*Asymptotic Expansion of Steady-State Solutions of Symmetric Hyperbolic Linear Differential Equations**

ROBERT M. LEWIS

Communicated by J. KELLER

INTRODUCTION

In a series of lectures [9] given at New York University during 1947–48, R. K. LUNEBURG introduced a method for obtaining asymptotic expansions of solutions of Maxwell's equations. The theory was refined and extended to linear second order hyperbolic partial differential equations by M. KLINE [10]. Similar asymptotic expansions have been studied by several authors† and have been applied to the solution of diffraction problems [11]. These methods have the virtue of providing the asymptotic expansion of the solution even where it is difficult or impossible to obtain the solution itself. In cases where the exact solution is known, the expansion is frequently more useful and can be obtained more easily by these methods than by expanding the known solution.

The present paper is concerned with extending the Luneburg-Kline theory in two directions: First, we will consider more general partial differential equations. Second, we will attempt to provide proofs of certain assertions for which LUNEBURG and KLINE use plausibility arguments. Our goal is a rigorous proof of the asymptotic nature of the formal expansions which will be derived. In this proof the results of Part I** play a key role but nevertheless we have attempted to make the present discussion self-contained.

The plan of presentation is as follows: In the remainder of this introduction we discuss notation and the formulation of the problem. Then in Section 1 a

*The research reported in this article has been sponsored by the Department of Defense under Contract No. DA49-170-sc-2253.

†See the references listed in [11].

**LEWIS, ROBERT M., *Discontinuous Initial Value Problems for Symmetric Hyperbolic Linear Differential Equations*, in this issue. This paper will be referred to as Part I.

formal expansion is derived. In Section 2 this expansion is applied to Maxwell's equations. Thus Sections 1 and 2 contain our formal results. The remaining sections are concerned with the proof of the asymptotic nature of these formal results.

We now turn to the definitions and formulation of the problem. A fuller discussion of some of the following notions is contained in Part I. We denote by u a k component vector function of the $n + 1$ variables $x_0, x_1, \dots, x_n$, and by A^r and B $k \times k$ matrices with real entries. We will sometimes denote x_0 by t and $x_1, \dots, x_n$ by x . The matrices A^r and B are taken to be *smooth** functions of x in the entire x -space. We define a first order linear differential operator

$$(1) \quad Lu = \sum_{r=0}^n A^r u_{x_r} + Bu \quad \left(u_{x_r} = \frac{\partial u}{\partial x_r} \right).$$

The operator L is taken to be *symmetric hyperbolic*, i.e., the matrices A^r are assumed to be symmetric, and the matrix A^0 to be positive-definite. Without loss of generality we may take $A^0 = I$. Now we consider the initial value problem for the vector $w(t, x)$:

$$(2) \quad Lw = g(x)e^{-i\omega t}, \quad w(0, x) = h(x).$$

Here g and h are vector functions and ω is a (scalar) parameter which will be referred to as the (circular) frequency. In physical applications, the function $ge^{-i\omega t}$ represents a source with harmonic time dependence.

Under appropriate conditions (which will be made explicit later) the solution w of (2) may be decomposed uniquely into two terms:

$$(3) \quad w(t, x) = z(t, x) + e^{-i\omega t}v(x)$$

where $z(t, x) \rightarrow 0$ as $t \rightarrow \infty$. z is called the transient, and $v(x)$ will be referred to as the *steady-state function*. The problem we consider is that of obtaining the asymptotic expansion of v with respect to ω as $\omega \rightarrow \infty$. It will be shown that $v(x)$ is independent of the initial function h and is a solution of the *reduced equation*

$$(4) \quad Nv = \sum_{r=1}^n A^r v_{x_r} + (B - i\omega)v = g.$$

(Eq. (4) may be obtained formally by substituting $w = e^{-i\omega t}v$ into Eq. (2).) Although the reduced equation does not suffice to determine v uniquely, it can be used in formally deducing the expansion of v . In this deduction it is necessary to impose on the expansion an *outgoing condition* which will be explained later. In the later sections it will be shown that the formal expansion is indeed an asymptotic expansion (for $\omega \rightarrow \infty$) of $v(x)$.

*The term "smooth" will be used to describe functions which have continuous derivatives of sufficiently high order for the purpose at hand.

1. The formal expansion. We seek a formal expansion $\tilde{v}$ of the steady state function v defined in the introduction. This function satisfies the equation

$$(5) \quad Nv = g(x).$$

Our procedure will be formal; the results will be justified later. We begin by considering an expansion of the form

$$(6) \quad y = \sum_{r=0}^{\infty} \frac{y^{(r)}(x)}{(i\omega)^r}.$$

Here the $y^{(r)}(x)$ are vector functions to be determined. If we insert (6) in (5) and compare powers of ω , we obtain

$$(7) \quad y^{(0)} = 0, \quad y^{(1)} = -g, \quad y^{(r+1)} = \mathcal{L}y^{(r)}, \quad r = 1, 2, \dots.$$

The operator $\mathcal{L}$ is given by

$$(8) \quad \mathcal{L} = \sum_{\nu=1}^n A^{\nu} \frac{\partial}{\partial x_{\nu}} + B.$$

From (7) we obtain the functions $y^{(r)}$ explicitly in the form

$$(9) \quad y^{(0)} = 0, \quad y^{(r)} = -\mathcal{L}^{(r-1)}g, \quad r = 1, 2, \dots.$$

Here $\mathcal{L}^0 = I$. Thus

$$(10) \quad y = -\sum_{r=0}^{\infty} \frac{\mathcal{L}^r g}{(i\omega)^{r+1}}.$$

If $g(x)$ is infinitely differentiable, the desired formal solution is simply

$$(11) \quad \tilde{v} = y.$$

If the source region is bounded (*i.e.*, if $g \equiv 0$ outside a bounded region) the formal solution vanishes identically outside that region.

Let us now consider the more interesting case in which g or one of its derivatives is discontinuous. More precisely, let Γ be a smooth $n - 1$ dimensional manifold in x -space, and let $g(x)$ be smooth on both sides of Γ , while g and its derivatives have jump discontinuities across Γ . Later, in considering problems with boundaries, we shall require that $g(x)$ vanish in a neighborhood of the boundary. A function $g(x)$ with all these properties will be said to be of *type g*. We require that our formal solution v shall be continuous. (It will be seen later that the exact solution v is also continuous.) To achieve this we may begin with (10) but we must repair the discontinuities in it without disturbing the equation (5). We may hope to accomplish this by adding to y solutions of the homogeneous equation

$$(12) \quad Nv = 0.$$

For this purpose we shall study functions of the form

$$(13) \quad z^{(\alpha)} = e^{i\omega \Psi^{(\alpha)}(x)} \sum_{r=0}^{\infty} \frac{z^{(\alpha)(r)}(x)}{(i\omega)^r}.$$

Here α is an index, $\Psi^{(\alpha)}$ is a scalar function, $z^{(\alpha)}$ and $z^{(\alpha)(r)}$ are vector functions.*

If we apply the operator N to (13) and equate the coefficient of each power of $i\omega$ to zero we obtain

$$(14) \quad G^{(\alpha)} z^{(\alpha)(0)} = 0,$$

$$(15) \quad G^{(\alpha)} z^{(\alpha)(r+1)} - \mathfrak{L} z^{(\alpha)(r)} = 0, \quad r = 0, 1, 2, \dots,$$

where

$$(16) \quad G^{(\alpha)} = I - \sum_{\nu=1}^n \Psi_{x_\nu}^{(\alpha)} A^\nu.$$

If the matrix $G^{(\alpha)}$ were non-singular, these equations would have only the trivial solution $z^{(\alpha)(r)} \equiv 0$, $r = 0, 1, \dots$. Hence we assume that

$$(17) \quad \det G^{(\alpha)} = 0.$$

Before taking a closer look at the equations (14), (15), it is necessary to impose an important condition on the class of operators L (or N) under consideration, which we shall call the

Multiplicity Condition: If $P(x, z) = \sum_{\nu=1}^n z_\nu A^\nu$, where the z_ν are real numbers and not all zero, then the multiplicities of the eigenvalues of the matrix P are independent of x and $z = z_1, \dots, z_n$.

For convenience, we shall temporarily omit the index (α) in our consideration of a particular $z^{(\alpha)}$. From (17) we see that the matrix G is singular, so it has at least one zero eigenvalue at every point. At some point, let $\lambda_1, \dots, \lambda_q$ denote the zero eigenvalues of G (i.e., $\lambda_1 = \lambda_2 = \dots = \lambda_q = 0$). Then the multiplicity condition assures us that the eigenvalues $\lambda_1(x), \dots, \lambda_q(x)$ of $G(x)$ are zero everywhere, and all the others are non-zero.† Let $R^1, \dots, R^q$ be an orthonormal set of null eigenvectors of G , i.e., those eigenvectors out of the complete set $R^1, \dots, R^k$ of orthonormal eigenvectors which are associated with the eigenvalue $\lambda_1 \equiv \dots \equiv \lambda_q \equiv 0$. From (14) we see that $z^{(0)}$ is in the null-space of G , so it has an expansion in terms of the null eigenvectors:

$$(18) \quad z^{(0)} = \sum_{m=1}^q \sigma_m^0 R^m$$

*The formal expansion y defined by (6) is a special case of the form (13). In fact, if one attempts to satisfy the inhomogeneous equation (5) by inserting (13) one is forced to set $\Psi^{(\alpha)} \equiv 0$. Thus, our formal procedure requires the single *ansatz* (13).

†For a clarification of this assertion, see Part I, Section 2, the second footnote.

(the $\sigma_m^0 = \sigma_m^0(x)$ are scalars). We obtain a differential equation for $z^{(0)}$ from (15) for $r = 0$ by multiplying (15) by a null eigenvector:

$$(19) \quad \sum_{\nu=1}^n R^l A^{\nu} z_{x_{\nu}}^{(0)} + R^l B z^{(0)} = 0, \quad l = 1, \dots, q.$$

Inserting (18) in (19) we obtain

$$(20) \quad \sum_{\nu=1}^n \sum_{m=1}^q R^l A^{\nu} R^m (\sigma_m^0)_{x_{\nu}} + \sum_{\nu=1}^n \sum_{m=1}^q R^l A^{\nu} R_{x_{\nu}}^m (\sigma_m^0) + \sum_{m=1}^q R^l B R^m \sigma_m^0 = 0, \\ l = 1, \dots, q.$$

Equation (20) is a first order system of linear differential equations for the scalar coefficients σ_m^0 in the expansion of $z^{(0)}$. Before discussing it further, we will derive the corresponding equations for $z^{(r)}$, $r > 0$.

To do this we expand $z^{(r)}$ in terms of *all* the eigenvectors of G :

$$(21) \quad z^{(r)} = \sum_{m=1}^h \sigma_m^r R^m.$$

From (15) and (21) we have

$$(22) \quad G z^{(r)} = \sum_{\nu=1}^n A^{\nu} z_{x_{\nu}}^{(r-1)} + B z^{(r-1)} = \sum_{m=q+1}^h \sigma_m^r \lambda_m R^m, \quad r = 1, 2, \dots.$$

Here the eigenvalue λ_m is simply

$$(23) \quad \lambda_m = R^m G R^m.$$

Multiplying (22) by an eigenvector which is not null, we obtain

$$(24) \quad \sigma_s^r = \frac{1}{\lambda_s} \left(\sum_{\nu=1}^n R^s A^{\nu} z_{x_{\nu}}^{(r-1)} + R^s B z^{(r-1)} \right), \\ s = q+1, \dots, h, \quad r = 1, 2, \dots.$$

Thus we see that the coefficients of the non-null eigenvectors in the expansion (21) or $z^{(r)}$ are given explicitly by (24) in terms of $z^{(r-1)}$.

To obtain the coefficients of the null eigenvectors, we multiply (15) by a null eigenvector

$$(25) \quad \sum_{\nu=1}^n R^l A^{\nu} z_{x_{\nu}}^{(r)} + R^l B z^{(r)} = 0, \quad l = 1, \dots, q, \quad r = 0, 1, 2, \dots.$$

Inserting (21) in (25) and rearranging terms we obtain

$$(26) \quad \sum_{\nu=1}^n \sum_{m=1}^q R^l A^{\nu} R^m (\sigma_m^r)_{x_{\nu}} + \sum_{m=1}^q \tau_{lm} \sigma_m^r = g_l^r, \\ l = 1, \dots, q, \quad r = 0, 1, 2, \dots,$$

where

$$(27) \quad \tau_{lm} = \sum_{\nu=1}^n R^l A' R_{x_\nu}^m + R^l B R^m$$

and

$$(28) \quad g_l^r = - \sum_{\nu=1}^n \sum_{s=q+1}^k R^l A' R^s (\sigma_s^r)_{x_\nu} - \sum_{s=q+1}^k \tau_{ls} \sigma_s^r.$$

By virtue of (24), g_l^r is given explicitly in terms of $z^{(r-1)}$. If we establish the convention $z^{(-1)} \equiv 0$, (20) appears as a special case of (26). (This has been anticipated by including $r = 0$ in (26).)

It is possible to put (26) in a much simpler form. For this purpose we employ the following lemma which is a consequence of the multiplicity condition.*

Lemma: Let $R_0^1, \dots, R_0^q$ be eigenvectors associated with the same eigenvalue λ_0 of the matrix $P = \sum_{\nu=1}^n z_\nu A'$ (z_ν real). Then, for each j ,

$$(29) \quad R_0^l A' R_0^m = c_j \delta_{lm}$$

(i.e., $R_0^l A' R_0^m = 0$ if $l \neq m$, and is independent of l if $l = m$).

It follows immediately from the lemma that (26) may be written

$$(30) \quad \sum_{\nu=1}^n R^l A' R^l (\sigma_l^r)_{x_\nu} + \sum_{m=1}^q \tau_{lm} \sigma_m^r = g_l^r, \\ l = 1, \dots, q, \quad r = 0, 1, 2, \dots.$$

We now introduce a family of curves in the x -space defined parametrically by $x_\nu = x_\nu(s)$, where

$$(31) \quad \dot{x}_\nu = \frac{dx_\nu}{ds} = R^1 A' R^1, \quad \nu = 1, \dots, n.$$

These curves will be referred to simply as *rays*. Then Equation (30) may be written as an ordinary differential equation along a ray:

$$(32) \quad \dot{\sigma}_l^r + \sum_{m=1}^q \tau_{lm} \sigma_m^r = g_l^r, \quad l = 1, \dots, q, \quad r = 0, 1, 2, \dots.$$

For each fixed r , (32) is a system of q linear ordinary differential equations. The recursive system of differential equations (32), together with the algebraic equations (24), will be called the *transport equations*.

The function Ψ appearing in the expansion (13) of $z^{(\alpha)}$ is called the *phase function*. We have seen that it is a solution of the single first order non-linear partial differential equation (17), a special case of which is called the *eiconal equation*. It is interesting to note, and not difficult to verify, that our "rays" are the "characteristic curves" used in constructing the solution of the first order equation for Ψ . We also note that

*The proof appears in Part I, Section 2.

$$(33) \quad \frac{d\Psi}{ds} = \sum_{\nu=1}^n \Psi_{x_\nu} \dot{x}_\nu = \sum_{\nu=1}^n R^1 \Psi_{x_\nu} A^\nu R^1 = R^1 [I - G] R^1 = 1.$$

Thus Ψ may be identified (up to an additive constant on each ray) with the curve parameter s .

A formal series $z^{(\alpha)}$ of the form (13), where $\Psi^{(\alpha)}$ is a solution of the eiconal equation and the $z^{(\alpha)(r)}$ (or rather their scalar components) are solutions of the transport equations, will be called a *wave*. A wave is not completely determined until we specify initial conditions for the eiconal equation and for the transport equations. Thus we shall, in general, have some $n - 1$ dimensional manifold, on which the value of Ψ is prescribed. Because of the non-linearity of the eiconal equation (17) there may be several solutions with these prescribed initial values. We shall think of each solution Ψ as defined only on one side of the initial manifold (and zero on the other), and in its region of definition we associate with Ψ a corresponding wave (for which the initial values of the $z^{(r)}$ still need to be prescribed) which is also zero outside this region. Thus our waves will in general be discontinuous across their initial manifolds.

It is, by now, perhaps obvious what must be done to complete the construction of $\bar{v}$. We shall attempt to repair the discontinuities in y by adding waves to (10). Thus our formal solution will be

$$(34) \quad \bar{v} = y + \sum_{\alpha} z^{(\alpha)}$$

where the $z^{(\alpha)}$ must be appropriately specified. Since y is smooth except on the manifold Γ , we take Γ to be the initial manifold for each $z^{(\alpha)}$. At this point we can easily state the *outgoing condition* referred to in the introduction. First we note that with every solution Ψ , and therefore with every wave, is associated an $n - 1$ parameter family of rays. A wave will be said to be *outgoing (incoming)* with respect to the initial manifold Γ if on every one of its associated rays, Ψ increases (decreases) monotonically with the distance from Γ . In the expansion of (34) only waves $z^{(\alpha)}$ which are outgoing with respect to Γ are to be included.

The requirement that v shall be continuous may be stated simply as

$$(35) \quad [\bar{v}]^{\Gamma} = 0 \quad \text{on} \quad \Gamma,$$

where $[\bar{v}]^{\Gamma}$ denotes the jump in $\bar{v}$ across Γ (i.e., the difference of the limiting values as Γ is approached from each side) with some given orientation. Inserting (34) in (35), we have

$$(36) \quad [y]^{\Gamma} + \sum_{\alpha} e(\alpha) z^{(\alpha)} = 0 \quad \text{on} \quad \Gamma.$$

Here $e(\alpha) = \pm 1$ and it is determined by the orientation of $[\]^{\Gamma}$ and the side of Γ on which $z^{(\alpha)}$ is zero. Since $[y]^{\Gamma}$ contains no exponential factor in ω we see that (36) can be satisfied only if

$$(37) \quad \Psi^{(\alpha)} = 0 \quad \text{on} \quad \Gamma.$$

Inserting (6) and (13) in (36) and equating the coefficient of each power of $i\omega$ to zero, we obtain

$$(38) \quad \sum_{\alpha} e(\alpha) z^{(\alpha)(0)} = 0 \quad \text{on } \Gamma,$$

$$(39) \quad \sum_{\alpha} e(\alpha) z^{(\alpha)(r)} = [\mathfrak{L}^{(r-1)} g]^{\Gamma} \quad \text{on } \Gamma, \quad r = 1, 2, \dots.$$

Now we must determine the range of α , that is, we must determine how many outgoing waves appear in (34). In solving the eiconal equation and initial conditions (37), $\Psi^{(\alpha)}$ and its derivatives must satisfy the so-called "strip manifold condition." Geometrically this means that the solution surface $\Psi = \Psi^{(\alpha)}(x)$ in Ψx -space must be tangent to Γ , so the projection into x -space of the normal to the solution surface must be parallel to the normal to Γ . Thus

$$(40) \quad \Psi_{x_{\nu}}^{(\alpha)} = \rho(\alpha) \gamma_{\nu}, \quad \nu = 1, \dots, n,$$

where $\gamma_1, \dots, \gamma_n$ are components of the unit normal to Γ in x -space, and $\rho(\alpha)$ is a proportionality factor. It is easily seen from the eiconal equation that a solution surface can never have a tangent plane parallel to the plane $\Psi = 0$ (in Ψx -space). Hence each solution $\Psi^{(\alpha)}$ is increasing (with distance) on one side of Γ and decreasing on the other. Thus each $\Psi^{(\alpha)}$ provides us with two waves, one on each side of Γ . One is outgoing and one is incoming. We use only the outgoing one. The scalar factor $\rho(\alpha)$ is determined by the eiconal equation (17); thus

$$(41) \quad \det G^{(\alpha)} = \det \left(I - \sum_{\nu=1}^n \Psi_{x_{\nu}}^{(\alpha)} A^{\nu} \right) = \det (I - \rho(\alpha) E) = 0 \quad \text{on } \Gamma$$

where

$$(42) \quad E = \sum_{\nu=1}^n \gamma_{\nu} A^{\nu}.$$

There will be as many outgoing waves as there are distinct solutions $\rho(\alpha) = 1/\beta(\alpha)$ of (41), hence as many as there are distinct non-zero eigenvalues $\beta(\alpha)$ of E . (There will be equally many incoming waves, namely the waves associated with the continuation of each $\Psi^{(\alpha)}$ across Γ .)

Let us introduce the orthonormal eigenvectors B^i of E which are associated with the non-zero eigenvalues $\beta(j)$:

$$(43) \quad (I - \rho(j)E)B^i = G^{(j)}B^i = 0 \quad \text{on } \Gamma.$$

The $\rho(j)$ and $G^{(j)}$ in (43) are not necessarily distinct for distinct j , *i.e.*, $\rho(j) = \rho(i)$ and $G^{(j)} = G^{(i)}$ if $\beta(j) = \beta(i)$. Furthermore, each non-null eigenvector B^j of E is a null eigenvector of $G^{(\alpha)}$ for some α , and the totality of null eigenvectors for all the $G^{(\alpha)}$ is the totality of non-null eigenvectors B^j of E . This statement holds, of course, only on Γ where the B^j are defined. The matrix E , however, may well be singular. In this case there will also be null eigenvectors $J^1, \dots, J^p$ of E . In order to complete the construction of a continuous formal solution

v it is necessary to impose a new condition on the discontinuities of $g(x)$ which we call the

Source Condition: $J^\nu[g]^\Gamma = 0$, $\nu = 1, \dots, p$, on Γ .

(Here we have, of course, an inner-product of two vectors. If E is non-singular the condition is empty, *i.e.*, it does not restrict g .) It will be seen in the next section that the source condition has a very interesting interpretation for Maxwell's equations.

We must now proceed to select initial values in order to satisfy (38) and (39). In so doing we will make constant use of the fact that the set of eigenvectors B^i may be decomposed into disjoint subsets each of which is precisely the set of null eigenvectors of one of the $G^{(\alpha)}$. Since the B^i are independent and since each $z^{(\alpha)(0)}$ has an expansion in terms of the null eigenvectors of $G^{(\alpha)}$, it follows from (38) that each $z^{(\alpha)(0)}$ is zero on Γ . Therefore the initial values of the $\sigma_m^{0(\alpha)}$ are zero, and since Equation (32) is homogeneous for $r = 0$, we conclude that $z^{(\alpha)(0)} \equiv 0$ for each α .* We now consider (39) for $r = 1$. Again each $z^{(\alpha)(1)}$ has an expansion in terms of the null eigenvectors of $G^{(\alpha)}$, for, by (24), $\sigma_s^{1(\alpha)} = 0$ because $z^{(\alpha)(0)} \equiv 0$. It is clear now that we may satisfy (39) only because of the source condition, and that the initial values on Γ of each $z^{(\alpha)(1)}$ are uniquely determined. Now suppose we have determined the $z^{(\alpha)(\rho)}$ and have satisfied (39) for $\rho = 0, 1, \dots, r-1$. We recall that $z^{(\alpha)(r)}$ has an expansion in terms of all the eigenvectors of $G^{(\alpha)}$. By (24) the coefficients of the non-null eigenvectors are determined in terms of $z^{(\alpha)(r-1)}$. It remains to determine the coefficients of the null eigenvectors from (39). This can be done, and (39) will be satisfied, if and only if

$$(44) \quad \sum_{\alpha} e(\alpha) J^\nu z^{(\alpha)(r)} = J^\nu [\mathfrak{L}^{(r-1)} g]^\Gamma \quad \text{on } \Gamma, \quad \nu = 1, \dots, p.$$

We will now show that (44) is a consequence of the foregoing induction hypothesis. First, we consider the directional derivative

$$(45) \quad \frac{d}{d\tau} = \sum_{i=1}^n J^i A^i \frac{\partial}{\partial x_i},$$

for any $\nu = 1, \dots, p$. It is easily seen that the direction of $d/d\tau$ is tangential to Γ . Hence, for any vector u which is smooth on both sides of Γ ,

$$(46) \quad \frac{d}{d\tau} [u]^\Gamma = \left[\frac{d}{d\tau} u \right]^\Gamma.$$

Hence

$$(47) \quad \begin{aligned} J^\nu \mathfrak{L}[u]^\Gamma &= \frac{d}{d\tau} [u]^\Gamma + J^\nu B[u]^\Gamma \\ &= \left[\frac{d}{d\tau} u \right]^\Gamma + [J^\nu B u]^\Gamma = [J^\nu \mathfrak{L} u]^\Gamma = J^\nu [\mathfrak{L} u]^\Gamma. \end{aligned}$$

**I.e.*, we could have started the expansion (13) with $r = 1$.

It follows that

$$(48) \quad J^\nu \mathcal{L}[\mathcal{L}^{(r-2)} g]^\Gamma = J^\nu [\mathcal{L}^{(r-1)} g]^\Gamma.$$

We observe that each J^ν is a non-null eigenvector for every $G^{(\alpha)}$. Thus from (24)

$$(49) \quad J^\nu z^{(\alpha)(r)} = \frac{1}{J^\nu G^{(\alpha)} J^\nu} \{ J^\nu \mathcal{L} z^{(\alpha)(r-1)} \}.$$

But

$$(50) \quad J^\nu G^{(\alpha)} J^\nu = J^\nu (I - \rho(\alpha) E) J^\nu = 1.$$

Hence

$$(51) \quad J^\nu z^{(\alpha)(r)} = J^\nu \mathcal{L} z^{(\alpha)(r-1)}.$$

By the induction hypothesis we may assume the validity of (39) with r replaced by $r - 1$, *i.e.*,

$$(52) \quad \sum_\alpha e(\alpha) z^{(\alpha)(r-1)} = [\mathcal{L}^{(r-2)} g]^\Gamma.$$

Now, using (51), (52), and (48) (in that order), we have

$$(53) \quad \sum_\alpha e(\alpha) J^\nu z^{(\alpha)(r)} = J^\nu \mathcal{L} \sum_\alpha e(\alpha) z^{(\alpha)(r-1)} = J^\nu \mathcal{L} [\mathcal{L}^{(r-2)} g]^\Gamma = J^\nu [\mathcal{L}^{(r-1)} g]^\Gamma.$$

Thus we have verified (44) and the induction argument is complete.

It follows that the initial values for $z^{(\alpha)(r)}$ will be uniquely determined for all α and r and (39) will be satisfied for all r . We may then solve the transport equations recursively for $r = 0, 1, \dots$ along each ray, starting at Γ , and determine the $z^{(\alpha)(r)}$. With this, the formal construction is complete except for the following reservation: Examination of the transport equations reveals that the coefficients of the linear differential equation (32) depend on the eigenvectors of the matrix $G^{(\alpha)}$, hence on the function $\Psi^{(\alpha)}$. By our assumptions about Γ , $\Psi^{(\alpha)}$ will be smooth in a neighborhood of Γ , but as we proceed along a ray from Γ we may reach a point where $\Psi^{(\alpha)}$ or one of its derivatives becomes singular. We cannot, by the present construction, continue the solutions of the transport equations beyond such a point. Points of this type are said to lie on a *caustic*.* For convenience we state a condition which guarantees that the construction can be completed:

Caustic Condition: *The problem is such that the phase functions $\Psi^{(\alpha)}$, associated with each of the waves in the construction, are free of caustics.*

Having completed the formal construction for the problem as formulated in the introduction, we shall now show how to extend the construction to problems involving boundaries.

If $\beta(x) = 0$ is the equation of any smooth surface B in the x -space, the transformation $z_\nu = \alpha_\nu(x)$, $\nu = 1, \dots, n$, with $\alpha_n = \beta$ maps B into the hyperplane

*For a discussion of the formal construction for certain problems involving caustics, see [12].

$z_n = 0$, while preserving the symmetric hyperbolic nature of the operator L , and therefore the properties of the reduced operator N . Thus, without loss of generality, we may take our boundary B to be the plane $x_n = 0$. On this boundary we require our solution v (hence our formal solution $\bar{v}$) to satisfy the following type of homogeneous *boundary condition*.

The vector v is an element of a k dimensional Euclidean vector space V . Consider a subspace T with the following properties:

1. The matrix A^n is *non-positive over T* , i.e., $uA^n u \leq 0$ for every u in T .
2. T is *maximal*, i.e., the dimension of T is as large as the dimension of any subspace having property 1.

Our boundary condition may now be expressed simply as

$$(54) \quad v(x) \text{ in } T \text{ for } x_n = 0.$$

If $T^\perp$ is the orthogonal complement of T , and $T^\perp$ is generated by the vectors $b^1, \dots, b^m$, the condition may also be stated as

$$(55) \quad vb^s = 0, \quad s = 1, \dots, m, \quad \text{for } x_n = 0.$$

The vectors b^r are assumed to be smooth functions of $x_1, \dots, x_{n-1}$.

For problems with boundaries, we have assumed that the source region does not intersect the boundary, i.e., that $g(x) \equiv 0$ in a neighborhood of the boundary. Thus the term y in the expansion (20) of $\bar{v}$ automatically satisfies the boundary condition. Suppose now that a particular wave $z^{(\alpha)}$ (outgoing from Γ) is "incident" on the boundary B (i.e., any of the associated rays intersect the boundary). We observe first that $z^{(\alpha)}$ will be incoming with respect to B . Our procedure will be to add appropriate *reflected waves* to the expansion $\bar{v}$ which will be outgoing with respect to B , and which, together with $z^{(\alpha)}$, will satisfy the boundary condition. We emphasize that the condition will be satisfied individually for each $z^{(\alpha)}$ together with its associated reflected waves. We shall see that the boundary condition uniquely determines the reflected waves.

The incoming wave $z^{(\alpha)}$ will have some given phase (i.e., the value of $\Psi^{(\alpha)}$) on B . It is clear that the associated reflected waves must have the same phase on B . In geometrical language, if the *phase surface* $\Psi = \Psi^{(\alpha)}(x)$ in Ψx -space intersects the boundary B (i.e., the manifold $x_n = 0$ in Ψx -space) in the $n - 1$ dimensional manifold Δ , then the corresponding phase surfaces of the reflected waves must also intersect B in Δ . We shall sometimes denote by x_0 the coordinate Ψ in Ψx -space.

Let $\delta_0, \dots, \delta_{n-1}, 0$ denote components of the unit normal to Δ in the plane $x_n = 0$. Let $\Psi = \Psi^{(p)}(x)$ be the equation of a phase surface $C^{(p)}$ whose intersection with B is Δ . Then (by the strip manifold condition), since the components of the normal to $C^{(p)}$ are $(1, -\Psi_{x_1}^{(p)}, \dots, -\Psi_{x_n}^{(p)})$, we have $1 = \rho\delta_0$; $-\Psi_{x_r}^{(p)} = \rho\delta_r$, $r = 1, \dots, n - 1$; and $\Psi_{x_n}^{(p)}$ is a solution of

$$(56) \quad \det G^{(p)} = \det \left(I - \sum_{\nu=1}^n \Psi_{x_\nu}^{(p)} A^\nu \right) = \det (-\Psi_{x_n}^{(p)} A^n + \rho F) = 0 \quad \text{on } \Delta.$$

Here

$$(57) \quad F = \sum_{\nu=0}^{n-1} \delta_{\nu} A^{\nu}.$$

Equation (56) will, in general, have complex roots $\Psi_{x_n}^{(p)}$. For each distinct real root there will be a corresponding phase surface passing through Δ .

We now impose a new condition on the problem, the

Shadow Condition: *F shall be non-singular throughout B.*

In Part I, Section 6,* it is pointed out that, at the point of minimum $\Psi^{(\alpha)}$, $F = I$. Therefore it follows from the above condition that F is positive definite on B . It is also pointed out that the condition guarantees that no rays of either the incoming or reflected waves can be tangent to the boundary. Thus we can have no shadows in the problem.

Since F is positive definite we may set $F = JJ'$ where J is a non-singular matrix hence has an inverse S . Let $\mathcal{G} = SA^nS'$. Then $\mathcal{G}$ is symmetric and we may write

$$(58) \quad (\mathcal{G} - \alpha_p)Q^{(p)} = 0, \quad p = 1, \dots, k.$$

Here the $Q^{(p)}$ are a complete set of orthogonal eigenvectors of $\mathcal{G}$, which we may normalize by the condition

$$(59) \quad Q^{(p)}SS'Q^{(p)} = 1, \quad p = 1, \dots, k$$

The eigenvalues α_p are real. Set $B^{(p)} = S'Q^{(p)}$, $p = 1, \dots, k$. It follows from (59) that the $B^{(p)}$ are unit vectors. They are clearly linearly independent. From (58), $J(SA^nS' - \alpha_p)J'B^{(p)} = 0$, or

$$(60) \quad (A^n - \alpha_p F)B^{(p)} = 0, \quad p = 1, \dots, k.$$

For each non-zero α_p , set $\Psi_{x_n}^{(p)} = \rho/\alpha_p$; then

$$(61) \quad G^{(p)}B^{(p)} = (-\Psi_{x_n}^{(p)}A^n + \rho F)B^{(p)} = 0, \quad \text{if } \alpha_p \neq 0.$$

Equation (61) expresses the fact that for each distinct non-zero α_p there is a distinct phase surface $C^{(p)}$ passing through Δ , and $B^{(p)}$ is a null eigenvector of the associated matrix $G^{(p)}$. Since we may have multiple roots α_p , we may have several distinct $B^{(p)}$ associated with the same matrix $G^{(p)}$.

We have, up until now, not distinguished between waves $z^{(p)}$ which are incoming and outgoing with respect to B .† For convenience we simply assign the value zero to the coefficients of all incoming waves except the one $z^{(\alpha)}$

*The discussion of that section closely parallels this one if we identify Ψ with t , "phase surface" with "characteristic surface," etc.

†We are, of course, dealing only with waves on one side of B , i.e., within the region of the problem, which we take to be $x_n \geq 0$.

which "arises" at Γ . We will then attempt to determine the initial values for the outgoing waves from the boundary condition

$$(62) \quad b^\nu \sum_p z^{(p)} = 0 \quad \text{on } B, \quad \nu = 1, \dots, m.$$

Comparing powers of $i\omega$ in the expansions of $z^{(p)}$ it follows at once that

$$(63) \quad b^\nu \sum_p z^{(p)(r)} = 0 \quad \text{on } B, \quad \nu = 1, \dots, m, \quad r = 0, 1, 2, \dots.$$

We recall that each $z^{(p)(0)}$ has an expansion in terms of null eigenvectors of $G^{(p)}$. But this set of vectors is just a subset of the set of vectors $B^{(i)}$. Thus

$$(64) \quad \sum_p z^{(p)(0)} = \sum_{p=1}^k \sigma_p^0 B^{(p)}.$$

(The σ_p^0 may be identified with the σ_m^0 in (18), but we have renumbered them and (64) contains the coefficients for several different waves.) Inserting (64) in (63) for $r = 0$,

$$(65) \quad \sum_{p=1}^k b^\nu B^{(p)} \sigma_p^0 = 0, \quad \nu = 1, \dots, m.$$

Of course, some of the σ_p^0 are already determined, namely those associated with the incoming waves. It is easy to see that σ_p^0 will (will not) have a preassigned value at a point on B if Ψ decreases (increases) with x_n , *i.e.*, if

$$\dot{x}_n = \frac{dx_n}{ds} = \frac{dx_n}{d\Psi} = B^{(p)} A^n B^{(p)} \quad \text{is } < 0 \text{ } (> 0).$$

(The possibility $\dot{x}_n = 0$ is precluded by the shadow condition.) Let $B^{(p)} A^n B^{(p)}$ be > 0 for $p = 1, \dots, s$ and ≤ 0 for $p = s+1, \dots, k$. We rewrite (65) placing known quantities (which we denote simply by W_i) on the right side:

$$(66) \quad \sum_{p=1}^s b^\nu B^{(p)} \sigma_p^0 = W_1, \quad \nu = 1, \dots, m.$$

In Part I, Section 6, it is proved that the coefficient matrix $(b^\nu B^{(p)})$ of (66) is square (*i.e.*, $s = m$) and non-singular. Hence (66) can be solved uniquely for $\sigma_1^0, \dots, \sigma_m^0$; and these quantities provide the initial values for the transport equations of order zero ($r = 0$) for the reflected waves.

To obtain the initial conditions for the higher transport equations, we use (63) for $r = 1, 2, \dots$. We recall that each $z^{(p)(r)}$ has an expansion in terms of the eigenvectors of $G^{(p)}$, the coefficients of the non-null eigenvectors being known from (24) in terms of $z^{(p)(r-1)}$. Thus we may write

$$(67) \quad \sum_p z^{(p)(r)} = \sum_{p=1}^k \sigma_p^r B^{(p)} + W_2.$$

From (63)

$$(68) \quad \sum_{p=1}^k b^\nu B^{(p)} \sigma_p^r = W_3, \quad \nu = 1, \dots, m.$$

As before, those σ_p^r associated with incoming waves are known. Thus

$$(69) \quad \sum_{\nu=1}^s b^\nu B^{(\nu)} \sigma_p^r = W_4, \quad \nu = 1, \dots, m.$$

Since (66) and (69) have the same coefficient matrix it is clear that the latter also has a unique solution. Thus the initial values for the transport equations of all orders (for the reflected waves) are uniquely determined.

Assuming, now, that the caustic condition is satisfied, we may solve the transport equations and completely determine all of the reflected waves (for all of the incident waves $z^{(\alpha)}$). This completes the construction of the formal solution for problems with boundaries.* If we now use the index α to label all our waves, including reflected ones, the formal expansion may be written in the form

$$(70) \quad \bar{v} = - \sum_{r=0}^{\infty} \frac{\mathcal{L}^r g}{(i\omega)^{r+1}} + \sum_{\alpha} e^{i\omega \Psi^{(\alpha)}(x)} \sum_{r=1}^{\infty} \frac{z^{(\alpha)(r)}(x)}{(i\omega)^r}.$$

2. Maxwell's equations. In the present section we will demonstrate the applicability of the results of this paper to Maxwell's equations. We begin by stating the equations in their usual form:

$$(71) \quad \nabla \times H - \frac{1}{c} \frac{\partial}{\partial t} (\epsilon E) = \frac{4\pi}{c} I, \quad I = \sigma E + \frac{c}{4\pi} F(x, t),$$

$$(72) \quad \nabla \times E + \frac{1}{c} \frac{\partial}{\partial t} (\mu H) = 0,$$

$$(73) \quad \nabla \cdot (\epsilon E) = 4\pi \rho,$$

$$(74) \quad \nabla \cdot (\mu H) = 0.$$

Here the independent variables are $t = x_0$ and $x = (x_1, x_2, x_3)$; H, E, I, F are vectors with three components. The scalars ϵ, μ, σ will be taken to be smooth functions of x ; ϵ and μ are assumed to be everywhere positive. (Physically we are restricting consideration to isotropic media in which the electromagnetic properties do not change with time, and vary smoothly with position.) F represents impressed currents, *i.e.*, sources.

We shall show that Equations (71) and (72) form a symmetric hyperbolic system. As such, the usual existence and uniqueness theorems for initial value problems (see Part I) apply. In view of the completeness of this theory Equations (73), (74) appear to be somehow superfluous. The confusion is easily resolved, however, if we interpret them as additional conditions *on the initial data* of the problem. From this point of view, we begin with (71) and (72); then we

*For problems with multiple boundaries and multiple reflections the construction is an obvious extension of the foregoing.

define the function $\rho(x, t)$ by equation (73), and impose the following condition on the initial data:

$$(75) \quad \nabla \cdot (\mu H) = 0 \quad \text{at} \quad t = 0.$$

It follows easily from (72) that

$$(76) \quad \frac{\partial}{\partial t} (\nabla \cdot \mu H) = 0,$$

and (74) is an immediate consequence of (75) and (76). From (71) and (73) we obtain the "equation of continuity,"

$$(77) \quad \nabla \cdot I + \rho_t = 0.$$

We now set $E = (\phi_1, \phi_2, \phi_3)$, $H = (\phi_4, \phi_5, \phi_6)$, $F = (f_1, f_2, f_3)$,

$$\phi = \begin{bmatrix} \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_6 \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then (71) and (72) may be written as

$$(78) \quad C^0 \phi_t + \sum_{\nu=1}^3 C^\nu \phi_{x_\nu} + \mathfrak{M} \phi + f = 0$$

where the matrices C^0 and $\mathfrak{M}$ are given by

$$(79) \quad C^0 = \mathfrak{D}^2, \quad \mathfrak{D} = \begin{pmatrix} \sqrt{\frac{\epsilon}{c}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{\epsilon}{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\epsilon}{c}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{\mu}{c}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{\mu}{c}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{\mu}{c}} \end{pmatrix},$$

$$(79) \quad \mathfrak{M} = \frac{4\pi\sigma}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the matrices C'' may be conveniently written together in the form

$$\mathcal{C} \equiv \sum_{\nu=1}^3 z_{\nu} C'' = \begin{bmatrix} 0 & 0 & 0 & 0 & z_3 & -z_2 \\ 0 & 0 & 0 & -z_3 & 0 & z_1 \\ 0 & 0 & 0 & z_2 & -z_1 & 0 \\ 0 & -z_3 & z_2 & 0 & 0 & 0 \\ z_3 & 0 & -z_1 & 0 & 0 & 0 \\ -z_2 & z_1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To transform (78) into our standard form, we set $\phi = \mathfrak{D}^{-1}u$. If we insert this expression in (78) and multiply on the left by $\mathfrak{D}^{-1}$ we obtain

$$(80) \quad u_t + \sum_{\nu=1}^3 A'' u_{x_{\nu}} + Bu = \eta$$

where

$$(81) \quad A'' = \mathfrak{D}^{-1}C'' \mathfrak{D}^{-1},$$

$$(82) \quad B = \sum_{\nu=1}^3 \mathfrak{D}^{-1}C'' (\mathfrak{D}^{-1})_{x_{\nu}} + \mathfrak{D}^{-1}\mathfrak{M} \mathfrak{D}^{-1} = \sum_{\nu=1}^3 A'' \mathfrak{D} (\mathfrak{D}^{-1})_{x_{\nu}} + \mathfrak{D}^{-1}\mathfrak{M} \mathfrak{D}^{-1},$$

$$(83) \quad \eta = -\mathfrak{D}^{-1}f.$$

It is clear from either (78) or (80) that we have a symmetric hyperbolic system. Furthermore, if we define the matrix

$$(84) \quad P = \sum_{\nu=1}^3 z_{\nu} A'' = \mathfrak{D}^{-1}\mathcal{C} \mathfrak{D}^{-1} = e\mathcal{C}, \quad e = \frac{c}{\sqrt{\epsilon\mu}},$$

one may verify that the eigenvalues of P are

$$(85) \quad 0, 0, e d, e d, -e d, -e d, \quad d = \sqrt{z_1^2 + z_2^2 + z_3^2}.$$

It follows at once that the *multiplicity condition* is satisfied.

For convenience, we also list a set of orthonormal eigenvectors of P in the same order as their corresponding eigenvalues appear in (85):

$$\begin{aligned}
 R^1 &= \frac{1}{d\sqrt{2}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}, & R^2 &= \frac{1}{d\sqrt{2}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ -z_1 \\ -z_2 \\ -z_3 \end{bmatrix}, \\
 (86) \quad R^3 &= \frac{1}{d\sqrt{2(z_1^2 + z_2^2)}} \begin{bmatrix} -z_1 z_3 \\ -z_2 z_3 \\ z_1^2 + z_2^2 \\ dz_2 \\ -dz_1 \\ 0 \end{bmatrix}, & R^4 &= \frac{1}{d\sqrt{2(z_1^2 + z_3^2)}} \begin{bmatrix} dz_2 \\ -dz_1 \\ 0 \\ z_1 z_3 \\ z_2 z_3 \\ -(z_1^2 + z_2^2) \end{bmatrix}, \\
 R^5 &= \frac{1}{d\sqrt{2(z_1^2 + z_2^2)}} \begin{bmatrix} z_1 z_3 \\ z_2 z_3 \\ -(z_1^2 + z_2^2) \\ z_2 d \\ -z_1 d \\ 0 \end{bmatrix}, & R^6 &= \frac{1}{d\sqrt{2(z_1^2 + z_2^2)}} \begin{bmatrix} dz_2 \\ -dz_1 \\ 0 \\ -z_1 z_3 \\ -z_2 z_3 \\ z_1^2 + z_2^2 \end{bmatrix}.
 \end{aligned}$$

For Maxwell's equations, we shall take our boundaries to be "perfect conductors," *i.e.*, we shall apply the boundary condition

$$(87) \quad E_{\text{tan}} = 0$$

where E_{tan} denotes the components of E tangential to the boundary. If we denote the boundary by $y_3(x) = 0$, the transformation $y_\nu = y_\nu(x)$, $\nu = 1, 2, 3$, leads to an equation in which the coefficient matrix $\hat{A}_3$ multiplying u_ν is simply the matrix $P = e\mathcal{C}$, provided we set $z_\nu = \partial y_3 / \partial x_\nu$. Since $n = (z_1, z_2, z_3)$ now denotes a vector normal to the boundary, the condition may be expressed as

$$(88) \quad u_j = cz_j, \quad j = 1, 2, 3,$$

where u_j denotes components of u . It is clear that all vectors u satisfying (88) lie in a space T of dimension 4. Furthermore, for u in T it is easily verified that $u\hat{A}^3 u = 0$. Thus $\hat{A}^3$ is non-positive over T . Since $\hat{A}^3 = P$ has precisely 4 non-positive eigenvalues, it is easily shown (see Part I, Appendix 2) that T is maximal. Thus we have shown that the boundary condition (87) is of the form we have applied in Section 1.

We examine, now, the "source condition" which was introduced in Section 1. If $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ denotes a unit vector normal to the surface of discontinuity Γ in the source region, and J^μ are null eigenvectors of the matrix $E = \sum_{\nu=1}^3 \gamma_\nu A^\nu$, this condition is

$$(89) \quad J^\mu[g] = 0 \quad (\text{for all } J^\mu).$$

The function $g(x)$ is determined by

$$(90) \quad g e^{-i\omega t} = \eta = -\mathfrak{D}^{-1}f = -\sqrt{\frac{c}{\epsilon}} f.$$

We may identify the matrix E with the matrix P by setting $\gamma_\nu = z_\nu$, $\nu = 1, 2, 3$. Thus the J^μ are given by R^1 and R^2 of (86). Since ϵ is a continuous function, we may rewrite (89) as

$$R^\mu[f] = 0, \quad \mu = 1, 2,$$

or, in terms of 3 component vectors,

$$(91) \quad \gamma[F] = [\gamma F] = 0.$$

Thus, in physical terms, the source condition may be stated as follows: Across any surface, the normal component of the impressed current must be continuous.

In Section 6 we shall introduce the so-called *initial data condition*:

$$(92) \quad J^\mu[u(0, x)] = 0 \quad (\text{for all } J^\mu).$$

For Maxwell's equations this becomes

$$(93) \quad R^\mu[u(0, x)] = 0, \quad \mu = 1, 2.$$

Equation (93) is equivalent to the two equations

$$(94) \quad [\gamma H] = 0 \quad \text{at } t = 0,$$

$$(95) \quad [\gamma E] = 0 \quad \text{at } t = 0.$$

Equation (94) is a direct consequence of (74). Equation (95) must be interpreted as an additional condition on the initial data of the problem. In physical terms, (95) states that there shall be no "surface charges" present.

Summarizing the results of this section we may say that

(1) Maxwell's equations are symmetric hyperbolic and satisfy the multiplicity condition.

(2) The boundary condition for perfectly conducting boundaries is a special case of the type we considered in Section 1.

(3) The source condition is satisfied for impressed currents which satisfy (91).

(4) The initial data condition is a consequence of the divergence equations (73), (74) provided we exclude surface charges.

It follows from (1), (2) and (3) above that our formal expansion and the

entire contents of this paper are applicable to Maxwell's equations. The significance of (4) is discussed in Section 6.

3. The Duhamel integral theorem. In the introduction to Part II, we stated that the results of Part I play a key role in the proof of the asymptotic nature of our formal solution. In Part II we have been concerned with an inhomogeneous hyperbolic equation (2) (and its related reduced equation (4)) whereas in Part I we were concerned with the homogeneous equation $Lu = 0$. The connection between these two equations is established by the Duhamel integral theorem. Before proving this theorem, however, we must discuss the notion of "weak solutions."

As in Part I, we consider a space $\mathcal{S}_\Omega$ of smooth testing functions $\Omega(t, x)$ with compact support. More precisely, $\mathcal{S}_\Omega$ will be the collection of all smooth k component vector functions Ω each of which vanishes identically outside some compact region R which depends upon Ω and lies in the half space $t > 0$. (For problems with boundaries we assume that R does not intersect the boundary.) For any u and f in C^1 , the set of functions with continuous first derivatives, and any Ω in $\mathcal{S}_\Omega$ we have the identities

$$(96) \quad \iint \Omega Lu \, dx \, dt = \iint u M \Omega \, dx \, dt, \quad \iint \Omega f_t \, dx \, dt = - \iint f \Omega_t \, dx \, dt.$$

Here the operator M , the *adjoint* of L , is defined by

$$(97) \quad M \Omega = - \sum_{\nu=0}^n (A' \Omega)_{x_\nu} + B' \Omega.$$

B' denotes the transpose of the matrix B . The integration in (96) extends over the half space $t \geq 0$. The identities are obtained simply by integration by parts, the boundary terms vanishing because Ω has compact support. If now we have $Lu = f_t$, it follows trivially that

$$(98) \quad \iint u M \Omega \, dx \, dt + \iint f \Omega_t \, dx \, dt = 0 \quad \text{for all } \Omega \text{ in } \mathcal{S}_\Omega.$$

Conversely if u and f are in C^1 in the neighborhood of some point in the half space $t > 0$ and satisfy (98) it follows that $Lu = f_t$ in that neighborhood. We may, however, consider more general classes of functions than C^1 . Thus if u is any integrable function which satisfies (98) we shall say that u is a *weak solution* of the equation $Lu = f_t$. If, in addition, $u(0, x) = g(x)$ (and u is in T on $x_n = 0$ *) we say that u is a *weak solution* (I) of the problem

$$(99) \quad Lu = f_t, \quad u(0, x) = g(x) \quad (u \text{ in } T \text{ on } x_n = 0).$$

*This is the boundary condition defined in Section 1. In the following sections we shall frequently treat initial value problems and initial-boundary value problems simultaneously. Parentheses are used to denote the additional conditions or results associated with the problem with boundaries.

We shall also need a second definition of weak solutions. For this purpose we consider a second space S_Φ of smooth testing functions Φ , which have the same properties as the functions Ω except that we now require only that the function Φ vanishes identically outside some region R whose intersection with the half space $t \geq 0$ is compact (see Fig. 1). (For problems with boundaries we again

FIGURE 1

The support of typical testing functions Ω and Φ .

assume that R does not intersect the boundary.) It is clear that S_Ω is a subset of S_Φ . We now consider the analogue of (98)

$$(100) \quad \iint u M \Phi \, dx \, dt + \iint f \Phi_t \, dx \, dt = 0 \quad \text{for all } \Phi \text{ in } S_\Phi.$$

As before, if u and f are in C^1 in a neighborhood of any point in the half space $t > 0$, then $Lu = f_t$ in that neighborhood. If, now, u and f are in C^1 in a neighborhood of a point $(0, x)$ on the initial plane $t = 0$, we may integrate by parts to obtain

$$(101) \quad \begin{aligned} 0 &= \iint \Phi(Lu - f_t) \, dx \, dt + \int_{t=0} \Phi(0, x) \{u(0, x) - f(0, x)\} \, dx \\ &= \int_{t=0} \Phi(0, x) \{u(0, x) - f(0, x)\} \, dx. \end{aligned}$$

It follows that

$$(102) \quad u(0, x) = f(0, x).$$

If u and f are any integrable functions which satisfy (100) and f_t exists (and u is in T on $x_n = 0$) we say that u is a *weak solution* (II) of the problem

$$(103) \quad Lu = f_t, \quad u(0, x) = f(0, x) \quad (u \text{ in } T \text{ on } x_n = 0).^*$$

In Part I we introduced the class $\mathcal{K}$ of piece-wise smooth functions: For each vector function u in $\mathcal{K}$, defined in the region of interest $\mathfrak{D}$,** there are a finite number of smooth hypersurfaces which divide $\mathfrak{D}$ into subregions $\mathfrak{D}_\nu$, such that the first derivatives of u can be made continuous in the closure of each $\mathfrak{D}_\nu$ by appropriate definition of the derivative on the boundary.

Our two definitions of weak solutions are related by the following

Lemma: *If u and f are in $\mathcal{K}$, then u is a weak solution (I) of the problem (103) if and only if it is a weak solution (II) of the same problem.*

Proof: 1. Let u in $\mathcal{K}$ be a weak solution (II). Then, as before, $u(0, x) = f(0, x)$. Since S_Ω is a subset of S_Φ , (98) holds. Hence u is a weak solution (I).

2. Let u in $\mathcal{K}$ be a weak solution (I), and let C_ν denote the hypersurfaces subdividing $\mathfrak{D}$ (for both functions u and f). Then for any Φ in S_Φ

$$(104) \quad \iint u M \Phi \, dx \, dt + \iint f \Phi_t \, dx \, dt = \iint \Phi \{Lu - f_t\} \, dx \, dt \\ + \int_{t=0} \Phi(x, 0) \{u(0, x) - f(0, x)\} \, dx + \sum_\nu \int_{C_\nu} \{\Phi, u, f\}$$

where $\int_{C_\nu} \{\Phi, u, f\}$ denotes an integral over C_ν involving the jumps in u and f . As we have seen before, $Lu = f_t$ except on the C_ν . Also $u(0, x) = f(0, x)$, hence the first two terms on the right side vanish. Thus

$$(105) \quad \iint u M \Phi \, dx \, dt + \iint f \Phi_t \, dx \, dt = \sum_\nu \int_{C_\nu} \{\Phi, u, f\}.$$

For any Ω in S_Ω

$$(106) \quad \iint u M \Omega \, dx \, dt + \iint f \Omega_t \, dx \, dt = \sum_\nu \int_{C_\nu} \{\Omega, u, f\}.$$

It is clear that we may choose an Ω which is equal to Φ on the hypersurfaces C_ν . Since the left side of (106) is zero, u must satisfy (100). Hence u is a weak solution (II)

We come now to the Duhamel integral theorem. The theorem was deduced by R. K. LUNEBURG, for Maxwell's equations, using Laplace transforms. M. KLINE proved it for piecewise smooth solutions of Maxwell's equations and the second order linear hyperbolic equation by a process of direct substitution which is quite laborious. The following version is both more general and more direct:

Theorem: *Let u be a weak solution (II) of the problem*

$$(107) \quad Lu = 0, \quad u(0, x) = g(x) \quad (u \text{ in } T \text{ on } x_n = 0).$$

*By setting $f = \int_0^t g(x, \tau) \, d\tau + l(x)$, we may define a weak solution (72) of the initial value problem $Lu = g(x, t)$, $u(0, x) = l(x)$.

** $\mathfrak{D}$ is either the half space $t \geq 0$ or, in problems with boundaries, that portion of the half space in which the solution is defined.

Let $h(t)$ be a smooth scalar function of t and set

$$(108) \quad W(t, x) = \frac{\partial}{\partial t} \int_0^t u(s, x) h(t-s) ds.$$

Then W is a weak solution (II) of the problem

$$(109) \quad LW = g(x)h'(t), \quad W(0, x) = g(x)h(0) \quad (W \text{ in } T \text{ on } x_n = 0).$$

Proof: If we consider the boundary condition as in (55) it is easy to see that W satisfies the condition if u does. Now by definition

$$(110) \quad \iint u M \Phi dx dt + \iint g(x) \Phi_t dx dt = 0 \quad \text{for all } \Phi \text{ in } S_\Phi.$$

Using integration by parts we have

$$\begin{aligned} \int_0^\infty W M \Phi dt &= \int_0^\infty \frac{\partial}{\partial t} \left[\int_0^t u(s, x) h(t-s) ds \right] M \Phi(t, x) dt \\ &= - \int_0^\infty \left[\int_0^t u(s, x) h(t-s) ds \right] [M \Phi(t, x)]_t dt \\ (111) \quad &= - \int_0^\infty u(s, x) \int_s^\infty [M \Phi(t, x)]_t h(t-s) dt ds \\ &= - \int_0^\infty u(t, x) \int_t^\infty [M \Phi(s, x)]_t h(s-t) ds dt. \end{aligned}$$

Let

$$(112) \quad \Psi(t, x) = - \int_t^\infty \Phi_t(s, x) h(s-t) ds = - \int_0^\infty \Phi_t(z+t, x) h(z) dz.$$

It is easily seen that Ψ is in S_Φ . Also

$$(113) \quad M \Psi = - \int_0^\infty [M \Phi(z+t, x)]_t h(z) dz = - \int_t^\infty [M \Phi(s, x)]_t h(s-t) ds.$$

(Compare the last terms of (113) and (111).) Now

$$\begin{aligned} \int_0^\infty \Psi_t(t, x) dt &= - \int_0^\infty \int_0^\infty \Phi_{tt}(z+t, x) h(z) dz dt \\ (114) \quad &= - \int_0^\infty h(t) \int_0^\infty \Phi_{tt}(t+z, x) dz dt = \int_0^\infty h(t) \Phi_t(t, x) dt. \end{aligned}$$

From (111), (113), (110), (114),

$$\begin{aligned} \iint W M \Phi dx dt &= \iint u M \Psi dx dt = - \int g(x) \int \Psi_t dt dx \\ (115) \quad &= - \iint g(x) h(t) \Phi_t dt dx, \end{aligned}$$

and the proof is complete.

In what follows we shall deal only with functions u , w , f which are in $\mathcal{K}$. By the lemma, we may without ambiguity refer simply to "weak solutions." Setting $h(t) = -e^{-i\omega t}/i\omega$ we obtain the following

Corollary: *Let u in $\mathcal{K}$ be a weak solution of the problem*

$$(116) \quad Lu = 0, \quad u(0, x) = g(x) \quad (u \text{ in } T \text{ on } x_n = 0).$$

Set

$$(117) \quad w(t, x) = \frac{\partial}{\partial t} \left[\frac{-e^{-i\omega t}}{i\omega} \int_0^t u(s, x) e^{i\omega s} ds \right] \\ = -\frac{u(t, x)}{i\omega} + e^{-i\omega t} \int_0^t u(s, x) e^{i\omega s} ds.$$

Then w is in $\mathcal{K}$ and is a weak solution of the problem

$$(118) \quad Lw = g(x)e^{-i\omega t}, \quad w(0, x) = -\frac{g(x)}{i\omega} \quad (w \text{ in } T \text{ on } x_n = 0).$$

In the remainder of this paper, u and w will denote the functions defined by (116) and (117),* where $g(x)$ is of type g . Theorem II-B of Part I asserts that under appropriate conditions, which will be satisfied, the solution u of (116) exists and is unique in $\mathcal{K}$. We have shown above that (117) is a solution of (118). The uniqueness of the solution of (118) follows easily from the uniqueness assertion of Theorem II-B. In order to avoid confusion it is useful to remember that the function $g(x)$ plays three roles: It is the initial data for u , a factor in the inhomogeneous term of the equation for w , and the inhomogeneous term in the reduced equation $Nv = g$.

4. The steady state function. We begin by imposing a new condition on the operator L which guarantees that the function $u(t, x)$ decays to zero as $t \rightarrow \infty$:

Decay Condition: *If u is the solution of the problem*

$$(119) \quad Lu = 0, \quad u(0, x) = g(x) \quad (u \text{ in } T \text{ on } x_n = 0)$$

where $g(x)$ is any function of type g , there is a constant $\omega_0 > 0$ such that for every x in $\mathcal{D}$, and for $\omega \geq \omega_0$, the integral $\int_0^\infty u(t, x) e^{i\omega t} dt$ exists and is bounded independently of ω .**

Since, for any n , $u_{i,n}$ is the solution of a problem of the type (119) it follows immediately from the above condition that $\int_0^\infty u_{i,n}(t, x) e^{i\omega t} dt$ exists and is

*In the work of LUNEBURG and KLINE, u is called the "pulse solution."

**For certain operators L it can be proved that the decay condition is satisfied. This is done in the Appendix.

bounded independently of ω for $\omega \geq \omega_n > 0$. Now, by integrating by parts, it is easy to show that

$$u(T)e^{i\omega T} = i\omega \int_0^T u e^{i\omega t} dt + \int_0^T u_t e^{i\omega t} dt + c$$

where c is independent of T . Letting $T \rightarrow \infty$ we see that the right side of the above equation tends to a limit. Thus we may write

$$\lim_{t \rightarrow \infty} u(t, x) e^{i\omega t} = a.$$

But since the integral $\int_0^\infty u(t, x) e^{i\omega t} dt$ converges it must be that $a = 0$. We conclude that $u(t, x) \rightarrow 0$ as $t \rightarrow \infty$. Applying the decay condition, (117) may be rewritten

$$(120) \quad w(t, x) = z(t, x) + e^{-i\omega t} v(x)$$

where

$$(121) \quad z(t, x) = \frac{-u(t, x)}{i\omega} - e^{-i\omega t} \int_t^\infty u(s, x) e^{i\omega s} ds$$

and

$$(122) \quad v(x) = \int_0^\infty u(s, x) e^{i\omega s} ds.$$

It follows at once that $z(t, x) \rightarrow 0$ as $t \rightarrow \infty$. The function z is called the *transient* and $v(x)$ is called the *steady state function* of the problem (118). Note that v is uniquely determined by the decomposition (120) and the condition that $z \rightarrow 0$ as $t \rightarrow \infty$. Furthermore, it is easily seen that v is independent of the initial data of the problem (118), for we may modify that problem by adding to w the weak solution u' of any problem

$$(123) \quad Lu' = 0, \quad u'(0, x) = g'(x) \quad (u' \text{ in } T \text{ on } x_n = 0)$$

where g' is of type g . The resulting function $w' = w + u'$ will be the weak solution of the problem

$$(124) \quad Lw' = g(x)e^{-i\omega t}, \quad w'(0, x) = g'(x) - \frac{g(x)}{i\omega} \quad (w' \text{ in } T \text{ on } x_n = 0).$$

But problem (124) has a steady state function which is again given by (122).

The steady state function v , defined above, is our main object of interest in this paper. We shall attempt to show that the formal expansion $\tilde{v}$ derived in Section 1 is an asymptotic expansion of v for $\omega \rightarrow \infty$. To this end we shall employ the integral expression (122). In Section 6 we shall show that v is a solution of the reduced equation $Nv = g$.

5. Asymptotic expansion of the steady state function. In Part I we proved that under appropriate conditions the function $u(t, x)$ is piece-wise smooth,

i.e., u is smooth everywhere in $\mathfrak{D}$ except on certain hypersurfaces C^α where u and its derivatives have jump discontinuities. In Section 6 we will show that these hypersurfaces can never have a tangent plane parallel to the t -axis, so they can be defined by equations of the form $t = \Psi^{(\alpha)}(x)$. With this information, the asymptotic expansion of the integral (122) is obtained simply by repeated integrations by parts. At each step we obtain contributions from the lower end point of the integral and from the discontinuities of u and its derivatives. We shall assume that the discontinuities of u and its derivatives with respect to t all occur on a finite number of hypersurfaces $t = \Psi^{(\alpha)}(x)$, $\alpha = 1, \dots, p$. We write

$$(125) \quad v(x) = \int_0^{\Psi^{(1)}} + \int_{\Psi^{(1)}}^{\Psi^{(2)}} + \dots + \int_{\Psi^{(p)}}^{\infty} u(t, x) e^{i\omega t} dt.$$

Integrating by parts

$$(126) \quad v(x) = -\frac{u(0, x)}{i\omega} - \sum_{\alpha} \frac{e^{i\omega \Psi^{(\alpha)}}}{i\omega} [u]^\alpha - \left\{ \int + \dots + \int u_t(t, x) \frac{e^{i\omega t}}{i\omega} dt \right\}.$$

Here

$$(127) \quad [u]^\alpha = u(\Psi^{(\alpha)} + 0, x) - u(\Psi^{(\alpha)} - 0, x).$$

By repeated integrations by parts we obtain

$$(128) \quad v(x) = \sum_{r=0}^M \left(\frac{-1}{i\omega} \right)^{r+1} \{ u_{tr}(0, x) + \sum_{\alpha} e^{i\omega \Psi^{(\alpha)}(x)} [u_{tr}]^\alpha \} + \left(\frac{-1}{i\omega} \right)^{M+1} \int_0^{\infty} u_{t(M+1)} e^{i\omega t} dt.$$

Since u is a piecewise smooth solution of $Lu = u_t + \sum_{\nu=1}^n A^\nu u_{x_\nu} + Bu = 0$, and $u(0, x) = g(x)$, we have $u_t(0, x) = -\mathcal{L}g$, and in general

$$(129) \quad u_{tr}(0, x) = (-\mathcal{L})^r g.$$

The operator $\mathcal{L}$ is defined by (8).

Applying the decay condition, the integral in (128) may be estimated independently of ω . We substitute an order symbol for the last term in the sum together with the integral term:

$$(130) \quad v(x) = - \sum_{r=0}^{M-1} \frac{\mathcal{L}^r g}{(i\omega)^{r+1}} + \sum_{\alpha} e^{i\omega \Psi^{(\alpha)}(x)} \sum_{r=0}^{M-1} \left(\frac{-1}{i\omega} \right)^{r+1} [u_{tr}]^\alpha + O\left(\frac{1}{\omega^{M+1}} \right).$$

If $g(x)$ is sufficiently smooth, we know from Part I that u will be smooth and the terms involving $[u_{tr}]^\alpha$ will vanish. Now compare (130) with (10). We see that $\tilde{v} = y$ is indeed an asymptotic expansion to M terms of v . In the general case, where g and its derivatives are discontinuous across Γ , we must determine the functions $(-1)^{r+1}[u_{tr}]^\alpha$ which appear in (130). We shall identify them with the terms $z^{(\alpha)(r+1)}$ in the formal expansion (see (34) and (13)). We must also identify the functions $\Psi^{(\alpha)}$ appearing above with the $\Psi^{(\alpha)}$ in (13).

Thus we must study the discontinuities of u . This is done in the next section.

6. The discontinuities of u . The present section is essentially a repetition of Sections 2, 3 and 6 of Part I. There is, however, one important difference: In Part I we studied discontinuities in derivatives of u *normal* to the discontinuity surfaces. Any other non-tangential direction of differentiation would do as well. In particular, differentiation in the t -direction is permissible provided this direction is never tangential to a discontinuity surface.* In this section we shall use the t -direction. This, together with the fact that the coefficient matrices A^ν , B are independent of t , provides a considerable simplification. Furthermore, we shall need the results in terms of t -derivatives in order to determine the $[u_t]^\alpha$ of (130). The calculations will not be carried out fully, but only to the point where we may compare them with Section 1.

The function u is defined by the following:

$$(131) \quad \iint u M \Omega \, dx \, dt = 0 \quad \text{for all } \Omega \text{ in } \mathcal{S}_\Omega,$$

$$(132) \quad u(0, x) = g(x),$$

$$(133) \quad u \text{ is in } \mathcal{K},$$

and, for problems with boundaries,

$$(134) \quad (u \text{ in } T \text{ on } x_n = 0).$$

Suppose u is smooth on either side of the smooth hypersurface C , whose equation is $t = \Psi(x)$. We introduce the regular transformation

$$(135) \quad \phi = t - \Psi(x), \quad y = x.$$

Here $y = y_1, \dots, y_n$. We easily obtain the transformed operators

$$(136) \quad Lu = Gu_\phi + \sum_{\nu=1}^n A^\nu u_{\nu\phi} + Bu,$$

$$(137) \quad M\Omega = -(G\Omega)_\phi - \sum_{\nu=1}^n (A^\nu \Omega)_{\nu\phi} + B'\Omega$$

where

$$(138) \quad G = I - \sum_{\nu=1}^n \Psi_{x_\nu} A^\nu.$$

If we integrate by parts in (131) and use the fact that Ω is arbitrary we obtain

$$(139) \quad G[u] = 0.$$

The symbol $[u] = u(\Psi + 0, x) - u(\Psi - 0, x)$ denotes the jump in u across C .

*Discontinuity surfaces (*i.e.*, characteristic surfaces) may well have tangent planes parallel to the t -axis. This fact, which is true for Maxwell's equations (71) and (72), is commonly unrecognized. However, such surfaces will be eliminated by imposing conditions on the initial data $g(x)$.

Since A^ν and B are independent of ϕ , we may differentiate the equation $Lu = 0$ to obtain $Lu_{\phi(r-1)} = 0$ on both sides of C . Hence

$$(140) \quad G[u_{\phi^r}] + \sum_{\nu=1}^n A^\nu [u_{\phi(r-1)}]_{\nu^r} + B[u_{\phi(r-1)}] = 0, \quad r = 1, 2, \dots.$$

Since $[u_{\phi^r}] \neq 0$ for some r , we must have

$$(141) \quad \det G = 0.$$

It follows from (141) that the surface C is a *characteristic surface* for the operator L . Let $R^1, \dots, R^k$ be a complete set of orthonormal eigenvectors of G . We now impose the *multiplicity condition* of Section 1. It follows that on C certain eigenvalues $\lambda_1, \dots, \lambda_a$ of G are identically zero and all the others are non-zero. Let $R^1, \dots, R^a$ denote the associated null eigenvectors.

Now observe that $u_{\phi^r} = u_{t^r}$, and that the function $[u_{\phi^r}]$ is independent of ϕ .
et

$$(142) \quad [u_{\phi^r}] = [u_{t^r}] = (-1)^{r+1} z^{(r+1)}(y) = (-1)^{r+1} z^{(r+1)}(x), \\ r = 0, 1, 2, \dots, \quad z^{(0)} = 0.$$

Then (139) and (140) may be written

$$(143) \quad Gz^{(r+1)} - \mathcal{L}z^{(r)} = 0, \quad r = 0, 1, 2, \dots.$$

Now compare (143) with (15), (138) with (16), (141) with (17). It is clear that we may expand our $z^{(r)}$ as in (21) and obtain transport equations for the expansion coefficients which are identical to the transport equations of Section 1, namely (32) and (24).^{*} It follows that we may identify our functions Ψ and $z^{(r)}$ with those of Section 1 provided we can show that they are determined by the same initial conditions.

The function u will be discontinuous across those characteristic surfaces $C^{(\alpha)}$ which intersect the initial plane $t = 0$ in the manifold Γ , across which g is discontinuous. Let the equation of $C^{(\alpha)}$ be $t = \Psi^{(\alpha)}(x)$. Then

$$(144) \quad \Psi^{(\alpha)} = 0 \quad \text{on } \Gamma.$$

Since $C^{(\alpha)}$ must be tangent to Γ , the projection into the plane $t = 0$ of the normal to $C^{(\alpha)}$ must be parallel to the normal to Γ . Thus

$$(145) \quad \Psi_{x_\nu}^{(\alpha)} = \rho(\alpha)\gamma_\nu, \quad \nu = 1, \dots, n.$$

Here $\gamma_1, \dots, \gamma_n$ are components of the unit normal to Γ and $\rho(\alpha)$ is a proportionality factor which is determined by the equation

$$(146) \quad \det G^{(\alpha)} = \det \left(I - \sum_{\nu=1}^n \Psi_{x_\nu}^{(\alpha)} A^\nu \right) = \det (I - \rho(\alpha)E) = 0 \quad \text{on } \Gamma$$

^{*}Some confusion might result from our definition of *rays*. The rays of Part II are the projections into the plane $t = 0$ of the rays of Part I.

where

$$(147) \quad E = \sum_{\nu=1}^n \gamma_{\nu} A^{\nu}.$$

There will be as many characteristic surfaces $C^{(\alpha)}$ emanating from Γ as there are distinct solutions $\rho(\alpha) = 1/\beta(\alpha)$ of (146), hence as many as there are distinct non-zero eigenvalues $\beta(\alpha)$ of E . Precisely as in Section 1, we introduce the eigenvectors B^i and J^{ν} of E . Instead of a "source condition" it is more appropriate to define an

Initial Data Condition: $J^{\nu}[g] = 0$, $\nu = 1, \dots, p$, on Γ .

(The conditions are, of course, identical.*)

By using the fact that $Lu = 0$, we may obtain the values of $u_{i,r}$ on the initial plane $t = 0$. Thus

$$(148) \quad u_{i,r}(0, x) = (-\mathcal{L})^r g(x), \quad r = 0, 1, 2, \dots.$$

The initial values for the transport equations are obtained from the fact that

$$(149) \quad [u_{i,r}(0, x)]^{\Gamma} = - \sum_{\alpha} e(\alpha) [u_{i,r}]^{\alpha} \quad \text{on } \Gamma, \quad r = 0, 1, 2, \dots.$$

Here $[\]^{\Gamma}$ and $[\]^{\alpha}$ denote jumps across Γ and across the characteristic surface $C^{(\alpha)}$ and $e(\alpha) = \pm 1$. The orientation of $[\]^{\alpha}$ is already defined. We choose the orientation of $[\]^{\Gamma}$ to coincide with the orientation of $[\]^{\Gamma}$ as defined in Section 1. By appropriately indexing the characteristic surfaces $C^{(\alpha)}$ to correspond to the waves $z^{(\alpha)}$ of Section 1 it is not difficult to see that the numbers $e(\alpha)$ will have the same values here as in Section 1. Using (148) and (142) we may rewrite (149) as

$$(150) \quad [\mathcal{L}^{(r-1)} g]^{\Gamma} = \sum_{\alpha} e(\alpha) z^{(\alpha)(r)} \quad \text{on } \Gamma, \quad r = 1, 2, \dots.$$

Now compare (144) with (37), (146) with (41), (150) with (39). It is clear that the functions $z^{(\alpha)(r)}$ and $\Psi^{(\alpha)}$ are determined by the same initial conditions as they are in Section 1. Hence the functions in the two sections are identical.

For problems with boundaries we must apply the boundary condition (134). Suppose a characteristic surface $C^{(\alpha)}$, emanating from the manifold Γ in the initial hyperplane, intersects the boundary B ($x_n = 0$) in an $n - 1$ dimensional manifold Δ . We now consider all the characteristic surfaces $C^{(\nu)}$ which intersect B in Δ and attempt to choose initial conditions for the transport equations

*The initial data function g is identical with the source function g of Section 1. Hence the introduction of a "new" condition on g would appear to be superfluous. However, the results of this section are applicable to initial value problems in general, just as is the whole of Part I. From this point of view we really have a condition on initial data. It is this condition which makes an analysis in terms of t -derivatives possible. For Maxwell's equations the condition is particularly interesting for, as we have seen, it sheds new light on the "divergence equations" (73), (74).

on these surfaces in such a way that the jumps in u and its derivatives will be compatible with the boundary condition.

The analysis in Section 1 is directly applicable here. We need only identify Ψx -space with tx -space; "phase surface" with "characteristic surface," *etc.* Again we require that the *shadow condition* shall hold. The phase surfaces which are incoming (outgoing) with respect to B must be identified with the characteristic surfaces on which t decreases (increases) as we proceed along a ray from Δ . For convenience we shall also apply the terms "incoming" and "outgoing" to the characteristic surfaces $C^{(p)}$. Thus one of the incoming characteristic surfaces $C^{(p)}$ is our $C^{(\alpha)}$. The rest of them may be assumed to carry no discontinuities.*

The boundary condition may be stated in the form

$$(151) \quad b^\nu u = 0 \quad \text{on} \quad x_n = 0, \quad \nu = 1, \dots, m.$$

The vectors b^ν are independent of t . Differentiating repeatedly with respect to t , we obtain

$$(152) \quad b^\nu u_{t^r} = 0 \quad \text{on} \quad x_n = 0, \quad \nu = 1, \dots, m, \quad r = 0, 1, 2, \dots.$$

Now, in obvious notation,

$$(153) \quad \sum_p b^\nu [u_{t^r}]^p = b^\nu [u_{t^r}]^\Delta = [b^\nu u_{t^r}]^\Delta = 0, \\ \nu = 1, \dots, m, \quad r = 0, 1, 2, \dots.$$

Using the notation defined in (142), (153) may be written

$$(154) \quad \sum_p b^\nu z^{(p)(r)} = 0 \quad \text{on} \quad B, \quad \nu = 1, \dots, m, \quad r = 0, 1, 2, \dots.$$

We observe that (154) is identical with (63). It follows that the $z^{(p)(r)}$ of this section and the corresponding functions of Section 1 have the same initial values on B , hence are identical.

Having determined all the discontinuity surfaces† of u and all the discontinuities of u_{t^r} ($r = 0, 1, 2, \dots$) on these surfaces, we renumber them with the index α . Now, using the notation of (142), we may rewrite (130) as

$$(155) \quad v(x) = - \sum_{r=0}^{M-1} \frac{\mathcal{L}^r g}{(i\omega)^{r+1}} + \sum_{\alpha} e^{i\omega \Psi^{(\alpha)}(x)} \sum_{r=1}^M \frac{z^{(\alpha)(r)}(x)}{(i\omega)^r} + O\left(\frac{1}{\omega^{M+1}}\right).$$

Comparing this equation with (70) we see that *under appropriate conditions, the formal expansion $\tilde{v}$ is indeed an asymptotic expansion to M terms of the steady state function v .* These conditions will be summarized in Section 7.

We are now in a position to show that the steady state function v is a solution

*If, by coincidence, two of the $C^{(\alpha)}$ emanating from Γ intersect B in the same Δ , we may still treat them separately.

†We assume that they are finite in number.

of the reduced equation. By this we mean that for any point x , not on Γ , v is in C^1 in a neighborhood of x and satisfies the equation

$$(156) \quad Nv = \sum_{\nu=1}^n A^\nu v_{x_\nu} + (B - i\omega)v = g.$$

To show this, we use the integral expression (122), and split up the integral at a typical discontinuity surface $t = \Psi(x)$:

$$(157) \quad v(x) = \int_0^{\Psi(x)} + \int_{\Psi(x)}^\infty u(t, x) e^{i\omega t} dt.$$

Differentiating

$$(158) \quad v_{x_\nu} = \int_0^\infty u_{x_\nu} e^{i\omega t} dt - e^{i\omega\Psi} \Psi_{x_\nu}[u].$$

By integration by parts,

$$(159) \quad \int_0^\infty u e^{i\omega t} dt = -e^{i\omega\Psi}[u] - u(0, x) - i\omega \int_0^\infty u e^{i\omega t} dt.$$

Using (158) and (159) we have

$$(160) \quad \begin{aligned} Nv &= \int_0^\infty \left\{ \sum_{\nu=1}^n A^\nu u_{x_\nu} + Bu + u_t \right\} e^{i\omega t} dt \\ &\quad + e^{i\omega\Psi} \left\{ I - \sum_{\nu=1}^n \Psi_{x_\nu} A^\nu \right\} [u] + u(0, x) \\ &= \int_0^\infty \{Lu\} e^{i\omega t} dt + e^{i\omega\Psi} G[u] + g(x). \end{aligned}$$

But $Lu = 0$ for $t \neq \Psi(x)$ and from (139) $G[u] = 0$. Hence

$$(161) \quad Nv = g \quad \text{for } x \text{ not on } \Gamma.$$

Since g is discontinuous on Γ , it is too much to expect that v should be continuously differentiable on Γ . However, it is easy to see that v is *everywhere continuous*. This follows from the integral expression (122) and the fact that the discontinuity surfaces of u are never tangential to the t -axis.*

7. Summary. In this section we summarize our main results. Statements which refer to problems with boundaries are enclosed in parentheses.

Given the problem

$$Lw = g(x)e^{-i\omega t}, \quad w(0, x) = h(x) \quad (w \text{ in } T \text{ on } x_n = 0)$$

*This last assertion is true because we were able to carry out the construction of the discontinuities of u without using characteristic surfaces tangential to the t -axis. This was possible only because of the initial data condition. Had we used the formulation of Part I we would have discovered that the initial conditions for the transport equations on such surfaces were zero.

where

(1) L is a symmetric hyperbolic operator with coefficients that are smooth functions of x ,

(2) $g(x)$ and $h(x)$ are of type g ,

(3) $g(x)$ satisfies the source condition,

(4) the multiplicity condition is satisfied,

(5) the caustic condition is satisfied,

(6) (the shadow condition is satisfied),

then there exists a weak solution $w(t, x)$ defined in the region $t \geq 0$ which is unique in $\mathcal{K}$. If, in addition,

(7) the decay condition is satisfied

then w has the decomposition

$$w(t, x) = z(t, x) + e^{-i\omega t}v(x)$$

where $z(t, x) \rightarrow 0$ as $t \rightarrow \infty$. The steady state function $v(x)$ is uniquely determined by this decomposition and is independent of the choice of the function $h(x)$. It is continuous and, except on the discontinuities of $g(x)$, is a solution of the reduced equation $Nv = g$. The formal expansion $\bar{v}$ derived in Section 1 is an asymptotic expansion to M terms (for $\omega \rightarrow \infty$) of v . M will be as large as we please if the coefficients of L , (the boundary condition), and, except on the manifold Γ , the function $g(x)$ are sufficiently smooth.

APPENDIX

The behavior of u for large t . In Section 4 we introduced the decay condition. This condition, unlike the others, cannot be verified simply by examination of the operator L , or by a study of the geometry of the rays of the problem. Thus, without further investigation, we cannot even be sure that the class of operators which satisfy the decay condition is not empty. Unfortunately, the mathematical literature on the subject of the behavior for large t of solutions of hyperbolic equations appears to be practically non-existent. In this section, we shall attempt to examine the subject.

Our approach to the problem is by the well known method of energy inequalities.* Unfortunately, we shall have to place severe restrictions on the coefficient matrices of the operator L in order to arrive at the desired result. It appears likely that these restrictions are not inherent in the problem itself, but merely reflect the inadequacy of the method. Perhaps this discussion will stimulate further interest in the question.

The energy method leads to an estimate of the L^2 norm of the solution u . Thus, if we introduce the function

*This method plays an important role in uniqueness and existence theorems for hyperbolic equations. See, e.g., [4], [5].

$$(162) \quad N(t) = \sqrt{\int u^2(t, x) dx},$$

we can, under appropriate conditions, show that $N(t) \rightarrow 0$ as $t \rightarrow \infty$. Unfortunately, this does not suffice to guarantee pointwise convergence of u to zero as $t \rightarrow \infty$. However, convergence in the L^2 norm of u and certain of its derivatives is a sufficient condition for pointwise convergence of y . This is a consequence of the following

Lemma (SOBOLEV) [13]: *Let $v(x)$ be a vector function of $x = x_1, \dots, x_n$. Let v and its derivatives up to order n be square integrable over the entire x -space. Then*

$$(163) \quad |v(x_1, \dots, x_n)|^2 \leq 2^n (||v||^2 + ||v_{x_1}||^2 + \dots + ||v_{x_n}||^2 + ||v_{x_1 x_2}||^2 + \dots + ||v_{x_1 x_2 \dots x_n}||^2).$$

Here $||v||^2 = \int v^2 dx$ and $|v| = \max_i |v_i|$, where v_i denotes a component of v . The terms on the right side of the inequality include all derivatives of v up to order n , excluding those which involve differentiation with respect to any of the x_i more than once.

We shall write the equation for u in the form

$$(164) \quad Lu = u_i + A^v u_{x_v} + Bu = 0.$$

Here we employ the summation convention. In order to apply SOBOLEV's lemma, we must first enlarge the system (164) to include not only u , but also the appropriate derivatives of u . This is done simply by differentiation of (164) with respect to the required x variables. By introducing vectors and matrices with many more components we may express the enlarged system in vector form. Thus we introduce the vectors

$$(165) \quad \tilde{u} = \begin{bmatrix} u \\ u_{x_1} \\ \vdots \\ u_{x_n} \\ u_{x_1 x_2} \\ \vdots \\ u_{x_1 x_2 \dots x_n} \end{bmatrix}, \quad \tilde{g} = \begin{bmatrix} g \\ g_{x_1} \\ \vdots \\ g_{x_n} \\ g_{x_1 x_2} \\ \vdots \\ g_{x_1 x_2 \dots x_n} \end{bmatrix}.$$

The enlarged system now takes the form

$$\tilde{L}\tilde{u} = \tilde{u}_i + \tilde{A}^v \tilde{u}_{x_v} + \tilde{B}\tilde{u} = 0.$$

It is tedious to calculate the matrix $\tilde{B}$ explicitly. However, it is easy to see that

$$(166) \quad \tilde{A}^\nu = \begin{bmatrix} A^\nu & 0 & \cdot & \cdot & 0 & 0 \\ 0 & A^\nu & \cdot & \cdot & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdot & \cdot & A^\nu & 0 \\ 0 & 0 & \cdot & \cdot & 0 & A^\nu \end{bmatrix}.$$

Thus the $k \times k$ submatrices appearing along the diagonal of $\tilde{A}^\nu$ are all identical to A^ν , and the other entries of $\tilde{A}^\nu$ are zero. For our purposes it is sufficient to observe that the top row of $k \times k$ submatrices of the matrix $\tilde{B}$ consists of the matrix B on the left followed by zero matrices. This too is easily seen. We now define a space $\tilde{\mathcal{S}}$ of testing function $\tilde{\Omega}(t, x)$. The space $\tilde{\mathcal{S}}$ is defined exactly as is the space $\mathcal{S}$ in Part I, except that the vectors $\tilde{\Omega}$ have as many components as does $\tilde{u}$. It is convenient to introduce a special type of element of $\tilde{\mathcal{S}}$. Let

$$(167) \quad \hat{\Omega} = \begin{bmatrix} \Omega \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}.$$

Here Ω is any element of $\mathcal{S}$. As before, we introduce the adjoint operator

$$\tilde{M}\tilde{\Omega} = -(\tilde{A}^\nu\tilde{\Omega})_{x_\nu} + \tilde{B}'\tilde{\Omega}.$$

We say that $\tilde{u}$ is a *weak solution* of the initial value problem

$$(168) \quad \tilde{L}\tilde{u} = 0, \quad \tilde{u}(0, x) = \tilde{g}(x)$$

if and only if

$$(169) \quad \iint \tilde{u}\tilde{M}\tilde{\Omega} \, dx \, dt = 0 \quad \text{for all } \tilde{\Omega} \text{ in } \tilde{\mathcal{S}}, \quad \tilde{u}(0, x) = \tilde{g}(x).$$

We assume that $g(x)$ is of type g and vanishes identically outside a compact region, and that L satisfies the multiplicity condition. We also assume that the caustic condition is satisfied for the problem $Lu = 0$, $u(0, x) = g(x)$. It is then easy to show that $\tilde{g}(x)$ is of type g , and vanishes outside a compact region, that $\tilde{L}$ satisfies the multiplicity condition, and that the caustic condition is satisfied for the problem (168). It is obvious that $\tilde{L}$ is a symmetric hyperbolic operator. It follows from Theorem 2 of Part I that there exists a unique solution $\tilde{u}$ of (168) in the class $\mathcal{K}$ of piecewise smooth functions. $\tilde{u}$ is defined for all $t \geq 0$.

We observe from (166), (167) and our observations about $\tilde{B}$ that

$$(170) \quad \tilde{u}\tilde{M}\hat{\Omega} = uM\Omega.$$

Here u denotes the first k components of $\tilde{u}$. Hence, from (169),

$$\iint u M \Omega \, dx \, dt = 0 \quad \text{for all } \Omega \text{ in } \mathcal{S}.$$

Thus u is the solution of the problem

$$(171) \quad Lu = 0, \quad u(0, x) = g(x),$$

which is considered in the definition of the decay condition.

Before deriving the energy inequality, we wish to recall the first steps in the calculation of the discontinuities of a solution of a discontinuous initial value problem.* By integration by parts,

$$(172) \quad 0 = \iint \tilde{u} \tilde{M} \tilde{\Omega} \, dx \, dt = \iint \tilde{\Omega} \tilde{L} \tilde{u} \, dx \, dt + \int_c \tilde{\Omega} [\tilde{G} \tilde{u}] \, ds.$$

Here

$$(173) \quad \tilde{G} = I - \Psi_x \tilde{A}'$$

and $[f] = f(\Psi + 0, x) - f(\Psi - 0, x) = f_+ - f_-$ denotes the jump across the characteristic surface C whose equation is $t = \Psi(x)$. It follows that

$$(174) \quad [\tilde{G} \tilde{u}] = \tilde{G}[\tilde{u}] = 0.$$

From (174) we observe that

$$(175) \quad [\tilde{u} \tilde{G} \tilde{u}] = \tilde{u}_+ \tilde{G} \tilde{u}_+ - \tilde{u}_+ \tilde{G} \tilde{u}_- + \tilde{u}_- \tilde{G} \tilde{u}_+ - \tilde{u}_- \tilde{G} \tilde{u}_- = \tilde{u}_+ \tilde{G}[\tilde{u}] + \tilde{u}_- \tilde{G}[\tilde{u}] = 0.$$

Now, by integration by parts,

$$(176) \quad 0 = \int_0^T \int \tilde{u} \{ \tilde{L} \tilde{u} \} \, dx \, dt = \int_0^T \int \tilde{u} \tilde{M} \tilde{u} \, dx \, dt - \sum_p \int_{C_p} [\tilde{u} \tilde{G} \tilde{u}] \, ds + \int \tilde{u}^2 \, dx \Big|_0^T.$$

The C_p denote the discontinuity surfaces of u . We introduce the function

$$(177) \quad \tilde{N}^2(t) = \int \tilde{u}^2(t, x) \, dx$$

and rewrite the adjoint operator in the form

$$(178) \quad \tilde{M} \tilde{\Omega} = -\tilde{L} \tilde{\Omega} - \tilde{Q} \tilde{\Omega}.$$

Here

$$(179) \quad \tilde{Q} = \tilde{A}'_{x'} - \tilde{B} - \tilde{B}'.$$

*This is mentioned in Part II, Section 6, but is carried out in detail in Part I, Section 2.

†Here the integrals exist because, for each fixed t , $\tilde{u}(t, x)$ vanishes outside a compact region. This is a consequence of the same property of $\tilde{g}(x)$.

Now from (176), using (175), (177), (178),

$$(180) \quad 0 = - \int_0^T \int \tilde{u} \tilde{Q} \tilde{u} \, dx \, dt + \tilde{N}^2(T) - \tilde{N}^2(0).$$

By differentiating with respect to T ,

$$(181) \quad 2\dot{\tilde{N}}\tilde{N}(t) = \int \tilde{u} \tilde{Q} \tilde{u} \, dx.$$

We now impose the following restriction on the operator L :

Energy Condition: For every $\tilde{v}$, $\tilde{v} \tilde{Q} \tilde{v} \leq -2\gamma \tilde{v}^2$, $\gamma > 0$, γ independent of x .

Thus we require that the *energy matrix* $\tilde{Q}$ be negative definite. Since this matrix can be calculated, the energy condition is an algebraic condition on the coefficient matrices of the operator L . For a given operator one can easily determine whether or not the energy condition is satisfied.

Now (181) may be rewritten

$$(182) \quad \dot{\tilde{N}} \leq -\gamma \tilde{N}.$$

It follows that

$$(183) \quad \tilde{N}(t) \leq C_1 e^{-\gamma t}.$$

Here

$$(184) \quad C_1^2 = \tilde{N}^2(0) = \int \tilde{u}^2(0, x) \, dx = \int \tilde{g}^2(x) \, dx.$$

From SOBOLEV's lemma,

$$(185) \quad |u(t, x)|^2 \leq 2^n \int \tilde{u}^2(t, x) \, dx = 2^n \tilde{N}^2(t) \leq 2^n C_1^2 e^{-2\gamma t}.$$

Thus $|u(t, x)|$ decays exponentially as $t \rightarrow \infty$ for every x . This certainly suffices to guarantee that u satisfies the decay condition.

ACKNOWLEDGMENT

The author wishes to express his appreciation to the members of the faculty and staff of the Institute of Mathematical Sciences for their advice and criticism. In particular, he is indebted to Professor JOSEPH B. KELLER for his assistance and encouragement during the preparation of this paper.

REFERENCES

Note: References 1 through 8 are contained in LEWIS, ROBERT M., *Discontinuous Initial Value Problems for Symmetric Hyperbolic Linear Differential Equations*, this issue. This paper is referred to as Part I in the present publication.

- [9] LUNEBURG, R. K., *The Mathematical theory of optics*; Lecture notes, Brown University, 1944.
- , *Propagation of electromagnetic waves*; Lecture notes, New York University, 1948.
- [10] KLINE, M., An asymptotic solution of Maxwell's equations; *Comm. Pure and Appl. Math.*, **4**, No. 2-3, 225-263, 1951.
- , Asymptotic solution of linear hyperbolic partial differential equations; *J. Rational Mech. and Anal.*, **3**, No. 3, 315-342, 1954.
- , Asymptotic solutions of Maxwell's equations involving fractional powers of the frequency; *Comm. Pure Appl. Math.*, **8**, No. 4, 595-614, 1955.
- [11] KELLER, J. B., LEWIS, R. M., & SECKLER, B. D., Asymptotic solution of some diffraction problems; *Comm. Pure and Appl. Math.*, **9**, No. 2, 207-265, May, 1956.
- [12] KAY, I. & KELLER, J. B., Asymptotic evaluation of the field at a caustic; *J. Appl. Phys.*, **25**, No. 7, 876-883, 1954.
- [13] SOBOLEV, S. L., On a theorem of functional analysis; *Matematicheskii sbornik*, *N. S.* **4**, 471-497, 1938.

Institute of Mathematical Sciences
New York University