The Probabilistic Method for Problems of Radiative Transfer—III. Line Formation by Coherent Scattering

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ABSTRACT

In the present paper, based on the stochastic model of multiple scattering of photons, we treat exactly two transfer problems in a semi-infinite, coherently and isotropically scattering atmosphere: MILNE's problem in non-conservative cases and the formation of absorption lines by an emitting atmosphere.

The emergent intensity obtained for the first problem reduces to that given by CHANDRASEKHAR. For the second problem, developing the probability theory of radiative transfer given previously, we find a new formula for the emergent intensity in a more general and condensed form compared to those previously obtained. With the aid of this formula, the exact expressions for the radiation intensity are derived for the second problem, using as source function of the emitters an arbitrary polynomial in the optical depth, the n^{th} exponential integral, and the exponential term, respectively. After some minor but lengthy arguments, it is shown that they are alternative and somewhat condensed expressions for the exact solutions provided by Miss BUSBRIDGE.

I. INTRODUCTION

With the aid of the Laplace transformation, S. S. HUANG [8] has found the intensity formula in the Milne-Eddington (M-E) model, guessing the form for the emergent intensity and expressing the Planck function as a polynomial of the optical depth. Similarly, modifying somewhat his method, we obtained the same results in [13]. Miss IDA W. BUSBRIDGE [1] applied the invariance method to the above problem with the linear Planck function. In a recent paper [14] we extended AMBARZUMIAN's first method to the same problem.

Making use of the invariance method, H. G. HORAK and C. A. LUNDQUIST [6, 7, and 10] have treated the transfer problems for an emitting atmosphere (semi-infinite or finite) with a distribution of emission sources given by an arbitrary polynomial of the optical depth. Applying AMBARZUMIAN's first method to the above problem in a semi-infinite atmosphere, Miss IDA W. BUSBRIDGE [2]

found ingeniously the intensity formulæ when the source function of emitters is a polynomial of the optical depth and also the n^{th} exponential integral.

Introducing first the probability of emission into the theory of radiative transfer, V. V. SOBOLEV [11, 12] has dealt with the transfer problems in the M-E model under some restricted conditions. While the notion of the probability of emission may be implicitly traceable in AMBARZUMIAN's first method (*cf.* KOURGANOFF & BUSBRIDGE [9]), SOBOLEV's method may be considered as a new approach to the transfer of radiation.

In a recent paper [15], we modified SOBOLEV's method mathematically in a manner similar to the exact solution of MILNE's first integral equation, allowing for CHANDRASEKHAR's idea of an invariance arising from the asymptotic solution at infinity. We applied the method to the transfer problems in the M-E model, taking into account the fluorescence emission and using as the Planck function a quadratic polynomial in the optical depth and the exponential term, respectively.

When we treat some physical process mathematically, we must have recourse to some sort of simplified model of the physical situation. In a statistical model, the knowledge that the system is in a certain state at one definite time does not determine uniquely the state of the system at all times, but determines only a certain probability for each of the possible states. In other words, the random process is completely described by a set of probability distributions. It should be mentioned here that in our stationary case we use the optical depth parameter in place of the time parameter.

The characteristic feature of the probabilistic method will now be outlined. First, based on the stochastic model of multiple scattering of elementary particles, we construct the stochastic integro-differential equation which governs the probability distribution of a random process $\mu(t)$. Physically, the probability distribution corresponds to the probability of emission at a given point in a certain direction and it is a function of the properties of the medium, but not of the emission sources. The diffusion equation is similar in form to the Kolmogoroff-Feller equation. Using it and the functional relations obtained in our previous paper [15], we compute the exact solution of the transfer equation, without actually solving MILNE's first integral equation, by explicitly referring to the integral transformation of the diffusion equation.

In the present paper, with the aid of the probabilistic method, we treat two transfer problems in a semi-infinite, coherently and isotropically scattering atmosphere. First, we compute the exact solution of MILNE's problem in non-conservative cases. The result is the same as that given by S. CHANDRASEKHAR [3]. Second, we calculate the emergent intensities of absorption lines formed by an emitting atmosphere. In this case, modifying somewhat the method used previously by us, we find a new intensity formula in a more general and condensed form compared to those previously given. The source function of emitters is an arbitrary polynomial of the optical depth, the n^{th} exponential integral, and the exponential term, respectively. The results hold in modified form in

the conservative case. While the solutions obtained are similar in form to those provided by Miss IDA W. BUSBRIDGE [2], the former seem to be somewhat condensed in form compared with the latter. After some argument, we justify the coincidence of BUSBRIDGE's results with ours. The verification of the results can be done by inserting the intensities into the equation which is the Laplace transform of MILNE's first integral equation. Applications of the probabilistic method to problems of radiative transfer and of neutron diffusion in finite space will be discussed in our later papers.

II. THE STOCHASTIC INTEGRO-DIFFERENTIAL EQUATION

Let $p(\mu; t)$ be the probability distribution expressed in terms of a one-dimensional parameter t and a continuous stochastic variable $\mu(t)$. In our case t and μ denote respectively the optical depth and the cosine of the inclination to the outward normal. Mathematically, $p(\mu; t)d\mu$ is the probability of finding μ in the range $(\mu, \mu + d\mu)$ at depth t . Physically, it corresponds to the probability that a photon absorbed at a given point t will be re-emitted in a certain direction μ in the radiation emerging from the bounding plane $t = 0$.

In a manner similar to that used in the theory of stochastic processes of Markovian type (*cf.* DOOB [4]), the probability distribution can be accomplished either by simply remaining at the state μ without suffering any scattering process for a depth interval of length t or by starting from the state μ' at depth τ , the depth of the last jump, jumping to a state μ and remaining at the state through $t - \tau$ depth units. Hence we have

$$(2.1) \quad p(\mu; t) = p(\mu; 0)e^{-t/\omega} + \frac{1}{2}\omega p(\mu; 0) \int_0^t d\tau \int_0^1 e^{-(t-\tau)/\omega} p(\mu'; \tau) \frac{d\mu'}{\mu'},$$

where $\omega (0 < \omega \leq 1)$ is an albedo for a single scattering.

It should be mentioned here that $R(\mu|\mu')d\mu d\tau$ represents the probability that given μ' one finds μ in the range $(\mu, \mu + d\mu)$ in a parametric interval $d\tau$ of the depth τ . In the second term on the right hand side of (2.1) $R(\mu|\mu')$ arises explicitly in the integral and is given by

$$(2.2) \quad R(\mu|\mu') = \frac{1}{2}\omega \frac{p(\mu; 0)}{\mu'}.$$

If we put

$$(2.3) \quad p(\mu; 0) = H(\mu),$$

where $H(\mu)$ is the function introduced by CHANDRASEKHAR [3], it is later shown that

$$(2.4) \quad \int_0^1 R(\mu'|\mu) d\mu' \leq \frac{1}{\mu}.$$

The justification of (2.4) is given in our previous paper [15]. The equality holds in the conservative case alone. Physically this means that, on each scattering,

more radiation cannot be emitted than was incident. In our previous paper [16] concerning MILNE's problem it was shown that (2.1) in the conservative case can be derived from a modified Chapman-Kolmogoroff equation.

On differentiating (2.1) with respect to t , we obtain the stochastic integro-differential equation in $p(\mu; t)$:

$$(2.5) \quad \frac{\partial p(\mu; t)}{\partial t} = -\frac{p(\mu; t)}{\mu} + \frac{1}{2} \omega H(\mu) \int_0^1 p(\mu'; t) \frac{d\mu'}{\mu'}.$$

The equation is similar in form to the Kolmogoroff-Feller equation.

In our previous paper [15], starting from (2.5), we derived the following relations: Writing

$$(2.6) \quad \rho(\mu', \mu) = \mathfrak{L}_{1/\mu'}\{p(\mu; t)\} \equiv \int_0^\infty p(\mu; t) e^{-t/\mu'} \frac{dt}{\mu'},$$

we have

$$(2.7) \quad \rho(\mu', \mu) = \frac{\mu}{\mu + \mu'} H(\mu) H(\mu')$$

and

$$(2.8) \quad H(\mu) = 1 + \frac{1}{2} \omega \mu H(\mu) \int_0^1 \frac{H(\mu')}{\mu + \mu'} d\mu'.$$

Eq. (2.5) was first obtained by SOBOLEV [11] with the aid of a physical argument. Miss BUSBRIDGE [2] and I [14] derived it independently by an extension of AMBARZUMIAN's first method.

III. MILNE'S PROBLEM IN THE NON-CONSERVATIVE CASES

In this section we compute the exact solution of the transfer problem with no incident radiation and in non-conservative cases.

The Milne first integral equation appropriate to this case is

$$(3.1) \quad J(t) = \omega \Lambda_t \{J(\tau)\},$$

where the Hopf operator Λ is defined by

$$(3.2) \quad \Lambda_t \{f(\tau)\} = \frac{1}{2} \int_0^\infty f(\tau) E_1(|t - \tau|) d\tau.$$

Here $E_n(t)$ is the n^{th} exponential integral defined for a positive argument t :

$$(3.3) \quad E_n(t) = \int_0^1 e^{-t/\mu} \mu^{n-2} d\mu \quad (n = 1, 2, \dots).$$

Eq. (3.1) can be readily rewritten in the form

$$(3.4) \quad J(t) = J^*(t) + \omega G(t),$$

where $J^*(t)$ is the solution for an infinite unbounded atmosphere and $G(t)$ is

$$(3.5) \quad G(t) = -\frac{1}{\omega} J^*(t) + \Lambda_t \{J^*(\tau)\} + \omega \Lambda_t \{G(\tau)\}.$$

The second term on the right hand side of (3.4) represents the departure of the asymptotic solution at infinity as we approach $t = 0$.

According to CHANDRASEKHAR [3], (3.1) admits a solution of the form

$$(3.6) \quad J^*(t) = L_0 e^{k t},$$

where L_0 is a constant and k is the positive root, less than one, of the characteristic equation for $0 < \omega < 1$:

$$(3.7) \quad k = \omega \tanh^{-1} k.$$

The Laplace transform of (3.4) with respect to $1/\mu$ provides the emergent intensity. Then, recalling Eqs. (3.4) and (3.5), the emergent intensity is written in the form

$$(3.8) \quad I(0, \mu) = \mathfrak{L}_{1/\mu} \{J^*(t)\} + \omega \mathfrak{S}_\mu \left\{ -\frac{1}{\omega} J^*(t) + \Lambda_t \{J^*(\tau)\} \right\},$$

where $J^*(t)$ is given by (3.6). In (3.8) the scattering operator $\mathfrak{S}$ is defined by

$$(3.9) \quad \mathfrak{S}_\mu \{f(t)\} = \int_0^\infty f(t) p(\mu; t) \frac{dt}{\mu}.$$

Using the Λ transform of $e^{k\tau}$,

$$(3.10) \quad \Lambda_t \{e^{k\tau}\} = \frac{e^{k t}}{k} \tanh^{-1} k - \frac{1}{2} \int_0^1 \frac{e^{-t/\mu'}}{1 + k\mu'} d\mu',$$

and inverting the order of the integrations with respect to t and μ' in the second integral on the right hand side of (3.8), we have

$$(3.11) \quad I(0, \mu) = L_0 \left\{ \frac{1}{1 - k\mu} - \frac{1}{2} \omega \int_0^1 \frac{\mu' d\mu'}{1 + k\mu'} \frac{1}{\mu} \int_0^\infty p(\mu; t) e^{-t/\mu'} \frac{dt}{\mu'} \right\}.$$

With the aid of (2.6) and (2.7), (3.11) becomes

$$(3.12) \quad I(0, \mu) = L_0 \left\{ \frac{1}{1 - k\mu} - \frac{1}{2} \omega H(\mu) \int_0^1 \frac{\mu' H(\mu') d\mu'}{(\mu + \mu')(1 + k\mu')} \right\}.$$

Writing $\mu'/(\mu + \mu')(1 + k\mu')$ in partial fractions, after some minor manipulations, the required solution takes the form

$$(3.13) \quad I(0, \mu) = \frac{L_0}{H(1/k)} \frac{H(\mu)}{1 - k\mu}.$$

This solution is the same as that given by CHANDRASEKHAR (*cf.* [3], §88, Eq. (15)) based on the invariance method.

IV. LINE FORMATION BY AN EMITTING ATMOSPHERE

4.1. Milne's first integral equation. The integral equation governing the transfer of radiation by an emitting, semi-infinite atmosphere with coherent and isotropic scattering is expressible in the form

$$(4.1) \quad J(t) = \omega \Lambda_t \{J(\tau)\} + B(t),$$

where $J(t)$ is the source function and $B(t)$ is that of emitters.

Eq. (4.1) can be readily rewritten in the form

$$(4.2) \quad J(t) = B(t) + \omega G(t),$$

where

$$(4.3) \quad G(t) = \Lambda_t \{B(\tau)\} + \omega \Lambda_t \{G(\tau)\}.$$

Then, by an argument similar to that given in the previous section, the required emergent intensity takes the form

$$(4.4) \quad I(0, \mu) = \mathfrak{L}_{1/\mu} \{B(t)\} + \omega \mathfrak{S}_\mu \{ \Lambda_t \{B(\tau)\} \}.$$

On the other hand, modifying (4.2) slightly and assuming $\omega \neq 1$, we get

$$(4.5) \quad J(t) = \frac{B(t)}{1 - \omega} + \omega G^*(t),$$

where

$$(4.6) \quad G^*(t) = -\frac{B(t)}{1 - \omega} + \Lambda_t \left\{ \frac{B(\tau)}{1 - \omega} \right\} + \omega \Lambda_t \{G^*(\tau)\}.$$

Then the required emergent intensity becomes

$$(4.7) \quad I(0, \mu) = \mathfrak{L}_{1/\mu} \left\{ \frac{B(t)}{1 - \omega} \right\} + \omega \mathfrak{S}_\mu \left\{ -\frac{B(t)}{1 - \omega} + \Lambda_t \left\{ \frac{B(\tau)}{1 - \omega} \right\} \right\}.$$

Hence, combining Eqs. (4.4) and (4.7), the angular distribution of emergent radiation is expressed in the form

$$(4.8) \quad I(0, \mu) = \mathfrak{S}_\mu \{B(t)\}.$$

The intensity formula is new and more condensed in form than those previously given.

It should be mentioned here that this result holds in modified form even in the conservative case by an argument similar to that given by Miss BUSBRIDGE [2].

4.2. The solution with $B(t)$ a polynomial. Consider Eq. (4.1) for $0 < \omega < 1$ and with $B(t)$ an arbitrary polynomial of the optical depth t ,

$$(4.9) \quad B(t) = \sum_{i=0}^n \frac{a_{n-i}}{i!} t^i.$$

Then, using (4.8), the required exact solution takes the form

$$(4.10) \quad I_n(0, \mu) = \sum_{i=0}^n \frac{a_{n-i}}{i!} \int_0^\infty t^i p(\mu; t) \frac{dt}{\mu}.$$

Recalling (2.5), we are able to obtain readily the recurrence formula for the i^{th} moment of $p(\mu; t)$ with respect to t :

$$(4.11) \quad \phi_i(\mu) = i\mu \left\{ \phi_{i-1}(\mu) + \frac{1}{2} \frac{\omega}{(1-\omega)^{\frac{1}{2}}} H(\mu) \int_0^1 \phi_{i-1}(\mu) d\mu \right\} \quad \text{for } i \geq 1,$$

where

$$(4.12) \quad \phi_i(\mu) = \int_0^\infty p(\mu; t) t^i dt.$$

On the other hand, the zero moment $\phi_0(\mu)$ is similarly given by

$$(4.13) \quad \phi_0(\mu) = \frac{\mu H(\mu)}{(1-\omega)^{\frac{1}{2}}}.$$

Solving (4.11) and allowing for (4.13), we have

$$(4.14) \quad \phi_i(\mu) = H(\mu) K_i(\mu) \mu i! \quad (i \geq 0),$$

where

$$(4.15) \quad K_i(\mu) = (1-\omega)^{-\frac{1}{2}(i+1)} \begin{vmatrix} \mu^i & -\frac{1}{2}\omega\alpha_1 & -\frac{1}{2}\omega\alpha_2 & \cdots & -\frac{1}{2}\omega\alpha_i \\ \mu^{i-1} & (1-\omega)^{\frac{1}{2}} & -\frac{1}{2}\omega\alpha_1 & \cdots & -\frac{1}{2}\omega\alpha_{i-1} \\ \mu^{i-2} & 0 & (1-\omega)^{\frac{1}{2}} & \cdots & -\frac{1}{2}\omega\alpha_{i-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & (1-\omega)^{\frac{1}{2}} \end{vmatrix}.$$

In (4.15) α_i is the i^{th} moment of $H(\mu)$:

$$(4.16) \quad \alpha_i = \int_0^1 H(\mu) \mu^i d\mu \quad (i \geq 0).$$

With the aid of (4.14), the required solution (4.10) becomes

$$(4.17) \quad I_n(0, \mu) = H(\mu) \sum_{i=0}^n a_{n-i} K_i(\mu) \quad (n \geq 0).$$

Rearranging (4.17) in the order of ascending integral powers of μ , we have

$$(4.18) \quad I_n(0, \mu) = H(\mu) \sum_{i=0}^n \mu^i J_{n-i} \quad (n \geq 0),$$

where

$$(4.19) \quad J_i = (1-\omega)^{-\frac{1}{2}(i+1)} \begin{vmatrix} a_i & -\frac{1}{2}\omega\alpha_1 & -\frac{1}{2}\omega\alpha_2 & \cdots & -\frac{1}{2}\omega\alpha_i \\ a_{i-1} & (1-\omega)^{\frac{1}{2}} & -\frac{1}{2}\omega\alpha_1 & \cdots & -\frac{1}{2}\omega\alpha_{i-1} \\ a_{i-2} & 0 & (1-\omega)^{\frac{1}{2}} & \cdots & -\frac{1}{2}\omega\alpha_{i-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_0 & 0 & 0 & \cdots & (1-\omega)^{\frac{1}{2}} \end{vmatrix}.$$

While the expression (4.17) is new, the alternative one (4.18) is in agreement with that given by Miss BUSBRIDGE [2]. She showed that the constants J_n , mentioned above, are the solutions $J_n(0)$ of the equations which are derived from (4.1) when $B(t)$ is chosen as in (4.9) and t is put equal to 0.

By means of the invariance method, HORAK and LUNDQUIST [6, 7, 10] have obtained explicitly the emergent intensities for the above problem with a uniform, linear, and quadratic distribution of emission sources, respectively. The results obtained by them are equal to those given by us.

Thus we have the slightly modified form of the Theorem V due to Miss BUSBRIDGE [2] as follows: *When $0 < \omega < 1$, a solution $J_n(t)$ of (4.1), with $B(t)$ as in (4.9), is continuous for $t > 0$, tends to a finite limit as $t \rightarrow +0$, and is $O(t^n)$ as $t \rightarrow \infty$. If $I_n(0, \mu) = \mathfrak{L}_{1/\mu} \{J_n(t)\}$, then $I_n(0, \mu)$ ($\mu \geq 0$) is given by (4.17) and (4.18), respectively.*

4.3. The solution when $B(t)$ is the n^{th} exponential integral. Let

$$(4.20) \quad B(t) = E_n(t),$$

where the n^{th} exponential integral $E_n(t)$ is defined by (3.3). Then, using (4.8), the required solution is expressible in the form

$$(4.21) \quad I_n(0, \mu) = \int_0^\infty E_n(t) p(\mu; t) \frac{dt}{\mu}.$$

First, we consider the case $n = 1$. Recalling Eqs. (2.6), (2.7), and (2.8), the solution can be expressed straightforwardly in the form

$$(4.22) \quad I_1(0, \mu) = \frac{2}{\omega\mu} \{H(\mu) - 1\}.$$

When $n \geq 2$, using an argument similar to that mentioned above and writing $\mu'^i/(\mu + \mu')$ ($i \geq 1$) in descending integral powers of μ' , we have

$$(4.23) \quad \begin{aligned} I_n(0, \mu) = H(\mu) \left[\sum_{i=0}^{n-2} (-1)^i \mu^i \alpha_{n-2-i} + (-1)^{n-1} \frac{2}{\omega} \mu^{n-2} \right] \\ + (-1)^n \frac{2}{\omega} \mu^{n-2} \quad (n \geq 2). \end{aligned}$$

Eq. (4.22) is the same as that given by Miss BUSBRIDGE [2]. Eq. (4.23) is similar in form to Eq. (5.11) [2] and, after some argument, the former reduces to the latter.

Putting $t = 0$ in (4.1) with $B(t) = E_n(t)$ and recalling $E_n(0) = (n-1)^{-1}$, we have

$$(4.24) \quad \begin{aligned} J_n(0) &= \frac{1}{2} \omega \int_0^\infty J_n(t) E_1(t) dt + \frac{1}{n-1} \\ &= \frac{1}{2} \omega \int_0^1 I_n(0, \mu) d\mu + \frac{1}{n-1} \quad (n \geq 2). \end{aligned}$$

Substituting from (4.23) into (4.24), we get

$$(4.25) \quad J_n(0) = \frac{1}{2} \omega \sum_{i=0}^{n-2} (-1)^i \alpha_i \alpha_{n-2-i} + (-1)^{n-1} \alpha_{n-2} + \frac{1 + (-1)^n}{n-1}.$$

Using the relation between the first $2n$ moments of $H(\mu)$ derived by HUANG [8] and allowing for

$$(4.26) \quad \alpha_0 = 2\omega^{-1} \{1 - (1 - \omega)^{\frac{1}{2}}\},$$

Eq. (4.25) becomes

$$(4.27) \quad J_n(0) = \alpha_{n-2}.$$

Then, inserting (4.27) into (4.23), we get alternative forms of the required solution expressed in terms of $J_{n-i}(0)$ instead of α_{n-2-i} :

$$(4.28) \quad I_n(0, \mu) = H(\mu) \left[\sum_{i=0}^{n-2} (-1)^i \mu^i J_{n-i}(0) + (-1)^{n-1} \frac{2}{\omega} \mu^{n-2} \right] + (-1)^n \frac{2}{\omega} \mu^{n-2} \quad (n \geq 2).$$

Replacing the subscript n in (4.27) by $n - i$, substituting the resulting equation in (4.25), and solving it, we have

$$(4.29) \quad J_i(0) = (1 - \omega)^{-\frac{1}{2}(i-1)} \begin{vmatrix} c_i & \frac{1}{2}\omega\alpha_1 & -\frac{1}{2}\omega\alpha_2 & \cdots & (-1)^{i-1}\frac{1}{2}\omega\alpha_{i-2} \\ c_{i-1} & (1 - \omega)^{\frac{1}{2}} & \frac{1}{2}\omega\alpha_1 & \cdots & (-1)^{i-2}\frac{1}{2}\omega\alpha_{i-3} \\ c_{i-2} & 0 & (1 - \omega)^{\frac{1}{2}} & \cdots & (-1)^{i-3}\frac{1}{2}\omega\alpha_{i-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_2 & 0 & 0 & \cdots & (1 - \omega)^{\frac{1}{2}} \end{vmatrix},$$

where

$$(4.30) \quad c_i = \frac{1 + (-1)^i}{i-1} + (-1)^{i-1} \alpha_{i-2} \quad (i \geq 2).$$

The solution (4.28) was first found by Miss BUSBRIDGE [2]. Thus we have the slight modification of the Theorem VI due to Miss BUSBRIDGE [2] as follows: *When $0 < w < 1$ and $n \geq 2$, the equation (4.1), with $B(t)$ as in (4.20), has a continuous solution $J_n(t)$ for $t > 0$ which tends to a finite limit as $t \rightarrow +0$ and is $O(t^{n-2})$ as $t \rightarrow \infty$. If $I_n(0, \mu) = \mathfrak{L}_{1/\mu} \{J_n(t)\}$, then $I_n(0, \mu)$ ($\mu \geq 0$) is provided by Eqs. (4.23) and (4.28), respectively.*

4.4 The solution when $B(t) = e^{-at}$.

Let

$$(4.31) \quad B(t) = e^{-at} \quad (a \geq 0).$$

Then, recalling Eqs. (2.6), (2.7), and (4.8), the required solution straight-

In a manner similar to that given in the previous section, an alternative expression of (4.38) becomes

$$(4.40) \quad I_n(0, \mu) = H(\mu) \left[J'_n + \sum_{i=0}^n C_{n-i} \mu^{i+1} \right],$$

where

$$(4.41) \quad C_i = -\frac{2}{\alpha_1^{i+1}} \begin{vmatrix} a_i & \alpha_2 & \alpha_3 & \cdots & \alpha_{i+1} \\ a_{i-1} & \alpha_1 & \alpha_2 & \cdots & \alpha_i \\ a_{i-2} & 0 & \alpha_1 & \cdots & \alpha_{i-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_0 & 0 & 0 & \cdots & \alpha_1 \end{vmatrix}.$$

While the expression (4.38) is new, the alternative one (4.40) is the same as that given by Miss BUSBRIDGE [2]. In this context, HORAK's assertion [6] that for a semi-infinite atmosphere the emergent field becomes infinitely large in the conservative case can be interpreted as an infinite increase of the solution as ω tends to unity, apart from the solution at $\omega = 1$.

When $B(t) = E_1(t)$, by an argument similar to that mentioned above and with the aid of (4.22), the required solution is expressible in the form

$$(4.42) \quad I_1(0, \mu) = \mathfrak{S}_\mu \{E_1(t)\} + C_1 H(\mu) = \frac{1}{2\mu} \{H(\mu) - 1\} + C_1 H(\mu),$$

where the constant C_1 is arbitrary.

Furthermore, when $B(t) = E_n(t)$ ($n \geq 2$), the solution (4.23) should be replaced by

$$(4.43) \quad \begin{aligned} I_n(0, \mu) &= \mathfrak{S}_\mu \{E_n(t)\} + J_n H(\mu) \\ &= H(\mu) \left[J_n + \sum_{i=1}^{n-2} (-1)^i \alpha_{n-2-i} \mu^i + 2(-1)^{n-1} \mu^{n-2} \right] \\ &\quad + 2(-1)^n \mu^{n-2} \quad (n \geq 2), \end{aligned}$$

where the constant J_n is arbitrary. This expression is new. Inserting (4.43) into (4.24) for $\omega = 1$ and recalling $\alpha_0 = 2$, we have

$$(4.44) \quad \sum_{i=1}^{n-2} (-1)^i \alpha_{n-2-i} \alpha_i = -2c_n \quad (n \geq 2),$$

where the constant c_n is given by (4.30).

Putting

$$(4.45) \quad A_{n-i+1} = -\alpha_{n-i-2}, \quad A_2 = 0,$$

and substituting (4.45) in (4.44), we have

$$(4.46) \quad \sum_{i=1}^{n-1} (-1)^{i-1} A_{n-i+1} \alpha_i = -2c_n \quad (n \geq 2).$$

Solving (4.46), we get

$$(4.47) \quad A_i = -\frac{2}{\alpha_1^{i-1}} \begin{vmatrix} c_i & -\alpha_2 & \alpha_3 & \cdots & (-1)^i \alpha_{i-1} \\ c_{i-1} & \alpha_1 & -\alpha_2 & \cdots & (-1)^{i-1} \alpha_{i-2} \\ c_{i-2} & 0 & \alpha_1 & \cdots & (-1)^{i-2} \alpha_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_2 & 0 & 0 & \cdots & \alpha_1 \end{vmatrix}.$$

Inserting (4.45) into (4.43), an alternative expression for the solution becomes

$$(4.48) \quad I_n(0, \mu) = H(\mu) \left[J_n + \sum_{i=1}^{n-1} (-1)^{i-1} A_{n-i+1} \mu^i + 2(-1)^{n-1} \mu^{n-2} \right] + 2(-1)^n \mu^{n-2} \quad (n \geq 2).$$

This solution is the same as that given by Miss BUSBRIDGE [6].

When $B(t) = e^{-at}$, the solution holds even in the conservative case, since the diffusion equation (2.5) still exists (*cf.* [9], §28.4).

In conclusion, I wish to express my cordial gratitude to Professor S. MIYAMOTO for his valuable advice and encouragement during this investigation.

Added in proof. During the printing of this paper Professor V. V. SOBOLEV kindly presented me with his recent book (*Transfer of Radiative Energy in Atmospheres of Stars and Planets*, Moscow, 1956) in which he has also dealt with several problems of coherent line formation in a manner similar to our procedure.

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