{-1} \equiv 0$ and $\psi_0 \equiv 1$ ($\alpha > -1$), $n \geq 0$, and are orthogonal with respect to a density function defined on the positive real axis given by

$$(13) \quad \rho(x) = ce^{-x/\kappa} x^\alpha$$

where c is a normalizing constant.

Some pertinent generating functions associated with Laguerre systems of polynomials are

$$(14) \quad \sum L_n^\alpha(x) s^n = (1 - s)^{-\alpha-1} e^{xs/(s-1)},$$

$$(15) \quad \sum \frac{L_n^\alpha(x) w^n}{\Gamma(n + \alpha + 1)} = e^w (xw)^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{xw})$$

where J_α stands for the usual Bessel function of order α . Other formulas, as needed, will be cited later in the text.

2. Linear growth processes connected with various kinds of Meixner polynomial systems. Starting with the relation (9), and employing appropriate

renormalizations of ϕ_n and affine transformations of x , we may derive a series of four different recurrence laws which correspond to different kinds of processes.

Case I: For $\lambda < \mu$, define

$$(16) \quad \phi_n^{(1)}(x) = \phi_n\left(\frac{x}{\mu - \lambda}; \beta, \frac{\lambda}{\mu}\right).$$

It follows readily from (9) that for $n \geq 0$

$$-x\phi_n^{(1)}(x) = n\mu\phi_{n-1}^{(1)} - (n\lambda + n\mu + \beta\lambda)\phi_n^{(1)} + (n\lambda + \beta\lambda)\phi_{n+1}^{(1)}.$$

The polynomials $\phi_n^{(1)}$ constitute a polynomial system orthogonal with respect to the discrete mass measure $\rho(x)$ which concentrate masses

$$\left(1 - \frac{\lambda}{\mu}\right)^{-\beta} \frac{(\beta)_n}{n!} \left(\frac{\lambda}{\mu}\right)^n$$

at the points $(\mu - \lambda)n$ for $n = 0, 1, 2, \dots$ (see Equation (8)). We see, because of (9), that the process with birth rates $\lambda_n = n\lambda + \beta\lambda$ and death rates $\mu_n = n\mu$ for $n \geq 0$, where $\lambda < \mu$, is associated with the polynomial system $\phi_n^{(1)}$ (where β is an arbitrary positive parameter).

Case II: Multiplying $\phi_n^{(1)}(x)$ by the factor $(\lambda/\mu)^n$ and suitably translating the variable x we arrive at a set of polynomials

$$(17) \quad \phi_n^{(2)}(x) = \left(\frac{\lambda}{\mu}\right)^n \phi_n\left(\frac{x}{\mu - \lambda} - \beta; \beta, \frac{\lambda}{\mu}\right)$$

which satisfy

$$-x\phi_n^{(2)}(x) = n\lambda\phi_{n-1}^{(2)} - (n\mu + n\lambda + \beta\mu)\phi_n^{(2)} + (n + \beta)\mu\phi_{n+1}^{(2)}$$

for $n \geq 0$ with $\lambda < \mu$.

The effect of the above transformation is, in substance, to interchange the parameters λ and μ . The birth rates are now identified with $(n + \beta)\mu$ and the death rate is seen to be $n\lambda$. The corresponding polynomials $\phi_n^{(2)}(x)$ are orthogonal with respect to the measure which consists of the discrete masses $(1 - \gamma)^{-\beta} (\beta)_n \gamma^n / n!$ located at the points

$$(n + \beta)(\mu - \lambda), \quad n = 0, 1, 2, \dots,$$

where $\gamma = \lambda/\mu$.

Cases I and II are related to growth processes for which the state 0 was a reflecting barrier. By means of further normalizations of the Meixner polynomials we can obtain two additional classes of growth processes where there exists an absorbing state at -1 .

With $\lambda < \mu$ we have the following:

Case III: The polynomials

$$(16a) \quad \phi_n^{(3)}(x) = \frac{(\beta)_n}{n!} \phi_n\left(\frac{x}{\mu - \lambda} - \beta + 1; \beta; \frac{\lambda}{\mu}\right)$$

satisfy

$$\begin{aligned} -x\phi_n^{(3)}(x) &= (n + \beta - 1)\mu\phi_{n-1}^{(3)} \\ &\quad - [(n + \beta - 1)\mu + (n + 1)\lambda]\phi_n^{(3)} + (n + 1)\lambda\phi_{n+1}^{(3)}. \end{aligned}$$

Case IV: The polynomials

$$(17a) \quad \phi_n^{(4)}(x) = \frac{(\beta)_n}{n!} \left(\frac{\lambda}{\mu}\right)^n \phi_n\left(\frac{x}{\mu - \lambda} - 1; \beta; \frac{\lambda}{\mu}\right)$$

satisfy

$$\begin{aligned} -x\phi_n^{(4)}(x) &= (n + \beta - 1)\lambda\phi_{n-1}^{(4)} \\ &\quad - [(n + \beta - 1)\lambda + (n + 1)\mu]\phi_n^{(4)} + (n + 1)\mu\phi_{n+1}^{(4)}. \end{aligned}$$

The results are summarized in the following table. In order to avoid any ambiguities in interpretation of the various entres in the table, we review the meaning of some of the notation used:

- (i) $\gamma = \frac{\lambda}{\mu}$ and always $\lambda < \mu$.
- (ii) $\phi_n^{(k)}(\cdot; \beta, \gamma)$ is an n^{th} degree polynomial defined by one of (16), (17), (16a), (17a).
- (iii) $\rho_n = (1 - \gamma)^{-\beta} \frac{(\beta)_n \gamma^n}{n!}$.

All the results of the table are easily deduced by beginning with the recursion formula (9) and making the appropriate transformations. It is easy to show that in this manner we have obtained, in fact, all possible linear growth models connected with the classical Meixner polynomials.

It is worthwhile at this point to highlight some obvious qualitative consequences concerning the description of processes C through F. Several of these results depend on a knowledge of the asymptotic growth of π_n and $1/\lambda_n\pi_n$. The familiar asymptotic relations $(\beta)_n/n! \sim \Gamma(\beta)n^{\beta-1}$ are useful in this connection.

Process C.

Since $\sum 1/\lambda_n\pi_n = \infty$ we find that absorption into the -1 state takes place with certainty in finite time [7]. Moreover, the distribution of the time until absorption from an initial prescribed state has all moments finite. This follows because of the existence of the integrals $\int_0^\infty x^{-n-1}d\psi(x)$ for each n .

Process D.

In this case absorption into the -1 state is not a certain event as $\sum 1/\lambda_n\pi_n < \infty$. However, since

$$\sum_{n=0}^{\infty} \pi_n \sum_{k=n}^{\infty} \frac{1}{\lambda_k\pi_k} > C \sum \frac{1}{n} = \infty$$

TABLE 1

Process Label	Birth Rates	Death Rates	Character of Zero State	Polynomial System	Spectral Measure	π_n	$\frac{1}{\lambda_n \pi_n}$
C	$(n+1)\lambda$	$(n+\beta-1)\mu$ $\beta \geq 1$	If $\beta > 1$ then a permanent absorbing state exists at -1 and $\mu_0 = (\beta-1)\mu$.	$\phi_n^{(3)}(x)$	Masses ρ_n located at $(\mu-\lambda)(n+\beta-1)$, $n=0, 1, \dots$	$\left(\frac{\lambda}{\mu}\right)^n \cdot \frac{n!}{(\beta)_n}$	$\left(\frac{\mu}{\lambda}\right)^n \cdot \frac{(\beta)_n}{\lambda(n+1)!}$
D	$(n+1)\mu$	$(n+\beta-1)\lambda$ $\beta \geq 1$	For $\beta > 1$ a permanent absorbing state exists at -1 and $\mu_0 = (\beta-1)\lambda$.	$\phi_n^{(4)}(x)$	Masses ρ_n located at $(\mu-\lambda)(n+1)$, $n=0, 1, 2, \dots$	$\left(\frac{\mu}{\lambda}\right)^n \cdot \frac{n!}{(\beta)_n}$	$\left(\frac{\lambda}{\mu}\right)^n \cdot \frac{(\beta)_n}{\mu(n+1)!}$
E	$(n+\beta)\mu$ $\beta > 0$	$n\lambda$	The state 0 is a reflecting barrier ($\mu_0 = 0$).	$\phi_n^{(2)}(x)$	Masses ρ_n located at $(n+\beta)(\mu-\lambda)$, $n=0, 1, 2, \dots$	$\left(\frac{\mu}{\lambda}\right)^n \cdot \frac{(\beta)_n}{n!}$	$\left(\frac{\lambda}{\mu}\right)^n \cdot \frac{n!}{\mu(\beta)_{n+1}}$
F	$(n+\beta)\lambda$ $\beta > 0$	$n\mu$	The state 0 is a reflecting barrier ($\mu_0 = 0$).	$\phi_n^{(1)}(x)$	Masses ρ_n located at $n(\mu-\lambda)$, $n=0, 1, 2, \dots$	$\left(\frac{\lambda}{\mu}\right)^n \cdot \frac{(\beta)_n}{n!}$	$\left(\frac{\mu}{\lambda}\right)^n \cdot \frac{1}{\lambda} \cdot \frac{n!}{(\beta)_{n+1}}$

we conclude that $\lim_{t \rightarrow \infty} \sum_i P_{ii}(t) > 0$. In more descriptive language this states that although the randomly moving particle is not with certainty absorbed into state -1 , there is positive probability that the particle does not drift to infinity [7].

Process E.

The state 0 is now a reflecting barrier. Since $\sum 1/\lambda_n \pi_n < \infty$, all states in the process are transient. However, since $\sum_{n=0}^{\infty} \pi_n \sum_{k=n}^{\infty} 1/\lambda_k \pi_k = \infty$, the process is weakly transient in the sense that $\sum_{i=0}^{\infty} P_{ii}(t) = 1$ for all $t \geq 0$.

Process F.

This process is strongly recurrent in the sense that all states have a recurrence time distribution possessing an infinite number of moments. Inspection of Table 1 shows that the rate of convergence of the transition probabilities $P_{ik}(t)$ to the limiting probabilities $C\pi_k$ is of the order of magnitude $e^{-(\mu-\lambda)t}$.

3. Linear growth processes associated with Laguerre polynomials. Associated with the Laguerre system of polynomials are two processes. The process whose polynomials are given by (11) determines a linear growth process with no absorbing state. The process whose polynomial system is $L_n^\alpha(x/\kappa)$ has a permanent absorbing state at -1 . The descriptions of some of the meaningful quantities of these processes are summarized in the following table (Table 2).

We append some additional facts about Processes A and B.

Process A. In view of the divergence of the series $\sum 1/\lambda_k \pi_k$ for all $\alpha > 0$, we infer that -1 is a certain absorbing state. We also observe that $\int x^{-n-1} d\rho(x) < \infty$ is equivalent to the requirement $\alpha > n$, so that the n^{th} moment of the absorption time distribution is finite if and only if $\alpha > n$. In particular, if $0 < \alpha \leq 1$ then absorption is certain but the expected time of absorption is infinite, while the expected time of absorption is finite if $\alpha > 1$.

Process B. Since $\sum \pi_n = \infty$ for all $\alpha > -1$ and $\sum 1/\lambda_n \pi_n$ also diverges for $-1 < \alpha \leq 0$, we deduce that the process is null recurrent. $\alpha > 0$ implies that $\sum 1/\lambda_n \pi_n < \infty$ so in that circumstance the process is transient. However $\sum_0^\infty \pi_k \sum_{r=k}^\infty 1/\lambda_r \pi_r = \infty$ and hence the process is only weakly transient. We know this for another reason, for if it were strongly transient the spectral measure of the process would have to be discrete which is evidently not true for the case of Process B.

In dealing with transient processes one can investigate the random variable N_{10} representing the first passage time from state 1 to state 0 even if the occurrence of first passage is not a certain event. The natural concept to consider becomes the conditional first passage time given that the event of passing from state 1 to state 0 happened. We may then analyze the properties of this conditional first passage random variable [7]. For example, the problem of the finiteness of $E(N_{10} \mid N_{10} < \infty)$ is known to be equivalent to the convergence

TABLE 2

Process Label	Birth Rates $\lambda_n, n > 0$	Death Rates $\mu_n, n \geq 0$	Poly- nomial System	Spectral Measure	π_n	$\frac{1}{\lambda_n \pi_n}$
A (-1 absorbing state exists, $\alpha > 0$) defined for all $\alpha \geq 0$	$(n+1)_\kappa$	$(n+\alpha)_\kappa$	$L_n^\alpha\left(\frac{x}{\kappa}\right)$	$Ce^{-x/\kappa}x^\alpha$	$\frac{n!}{(\alpha+1)_n} \sim \frac{\Gamma(\alpha+1)}{n^\alpha}$	$\frac{\lambda(\alpha+1)_n}{(n+1)!} \sim \frac{\lambda}{\Gamma(\alpha+1)}$
B (0 reflecting state) $\alpha > -1$	$(n+1+\alpha)_\kappa$	n_κ	$\frac{L_n^\alpha\left(\frac{x}{\kappa}\right)}{\binom{n+\alpha}{n}}$	$Ce^{-x/\kappa}x^\alpha$	$\frac{(\alpha+1)_n}{n!} \sim \frac{n^\alpha}{\Gamma(1+\alpha)}$	$\frac{n!}{\lambda(\alpha+1)_{n+1}} \sim \frac{\Gamma(1+\alpha)}{n^{1+\alpha}}$

of the series [7]

$$\sum_{n=0}^{\infty} \pi_r \left(\sum_{l=r}^{\infty} \frac{1}{\lambda_l \pi_l} \right)^2 \sim \sum \frac{1}{n^\alpha}.$$

Specifically for $\alpha > 1$, the series converges to a finite value. The higher order moments may also be thus evaluated. Other points of interest for these processes may be analyzed in terms of the elaborate structure theorems developed for birth and death processes [6].

4. Generating functions and explicit representations of probabilities. In this section we obtain explicit expressions for the transition probabilities of the various processes introduced previously. The classical approach in securing these formulas was to derive a first order partial differential equation for the generating function of the transition probabilities. This differential equation was solved whenever possible. By our methods the sought for representations and explicit formulas for the transition probabilities emerge as a result of simple manipulations of the corresponding generating function for the appropriate polynomials.

We illustrate the methods for the case of Process F. We first compute $P_{0i}(t)$ from the general theory:

$$(18) \quad P_{0i}(t) = \pi_i \int_0^\infty e^{-x t} Q_i(x) d\rho(x)$$

where Q_n is the polynomial system of Process F and $\rho(x)$ is the spectral measure. These quantities are explicitly given in Table 1. Writing out (18) we have

$$P_{0i}(t) = \left(1 - \frac{\lambda}{\mu}\right)^{-\beta} \pi_i \sum_{n=0}^{\infty} e^{-(\mu-\lambda)nt} \phi_i(n; \beta, \gamma) \frac{(\beta)_n}{n!} \left(\frac{\lambda}{\mu}\right)^n.$$

With the aid of identities $\phi_i(n; \beta, \gamma) = \phi_n(j; \beta, \gamma)$ and Equations (6) and (10), we have

$$(19) \quad \begin{aligned} P_{0i} &= \left(1 - \frac{\lambda}{\mu}\right)^{-\beta} \pi_i \sum \frac{(\beta)_n}{n!} \phi_n\left(j; \beta, \frac{\lambda}{\mu}\right) \sigma^n \\ &= \pi_i \left(1 - \frac{\lambda}{\mu}\right)^{-\beta} \left(1 - \frac{\mu\sigma}{\lambda}\right)^j (1 - \sigma)^{-i-\beta} \end{aligned}$$

where $\sigma = (\lambda/\mu) e^{-(\mu-\lambda)t}$.

Next we seek to obtain an explicit formula for $P_{ij}(t)$. This will depend upon evaluation of a series of the form

$$(20) \quad M_{ii} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \phi_n(i) \phi_n(j) s^n.$$

To this end, we form

$$\sum_{j=0}^{\infty} \frac{(\beta)_j}{j!} z^j M_{ij}$$

which, after using (10) twice and interchanging orders of summation, reduces to the expression

$$\sum_{i=0}^{\infty} \frac{(\beta)_i}{j!} z^i \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \phi_n(i) \phi_n(j) s^n$$

$$= \left(1 - \frac{s}{\gamma}\right)^i (1-s)^{-i-\beta} \left[1 - \frac{1 - \frac{s}{\gamma^2}}{1 - \frac{s}{\gamma}} z\right]^i \left[1 - \frac{1 - \frac{s}{\gamma}}{1 - s} z\right]^{-i-\beta}.$$

Expanding in powers of z and reading off the coefficient of z^i (where for definiteness we have assumed that $i \leq j$) we get

$$(21) \quad \frac{j!}{(\beta)_i} \sum_{k=0}^i \binom{i}{k} (-1)^k \left[\frac{1 - \frac{s}{\gamma^2}}{1 - \frac{s}{\gamma}} \right]^k \frac{(i + \beta)_{i-k}}{(j-k)!} \left[\frac{1 - \frac{s}{\gamma}}{1 - s} \right]^{i-k}$$

except for a factor $[1 - (s/\gamma)]^i (1-s)^{-i-\beta}$.

In order to apply this to Process F we observe that

$$(22) \quad P_{ij}(t) = (1 - \gamma)^{-\beta} \pi_i \sum_{n=0}^{\infty} \sigma^n \frac{(\beta)_n}{n!} \phi_n(i) \phi_n(j)$$

where

$$\sigma = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}.$$

Identifying (22) with (20) we obtain from (21) for $i \leq j$ that

$$(23) \quad P_{ij}(t) = (1 - \gamma)^{-\beta} \pi_i \frac{j!}{(\beta)_i} \left(1 - \frac{\sigma}{\gamma}\right)^i (1 - \sigma)^{-i-\beta}$$

$$\times \sum_{k=0}^i \binom{i}{k} (-1)^k \left[\frac{1 - \frac{\sigma}{\gamma^2}}{1 - \frac{\sigma}{\gamma}} \right]^k \left[\frac{1 - \frac{\sigma}{\gamma}}{1 - \sigma} \right]^{i-k} \frac{(i + \beta)_{i-k}}{(j-k)!},$$

where $\sigma = (\lambda/\mu) e^{-(\mu-\lambda)t}$ and $\gamma = \lambda/\mu$. The formula for P_{ij} when $i > j$ is easily obtained from the relation $P_{ij} = (\pi_j/\pi_i) P_{ji}$. In the following table we summarize the formulas for the transition probabilities for the process indicated.

$$\text{Table 3.} \quad \left[i \leq j, \sigma = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}, \gamma = \frac{\lambda}{\mu} \right]$$

Process C.

$$P_{ij} = (1 - \gamma)^{-\beta} \pi_i \left(1 - \frac{\sigma}{\gamma}\right)^i (1 - \sigma)^{-i-\beta} e^{-(\mu-\lambda)(\beta-1)t} \frac{(\beta)_i}{i!}$$

$$\times \sum_{k=0}^i \binom{i}{k} (-1)^k \left[\frac{1 - \frac{\sigma}{\gamma^2}}{1 - \frac{\sigma}{\gamma}} \right]^k \left[\frac{1 - \frac{\sigma}{\gamma}}{1 - \sigma} \right]^{i-k} \frac{(i + \beta)_{i-k}}{(j - k)!}.$$

Process D.

$$P_{ij}(t) = (1 - \gamma)^{-\beta} \pi_i \frac{(\beta)_i}{i!} \left(\frac{\lambda}{\mu} \right)^{i+i} \left(1 - \frac{\sigma}{\gamma} \right)^i (1 - \sigma)^{-i-\beta} e^{-(\mu-\lambda)t} \\ \times \sum_{k=0}^i \binom{i}{k} (-1)^k \left[\frac{1 - \frac{\sigma}{\gamma^2}}{1 - \frac{\sigma}{\gamma}} \right]^k \left[\frac{1 - \frac{\sigma}{\gamma}}{1 - \sigma} \right]^{i-k} \frac{(i + \beta)_{i-k}}{(j - k)!}.$$

Process E.

$$P_{ii}(t) = \left(\frac{\lambda}{\mu} \right)^{i+i} (1 - \gamma)^{-\beta} \pi_i \frac{j!}{(\beta)_i} \left(1 - \frac{\sigma}{\gamma} \right)^i (1 - \sigma)^{-i-\beta} e^{-(\mu-\lambda)t} \\ \times \sum_{k=0}^i \binom{i}{k} (-1)^k \left[\frac{1 - \frac{\sigma}{\gamma^2}}{1 - \frac{\sigma}{\gamma}} \right]^k \left[\frac{1 - \frac{\sigma}{\gamma}}{1 - \sigma} \right]^{i-k} \frac{(i + \beta)_{i-k}}{(j - k)!}.$$

Process F.

$$P_{ii}(t) = (1 - \gamma)^{-\beta} \pi_i \frac{j!}{(\beta)_i} \left(1 - \frac{\sigma}{\gamma} \right)^i (1 - \sigma)^{-i-\beta} \\ \times \sum_{k=0}^i \binom{i}{k} (-1)^k \left[\frac{1 - \frac{\sigma}{\gamma^2}}{1 - \frac{\sigma}{\gamma}} \right]^k \left[\frac{1 - \frac{\sigma}{\gamma}}{1 - \sigma} \right]^{i-k} \frac{(i + \beta)_{i-k}}{(j - k)!}.$$

We next deduce the explicit expressions for the transition probabilities of Processes A and B. For Process A

$$(24) \quad P_{0i}(t) = \kappa^{\alpha+1} \frac{\pi_i}{\Gamma(\alpha+1)} \int_0^\infty e^{-xt} L_i^\alpha \left(\frac{x}{\kappa} \right) x^\alpha e^{-x/\kappa} dx \\ = \frac{\pi_i}{\Gamma(\alpha+1)} \mathcal{L}(y^\alpha L_i^\alpha(y))$$

evaluated at $s = \kappa t + 1$ (where $\mathcal{L}$ denotes “the Laplace transform of”) (see [2])

$$= \frac{\pi\alpha}{\Gamma(\alpha+1)} \left(\frac{\Gamma(\alpha+j+1)}{j!} \right) \frac{(\kappa t)^i}{(1+\kappa t)^{\alpha+j+1}} = \frac{(\kappa t)^i}{(1+\kappa t)^{\alpha+j+1}}.$$

Also

$$\sum_i P_{ij} s^i = \frac{\pi_j}{\Gamma(\alpha+1)} (1-s)^{-\alpha-1} \int_0^\infty e^{-x\kappa t} e^{x/(s-1)} x^\alpha L_j^\alpha(x) dx,$$

and by virtue of (14) this reduces to

$$\begin{aligned} &= \frac{\pi_i}{\Gamma(\alpha + 1)} (1 - s)^{-\alpha-1} \int_0^\infty e^{-(\kappa t + 1 - s/(s-1))x} x^\alpha L_i^\alpha(x) dx \\ &= \frac{(\kappa t)^i}{(1 + \kappa t)^{\alpha+i+1}} \left[1 + \frac{1 - \kappa t}{\kappa t} s \right]^i \left[1 - \frac{\kappa t}{1 + \kappa t} s \right]^{-(\alpha+i+1)}. \end{aligned}$$

Hence for $i \leq j$

$$(25) \quad P_{ii}(t) = \frac{(\kappa t)^i}{(1 + \kappa t)^{\alpha+i+1}} \sum_{k=0}^i \binom{i}{k} \left(\frac{1 - \kappa t}{\kappa t} \right)^k \frac{(\kappa t)^{i-k}}{(1 + \kappa t)^{i-k}} \frac{(\alpha + j + 1)_{i-k}}{(i - k)!}.$$

For Process B similar methods apply and we get

$$\sum_{j=0}^\infty P_{ij}(t) s^j = \frac{(\kappa t)^i}{(1 + \kappa t)^{\alpha+i+1}} \frac{\left(1 + \frac{1 - \kappa t}{\kappa t} s \right)^i}{\left(1 - \frac{\kappa t}{1 + \kappa t} s \right)^{\alpha+i+1}},$$

and so for $i \leq j$

$$P_{ij}(t) = \frac{(\kappa t)^i}{(1 + \kappa t)^{\alpha+i+1}} \sum_{k=0}^i \binom{i}{k} \left(\frac{1 - \kappa t}{\kappa t} \right)^k \left(\frac{\kappa t}{1 + \kappa t} \right)^{i-k} \frac{(\alpha + i + 1)_{i-k}}{(j - k)!}.$$

5. Number of transitions until absorption. A very interesting concept that was introduced by D. KENDALL in connection with linear growth models is the number of transitions to the right which the diffusing particle undergoes prior to ultimate absorption [11]. This measures, for example, the intensity of the epidemic if the state of a system describes the number of people currently infected. The relevant probabilities of this random variable are readily evaluated once R_i^n ,

$$(26) \quad R_i^n = \text{probability of being absorbed on the } n^{\text{th}} \text{ transition given that initially the particle is in state } i,$$

is completely determined. The quantities R_i^n evidently satisfy the recurrence relations

$$R_0^n = p_0 R_1^{n-1}, \quad R_i^n = q_i R_{i-1}^{n-1} + p_i R_{i+1}^{n-1}$$

where $p_i = \lambda_i/(\lambda_i + \mu_i)$ and $q_i = \mu_i/(\lambda_i + \mu_i)$. The particular values of p_n and q_n corresponding to Process A are

$$(27) \quad p_n = \frac{n + 1}{2n + \alpha + 1} \quad \text{and} \quad q_n = \frac{n + \alpha}{2n + \alpha + 1}.$$

Consider the polynomial system

$$(28) \quad x T_n^\alpha(x) = \frac{n + \alpha}{2n + 1 + \alpha} T_{n-1}^\alpha(x) + \frac{n + 1}{2n + \alpha + 1} T_{n+1}^\alpha(x)$$

and $T_0 = 1$, $T_1 = (1 + \alpha)x$. These are recognized as the ultraspherical polynomials C_n^λ of order $\lambda = \frac{1}{2}(1 + \alpha)$ normalized so that $C_n^\lambda(1) = \binom{n + 2\lambda - 1}{n}$ [13, p. 80],

$$T_n^\alpha(x) = C_n^{\frac{1}{2}(1+\alpha)}(x).$$

It follows immediately from the theory of random walks that

$$(29) \quad R_i^M = \frac{\Gamma(\alpha + 3)}{\Gamma^2\left(\frac{\alpha}{2} + 1\right)} \frac{\alpha}{(1 + \alpha)} \frac{1}{2^{\alpha+1}} \int_{-1}^1 x^{M-1} C_i^{\frac{1}{2}(1+\alpha)}(x) (1 - x^2)^{\frac{1}{2}\alpha} dx$$

(see [9]).

Another linear growth process to which a modification of formula (29) may be applied is the case where $\lambda_n = (n + 1)\lambda$ and $\mu_n = (n + 1)\mu$, $n \geq 0$. We have $p_n = \lambda/(\lambda + \mu) = p$, $q_n = \mu/(\lambda + \mu) = q$. The associated polynomials

$$U_n(x) = \left(\sqrt{\frac{p}{q}}\right)^n T_n^{(1)}\left(\frac{x}{\sqrt{4pq}}\right)$$

satisfy

$$xU_n(x) = pU_{n+1}(x) + qU_{n-1}(x), \quad n \geq 0,$$

where $U_0 \equiv 1$, $U_{-1} \equiv 0$.

Hence, corresponding to $\lambda_n = (n + 1)\lambda$ and $\mu_n = (n + 1)\mu$, we have

$$(30) \quad R_i^M = \frac{\mu}{\lambda + \mu} \frac{1}{2\Gamma^2(\frac{3}{2})} \int_{-1}^1 x^{M-1} \left(\sqrt{\frac{\lambda}{\mu}}\right)^i C_i^{(1)}\left(\frac{x}{\sqrt{4pq}}\right) (1 - x^2)^{\frac{1}{2}} dx.$$

6. Recurrence distribution and related processes. Inspection of Tables 1 and 2 indicates that not all linear growth processes are directly related to the classical Meixner and Laguerre polynomial systems. However, from a study of the concept of the "related process" it is possible to compute the spectral measure of the remaining linear models. Based on the same analysis we can derive approximate expressions for the recurrence time distributions connected with Processes A through F.

From a given birth and death process with infinitesimal matrix (2) a new process is obtained by stopping the given process whenever the state 0 is reached. For this new process the state 0 is transformed into an absorbing state, and if we ignore this state the process is a birth and death process for which the parameter μ_0 is positive. The waiting time in any state $i \geq 1$ has the same distribution for both the original and the new process, and, moreover, both processes have the same post exit distributions for each state $i \geq 1$. Consequently the infinitesimal matrix of the new process (with the state 0 ignored) is

$$(31) \quad \begin{bmatrix} -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots \\ \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots \\ 0 & \mu_3 & -(\lambda_3 + \mu_3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which is obtained from (2) by removing the zero row and zero column. The polynomials defined by

$$\begin{cases} Q_0^{(0)}(x) = 0, & Q_1^{(0)}(x) = -\frac{1}{\lambda_0}, \\ -xQ_n^{(0)}(x) = \mu_n Q_{n-1}^{(0)}(x) - (\lambda_n + \mu_n)Q_n^{(0)}(x) + \lambda_n Q_{n+1}^{(0)}(x), & n \geq 1, \end{cases}$$

are called the associated polynomials of the system $\{Q_n(x)\}$. It is seen that, except for the constant factor $-1/\lambda_0$, they are the polynomials belonging to the new birth and death process. Consequently the transition probability matrix $(\tilde{P}_{ij}(t))$, $i, j \geq 1$, of the new process is given by

$$\tilde{P}_{ij}(t) = \frac{\mu_1}{\lambda_0} \pi_j \int_0^\infty e^{-x't} [-\lambda_0 Q_i^{(0)}(x)] [-\lambda_0 Q_j^{(0)}(x)] d\alpha(x)$$

where α is the spectral measure of the "related process". In [7] it is shown that Stieltjes transforms of the spectral measures ψ and α of the two processes,

$$(32) \quad B(s) = \int_0^\infty \frac{d\psi}{x-s}, \quad C(s) = \int_0^\infty \frac{d\alpha}{x-s},$$

are related by the identity

$$(33) \quad B(s) = \frac{1}{\lambda_0 + \mu_0 - s - \lambda_0 \mu_1 C(s)}.$$

Once the function $B(s)$ or $C(s)$ has been found, the measures ψ and α can be computed by means of known formulas for inverting the Stieltjes transform. See [7] for a discussion of this inversion relative to the identity (33) and [14], [15] for the general inversion problem. If ψ is a discrete distribution with masses $\rho_0, \rho_1, \rho_2, \dots$ located at the points $0 \leq x_0 < x_1 < x_2 < \dots, x_n \rightarrow \infty$, then

$$(34) \quad B(s) = \sum_{n=0}^{\infty} \frac{\rho_n}{x_n - s}$$

is a meromorphic function whose only poles are simple poles located at the points x_n . In each of the open intervals (x_n, x_{n+1}) , $B(s)$ strictly increases from $-\infty$ to $+\infty$ and consequently has precisely a simple zero y_n in this interval. $C(s)$ is therefore a meromorphic function whose only poles are at the zeros of $B(s)$. The measure α is also a discrete distribution with jumps of magnitude

$$(35) \quad \gamma_n = -\frac{1}{B'(y_n)} \frac{1}{\lambda_0 \mu_1},$$

located at y_n , which strictly interlock with x_n :

$$(36) \quad 0 \leq x_0 < y_0 < x_1 < y_1 < x_2 < y_2 < \cdots.$$

We tabulate $B(s)$ for each of the processes C-F:

$$(37) \quad \begin{aligned} B(s) &= (1 - \gamma)^{-\beta} \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \frac{\gamma^n}{(n + \beta - 1)(\mu - \lambda) - s}, & \text{Process C.} \\ B(s) &= (1 - \gamma)^{-\beta} \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \frac{\gamma^n}{(n + 1)(\mu - \lambda) - s}, & \text{Process D.} \\ B(s) &= (1 - \gamma)^{-\beta} \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \frac{\gamma^n}{(n + \beta)(\mu - \lambda) - s}, & \text{Process E.} \\ B(s) &= (1 - \gamma)^{-\beta} \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \frac{\gamma^n}{n(\mu - \lambda) - s}, & \text{Process F.} \end{aligned}$$

In any practical situation the zeros y_n of $B(s)$ may be readily computed with the aid of high speed computers. Only one of the four functions $B(s)$ need be dealt with in any detail. In fact, examination of (37) shows that once the zeros of $B(s)$ for any one of the four processes are found then the zeros of the other $B(s)$ are obtained by translation. Some special properties shared by the zeros y_n which aid in their calculation may be cited. One such characteristic is the inequality $y_{n-1} < y_n - (\mu - \lambda)$. The proof of this fact may be found in [8, Section 3].

We now indicate how knowledge regarding the spectral measure α enables us to compute the recurrence time distribution. For definiteness we concentrate attention on Process F. Suppose we seek to evaluate $F_{10}(t)$ which represents the distribution function of the first passage time for the transition of the particle from the state one to the state zero. The solution of the problem is given in terms of the related spectral measure [7] as follows:

$$(38) \quad F_{10}(t) = \mu_1 \int_0^t d\xi \int_0^\infty e^{-x\xi} d\alpha(x) = \mu \sum_{n=0}^{\infty} \left[\frac{1 - e^{-y_n t}}{y_n} \right] \gamma_n.$$

Since y_n tends to ∞ and $0 \leq \gamma_n$, $\sum \gamma_n = 1$, a good approximation may be obtained by taking an appropriate finite number of terms.

We next show how knowledge of α can be used to generate other cases of linear growth not encompassed by those reviewed in Tables 1 and 2. Restricting again attention to Process F we know that the associated spectral measure α has birth and death rates given by $\lambda_n = (n + 1 + \beta)\lambda$, $\mu_n = (n + 1)\mu$, $n \geq 0$. We construct the measure

$$(39) \quad \theta(x) = \begin{cases} \mu \frac{d\alpha(x)}{x}, & x > 0, \\ 1 - \mu \int_0^\infty \frac{d\alpha(x)}{x}, & x = 0+, \end{cases}$$

whose orthogonal polynomials are $(\lambda_n/\lambda_0)\pi_n[Q_{n+1}(x) - Q_n(x)]$ where $Q_n(x)$ is the polynomial system attached to $\alpha(x)$.

The birth and death rates corresponding to the spectral measure $\theta(x)$ can be shown to be respectively $\lambda_n = (n+1)\mu$, $n \geq 0$, $\mu_n = (n+\beta)\lambda$, $n > 1$, $\mu_0 = 0$ [6].

By applying similar transformations on the "related spectral measure" of Processes C through F we obtain all linear growth processes with λ distinct from μ . For further details concerning these transformations the reader is referred to [6] and [7]. It should be emphasized that none of the four additional processes so obtained may be represented simply in terms of known elementary functions or recognizable classical systems of function. The series representations in terms of the zeros y_n should be regarded as, in essence, the simplest expression of the formulas linked to the related linear growth process.

We close this section with a brief discussion of some of the features of the "related process" of Process B of Table 2. The fundamental equation (32) which relates the Stieltjes transform of ψ and the spectral measure α of the related process must be inverted. The mechanism of the inversion when ψ has a positive continuous density is discussed in Proposition A of [7]. It is shown that α has a density

$$(40) \quad \alpha'(\xi) = \frac{\psi'(\xi)}{\left(\text{P.V.} \frac{1}{\pi} \int_0^\infty \frac{\psi'(x)}{x-\xi} dx\right)^2 + \psi'(\xi)^2}, \quad \xi \geq 0,$$

(P.V. denotes principal value). For the special case $d\psi(\xi) = \Gamma(\beta+1)^{-1} e^{-\xi} \xi^\beta d\xi$

$$\text{P.V.} \int_0^\infty \frac{d\psi(x)}{x-y} = -\frac{(\pi \cot \pi\beta)}{\Gamma(\beta+1)} e^{-y} y^\beta - e^{-y} \int_0^1 e^{yt} t^{-\beta-1} dt$$

for $-1 < \beta < 0$ and (40) becomes

$$(41) \quad \alpha'(\xi) = \frac{1}{\pi^2} \frac{\frac{1}{\Gamma(\beta+1)} e^{-\xi} \xi^\beta}{\left(\frac{\pi \cot \pi\beta e^{-\xi} \xi^\beta}{\Gamma(\beta+1)} + e^{-\xi} \int_0^1 e^{\xi t} t^{-\beta-1} dt\right)^2 + \frac{1}{\Gamma^2(\beta+1)} e^{-2\xi} \xi^{2\beta}}.$$

With the aid of the explicit evaluation of $\alpha(\xi)$ as given by (41) we can now write out the first passage distribution $F_{10}(t)$ of Process B. In fact,

$$(42) \quad F_{10}(t) = \int_0^\infty \left[\frac{1 - e^{-xt}}{x} \right] \alpha'(x) dx.$$

With (41) and use of the device suggested in (39) (see Lemmas A and B of [7]) we can introduce the two remaining types of linear growth processes for which $\lambda = \mu$. The analysis has already been described in connection with the "related processes" to the Meixner system and so we do not spell out the corresponding details for the Laguerre polynomial systems. In any practical case of linear growth the methodology of this paper may be routinely applied.

7. Continuous Ehrenfest model and the Krawtchouk polynomials. We discuss briefly the classical continuous Ehrenfest process and determine its orthogonal polynomial system and the spectral measure. For a description of the physical background and the analysis of certain limit laws attached to special random variables of the Ehrenfest process we refer the reader to [1], [5], and [16]. The Ehrenfest model is a birth and death process for which $\lambda_n = (N - n)p$, $\mu_n = nq$, $0 \leq n \leq N$ ($0 < p < 1$, $q = 1 - p$), such that the state -1 and the state $N + 1$ constitute reflecting barriers. We recognize the polynomial system as the classical Krawtchouk polynomials

$$(43) \quad Q_n(x) = \binom{N}{n}^{-1} p^{-n} \sum_{v=0}^n (-1)^v \binom{N-x}{n-v} \binom{x}{v} p^{n-v} q^v$$

where x is a discrete variable ranging over the integers $0, 1, 2, \dots, N$. The spectral measure ψ is the familiar binomial distribution which places masses $\binom{N}{x} p^x q^{N-x}$ at $x = 0, 1, 2, \dots, N$. Because of the generating function relationship

$$\sum_{k=0}^N \pi_k Q_k(x) w^k = (1-w)^x \left(1 + \frac{p}{q} w\right)^{N-x}, \quad \pi_k = \binom{N}{k} \left(\frac{p}{q}\right)^k,$$

we find easily

$$\sum_{s=0}^N P_{r,s}(t) w^s = [pe^{-t}(1-w) + (q+pw)]^{N-r} [(q+pw) - q(1-w)e^{-t}]^r$$

where $P_{r,s}(t)$ is the transition function.

Another form of $P_{r,s}(t)$ is

$$P_{r,s}(t) = \binom{N}{s} \left(\frac{p}{q}\right)^s \int_0^\infty e^{-xt} Q_r(x) Q_s(x) d\psi(x)$$

which is in essence a finite series. Recurrence and absorption probabilities may be computed by utilizing the technique of the "related process" in a standard fashion.

REFERENCES

- [1] R. BELLMAN & T. E. HARRIS, Recurrence times for the Ehrenfest model, *Pacific Journal of Mathematics*, **1** (1951), pp. 184-188.
- [2] G. DOETSCH, *Theorie und Anwendung der Laplace Transformation*, Berlin, 1937.
- [3] A. ERDÉLYI, *Higher transcendental functions*, McGraw-Hill, 1953.
- [4] W. FELLER, *An introduction to probability theory and its applications*, John Wiley and Sons, 1950.
- [5] W. FELLER, Two singular diffusion problems, *Ann. of Math.*, **54** (1951), pp. 173-182.
- [6] S. KARLIN & J. L. MCGREGOR, The differential equations of birth and death processes and the Stieltjes moment problem, *Trans. Amer. Math. Soc.*, **86** (1957), pp. 489-546.
- [7] SAMUEL KARLIN & JAMES MCGREGOR, The classification of birth and death processes, *Trans. Amer. Math. Soc.*, **86** (1957), pp. 366-401.

- [8] SAMUEL KARLIN & JAMES MCGREGOR, Many server queueing processes with Poisson input and exponential service times, to appear in *Pacific Journal of Math.*
- [9] SAMUEL KARLIN & JAMES MCGREGOR, Random walks, to appear in *Ill. Journ. of Math.*
- [10] D. G. KENDALL, On the generalized birth and death processes, *Ann. Math. Stat.*, **19** (1948), pp. 1–16.
- [11] D. G. KENDALL, Les processus stochastiques de croissance en biologie, *Annales des L'institute Henri Poincaré*, **13** (1952), pp. 43–108.
- [12] G. E. H. REUTER & W. LEDERMAN, On the differential equations for the transition probabilities of Markov processes with enumerably many states, *Proc. Cambridge Philos. Soc.*, **49** (1953), pp. 247–262.
- [13] G. SZEGÖ, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, 23, 1939.
- [14] E. C. TITCHMARCH, *Theory of Fourier integrals*, Oxford, 1948.
- [15] D. V. WIDDER, *The Laplace transform*, Princeton, 1941.
- [16] M. KAC, Random walk and the theory of Brownian motion, *Amer. Math. Monthly*, **54** (1947), pp. 369–391.

Stanford University
Stanford, California