*Integral Equations of the Schwarzschild-Milne Type**

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Communicated by S. CHANDRASEKHAR

1. Introduction. The Schwarzschild-Milne integral equation of radiative transfer theory is

$$(1) \quad f(x) = \frac{1}{2} \int_0^\infty E_1(|x - y|) f(y) dy, \quad x \geq 0,$$

where

$$(2) \quad E_1(t) = \int_0^1 e^{-t/\mu} \mu^{-1} d\mu, \quad t > 0.$$

Auxiliary conditions are $f(x) \geq 0$ and $f(x) \equiv 0$. The integral equation was generalized by HOPF [1] and is further generalized in this paper.

Let K_1 be any function such that

$$(3) \quad K_1(t) \geq 0,$$

$$(4) \quad \int_{-\infty}^\infty K_1(t) dt = 1,$$

$$(5) \quad \int_{-\infty}^\infty K_1(t) t dt = 0,$$

$$(6) \quad \int_t^\infty K_1(\pm s) ds \leq CK_1(\pm t), \quad t \geq 0,$$

where C is some positive constant. These conditions are satisfied with $K_1(t) = \frac{1}{2}E_1(|t|)$ and $C = 1$. Let Λ denote the integral operator such that

$$(7) \quad (\Lambda f)(x) = \int_0^\infty K_1(x - y) f(y) dy, \quad x \geq 0,$$

whenever the right member exists for all $x \geq 0$. The problem defined by

$$(8) \quad f = \Lambda f, \quad f \geq 0, \quad f \equiv 0$$

is the announced generalization.

* Most of the material in this paper is taken from the author's 1957 Oregon State College doctoral dissertation.

The following is a summary of the principal results presented in this paper. The solutions of (8) consist of the positive multiples of a function $x + q(x)$, where q is continuous and $0 < q \leq C$. The function q is represented as a Neumann series. By analogy with the radiative transfer case, a normalization condition is derived by integrating $f = \Lambda f$. There is a unique normalized solution, $J(x) = (F/2M_2) [x + q(x)]$, where M_2 is the second moment of K_1 and F is a normalization constant. The adjoint operator $\bar{\Lambda}$ is defined in terms of the kernel $\bar{K}_1(t) = K_1(-t)$. The corresponding problem has a unique normalized solution of the form $\bar{J}(x) = (\bar{F}/2M_2) [x + \bar{q}(x)]$. The method used by HOPF to derive the Hopf-Bronstein relation yields, in the present case, $J(0)\bar{J}(0) = F\bar{F}/8M_2$.

Let K_{m1} , $m \geq 1$, satisfy (3)–(6) with the same constant C in (6). Define Λ_m , q_m , J_m and F_m by analogy with Λ , q , J and F . If $K_{m1} \rightarrow K_1$ uniformly on each compact subset of the complement of some finite set and if $\int_{-\infty}^x K_{m1}(t)dt \rightarrow \int_{-\infty}^x K_1(t)dt$ pointwise, then, for each bounded measurable function f , $\Lambda_m f \rightarrow \Lambda f$ uniformly. This result is closely related to the Helly-Bray theorem on completely convergent sequences of distribution functions of random variables (see LOÈVE [2], p. 182). If $\Lambda_m f \rightarrow \Lambda f$ uniformly for each bounded measurable function f , then $q_m(x) \rightarrow q(x)$; if also $F_m \rightarrow F$, then $J_m(x) \rightarrow J(x)$. In both cases the convergence is uniform on each finite interval. Thus there exist topologies on the set of kernels and on the set of bounded continuous functions with respect to which the mapping which sends K_1 into q is continuous. Similar statements apply to the mappings which send K_1 into J , Λ into q , and Λ into J .

The class of kernels defined by (3)–(6) includes various Gaussian numerical integration approximations to $\frac{1}{2}E_1(|t|)$. These kernels arise in connection with the work of WICK [3], CHANDRASEKHAR [4], KROOK [5], SYKES [6] and the author [7] on the one-dimensional integrodifferential equation of transfer. Let $I_n(\tau, \mu)$ and $J_n(\tau)$, $n \geq 1$, denote the approximations to the solutions $I(\tau, \mu)$ and $J(\tau)$ of the classical transfer problem which were derived by WICK and CHANDRASEKHAR. Then J satisfies (1) and J_n satisfies (8) with the kernel a particular Gaussian numerical integration approximation to $\frac{1}{2}E_1(|t|)$. The results of this paper will be used in a later report* to prove that $J_n \rightarrow J$ and $I_n \rightarrow I$ uniformly on compact sets. Analogous statements apply to the approximations derived by SYKES.

The present study was motivated in the first instance by the question of the convergence of the Wick-Chandrasekhar approximations. A second motivation came from a desire to determine to what extent the methods which HOPF first used to solve the Schwarzschild-Milne integral equation could be adapted to more general equations. Thus integral transform or Wiener-Hopf techniques are eschewed. For recent studies involving these techniques, see the paper by SPITZER [8] and the abstract of WIDOM [9].

* A paper entitled "Convergence of the Wick-Chandrasekhar Approximation Technique in Radiative Transfer" will appear in the *Astrophysical Journal*, 127 (1958).

2. Iterates of the kernel functions. Let K_1 be any kernel function which satisfies (3)–(6). Let

$$(9) \quad \begin{aligned} K_n(\pm t) &= \int_t^\infty K_{n-1}(\pm s) ds, & t > 0, \\ K_n(0) &= \int_0^\infty K_{n-1}(s) ds, \end{aligned} \quad n \geq 2.$$

By (3), (4) and (6), these integrals exist, $K_2(t) \leq 1$ and

$$(10) \quad 0 \leq K_n(t) \leq CK_{n-1}(t) \leq \cdots \leq C^{n-2}K_2(t) \leq C^{n-1}K_1(t), \quad n \geq 1.$$

For each $n \geq 2$, $K_n(t)$ is non-increasing for $t \geq 0$, non-decreasing for $t < 0$, continuous for $t \neq 0$, and bounded by C^{n-2} . Furthermore, $K_n(0\pm)$ exist and $K_n(0) = K_n(0+)$ for $n \geq 2$.

It follows from (9), by repeated integration by parts or by a well known identity, that

$$(11) \quad K_{n+2}(\pm t) = \frac{1}{n!} \int_t^\infty K_1(\pm x)(x - t)^n dx, \quad t > 0, \quad n \geq 0,$$

$$(12) \quad K_{n+2}(0\pm) = \frac{1}{n!} \int_0^\infty K_1(\pm t)t^n dt, \quad n \geq 0.$$

Therefore the moments of K_1 ,

$$(13) \quad M_n = \int_{-\infty}^\infty K_1(t)t^n dt, \quad n \geq 0,$$

exist and are given by

$$(14) \quad M_n = n![K_{n+2}(0+) + (-1)^n K_{n+2}(0-)], \quad n \geq 0.$$

From (4), (5), (12) and (14),

$$(15) \quad M_0 = K_2(0+) + K_2(0-) = 1,$$

$$(16) \quad M_1 = K_3(0+) - K_3(0-) = 0.$$

For each fixed t , $K_2(t) = 0$ iff (if and only if) $K_3(t) = 0$. Therefore, by (16), $K_2(0+) = 0$ iff $K_2(0-) = 0$ so that, by (15), $K_2(0+) > 0$ and $K_2(0-) > 0$. In view of these results and the remarks following (10), there exist α and β such that $-\infty \leq -\alpha < 0 < \beta \leq \infty$ and $K_2(t) > 0$ iff $t \in (-\alpha, \beta)$. Since $K_n(t) = 0$ iff $K_{n+1}(t) = 0$, for t fixed and $n \geq 2$,

$$(17) \quad K_n(t) > 0 \quad \text{iff} \quad t \in (-\alpha, \beta), \quad n \geq 2.$$

It now follows by (9) and (10) with $n = 2$ that

$$(18) \quad K_1(t) > 0 \quad \text{for} \quad t \in (-\alpha, \beta),$$

$$(19) \quad K_1(t) = 0 \quad \text{for almost all} \quad t \in (-\alpha, \beta).$$

Note that $K_2(t)$ is absolutely continuous and

$$\frac{d}{dt} [e^{t/c} K_2(t)] = \frac{1}{C} e^{t/c} [K_2(t) - CK_1(t)]$$

for almost all $t > 0$. Similar statements are valid for $t < 0$. It follows that (6) is satisfied iff the functions $e^{t/c} K_2(\pm t)$ are non-increasing for $t > 0$. This helps to characterize the class of kernels under discussion.

3. The integral operator Λ . The operator Λ is defined by (7), where K_1 is any kernel function satisfying (3)–(6). The domain of Λ , denoted by $D(\Lambda)$, consists of the functions f such that the right member of (7) exists for all $x \geq 0$. For each $f \in D(\Lambda)$,

$$(20) \quad (\Lambda f)(x) = \int_0^\infty K_1(x-y)f(y) dy = \int_{-\infty}^x K_1(t)f(x-t) dt, \quad x \geq 0.$$

Let I denote the function defined by $I(x) = x$, $x \geq 0$. Then, by (20) and the results of Section 2,

$$(21) \quad \begin{aligned} (\Lambda I^n)(x) &= \int_{-\infty}^\infty K_1(t)(x-t)^n dt - \int_x^\infty K_1(t)(x-t)^n dt, \quad x \geq 0, \\ \Lambda I^n &= I^n + \sum_{i=2}^n \binom{n}{i} (-1)^i M_i I^{n-i} - (-1)^n n! K_{n+2}, \end{aligned}$$

where the functions K_n are restricted to the non-negative axis. It follows that $D(\Lambda)$ contains all polynomials. The functions I^0 and I will also be denoted by 1 and x , respectively. Thus, from (21),

$$(22) \quad \Lambda 1 = 1 - K_2,$$

$$(23) \quad \Lambda x = x + K_3.$$

For $f \in D(\Lambda)$ and $g \in D(\Lambda)$ it follows from (18), (19) and (20) that

$$(24) \quad f \geq 0 \Rightarrow \Lambda f \geq 0, \quad f \geq g \Rightarrow \Lambda f \geq \Lambda g,$$

$$(25) \quad f > g \Rightarrow \Lambda f > \Lambda g.$$

If f is measurable and $|f| \leq g$ for some $g \in D(\Lambda)$, then $|f| \in D(\Lambda)$, $f \in D(\Lambda)$ and $|\Lambda f| \leq \Lambda |f| \leq \Lambda g$. Since $1 \in D(\Lambda)$, $D(\Lambda)$ contains all bounded measurable functions. By (22), $\Lambda 1 \leq 1$. Hence, for each measurable function $f(x)$, $x \geq 0$,

$$(26) \quad |f| \leq M \Rightarrow |\Lambda f| \leq M.$$

The identity

$$(\Lambda f)(y) - (\Lambda f)(x) = \int_{-\infty}^x K_1(t)[f(y-t) - f(x-t)] dt + \int_x^y K_1(y)f(y-t) dt$$

and the fact that all moments of K_1 are finite imply that

$$(27) \quad f \text{ non-decreasing, } f \geq 0 \Rightarrow \Lambda f \text{ non-decreasing;}$$

(28) f continuous, $f = O(x^n)$ for some $n \geq 0 \Rightarrow \Lambda f$ continuous.

Denote by E the set of all bounded, continuous functions $f(x)$, $x \geq 0$. For each $f \in E$, let $\|f\| = \text{l.u.b. } |f(x)|$. Then

$$(29) \quad \Lambda E \subset E \subset D(\Lambda)$$

and $\|\Lambda f\| \leq \|f\|$ for $f \in E$. Since $\|\Lambda 1\| = 1$, the restriction of Λ to E has norm unity:

$$(30) \quad \|\Lambda\| = 1 \quad \text{on } E.$$

The next result is of a rather special nature. In (20) let $f(y)$ be defined as zero for $y < 0$. Then, by (18) and (19),

$$(\Lambda f)(x) = \int_{-\alpha}^{\beta} K_1(t) f(x-t) dt, \quad x \geq 0,$$

where $K_1(t) > 0$ for $t \in (-\alpha, \beta)$. Therefore, for each $f \geq 0$ and each $x \geq 0$,

$$(31) \quad (\Lambda f)(x) = 0 \quad \text{iff} \quad f(y) = 0 \quad \text{for almost all } y \in (x-\beta, x+\alpha).$$

4. The equation $f = \Lambda f$. Throughout this section assume that $f = \Lambda f$, $f \geq 0$ and $f \equiv 0$. Repeated application of (31) gives $f(x) = 0 \Rightarrow f(y) = 0$ for almost all $y = 0 \Rightarrow \Lambda f \equiv 0 \Rightarrow f \equiv 0$. Therefore

$$(32) \quad f(x) > 0 \quad \text{for all } x \geq 0.$$

It follows from (7), (10) and $f = \Lambda f$ that

$$(33) \quad \int_0^{\infty} K_n(x-y) f(y) dy \leq C^{n-1} f(x), \quad x \geq 0, \quad n = 1.$$

The indefinite integration of $f(x) = (\Lambda f)(x)$ yields, by (9) and (15),

$$(34) \quad \int_0^{\infty} K_2(x-y) \operatorname{sgn}(y-x) f(y) dy = B_2, \quad x \geq 0,$$

where

$$(35) \quad B_n = B_n(f) = \int_0^{\infty} K_n(-y) f(y) dy, \quad n \geq 2.$$

By (17) and (32), each B_n is positive. The indefinite integration of (34) yields, by (9) and (16),

$$(36) \quad \int_0^{\infty} K_3(x-y) f(y) dy = B_2 x + B_3, \quad x \geq 0.$$

From (33) and (36)

$$(37) \quad f(x) \geq C^{-2}(B_2 x + B_3).$$

Thus g.l.b. $f(x) > 0$ and $f(x)$ is unbounded.

5. Existence and uniqueness theorems. The substitution of $f = x + \phi$ into $f = \Lambda f$ yields, by (23), $\phi - \Lambda\phi = K_3$, where $K_3(x)$, restricted to $x \geq 0$, belongs to the set E of bounded, continuous functions. Even though $\|\Lambda\| = 1$ on E , it is shown below that the Neumann series associated with $\phi - \Lambda\phi = K_3$ converges to a solution of this equation. For this purpose a few preliminary results are needed.

Lemma 1. Suppose that $\phi = \sum_{n=0}^{\infty} \Lambda^n \Phi$ exists and that $\Phi \geq 0$. Then $\phi - \Lambda\phi = \Phi$.

Proof. Since each term in the series is non-negative, Lebesgue's monotone convergence theorem yields $\Lambda\phi = \sum_{n=0}^{\infty} \Lambda^{n+1} \Phi = \phi - \Phi$.

Lemma 2. Suppose that $\phi - \Lambda\phi = \Phi$, where $\Phi \geq 0$, $\phi \geq 0$ and ϕ is bounded. Then $\phi = \sum_{n=0}^{\infty} \Lambda^n \Phi$ and $\Lambda^n \phi \rightarrow 0$ monotonely.

Proof. Since $0 \leq \Lambda\phi = \phi - \Phi \leq \phi$, (24) gives $0 \leq \Lambda^{n+1}\phi \leq \Lambda^n \phi \leq \phi$. It follows from this result and

$$\sum_{j=1}^{n-1} \Lambda^j \Phi = \sum_{j=1}^{n-1} \Lambda^j (\phi - \Lambda\phi) = \phi - \Lambda^n \phi$$

that $\theta = \sum_{n=0}^{\infty} \Lambda^n \Phi$ exists, $0 \leq \theta \leq \phi$ and $\Lambda^n \phi \rightarrow \phi - \theta$ monotonely. By Lemma 1, $\theta - \Lambda\theta = \Phi$. It follows that $\phi - \theta = \Lambda(\phi - \theta)$ and $0 \leq \phi - \theta \leq \phi$. Assume that $\phi - \theta \equiv 0$. Then (37) implies that $\phi - \theta$ and ϕ are unbounded, a contradiction. Therefore $\theta = \phi$ and the lemma is proved.

Lemma 3. $\sum_{n=0}^{\infty} \Lambda^n K_2 = 1$ and $\Lambda^n 1 \rightarrow 0$ monotonely. In both statements the convergence is uniform in each finite interval.

Proof. The first two assertions follow from (22) and Lemma 2. Since each function $\Lambda^n 1$ is continuous by (28), the proof is completed by using $\sum_{j=0}^{n-1} \Lambda^j K_2 = 1 - \Lambda^n 1$ and a well known theorem.

Lemma 4. Suppose that $\phi - \Lambda\phi = \Phi$, where ϕ is bounded. Then $\phi = \sum_{n=0}^{\infty} \Lambda^n \Phi$ and $\Lambda^n \phi \rightarrow 0$. In both statements the convergence is uniform in each finite interval.

Proof. Let $|\phi| \leq M$. Then

$$\left| \phi - \sum_{j=0}^{n-1} \Lambda^j \Phi \right| = |\Lambda^n \phi| \leq M \Lambda^n 1,$$

and the desired results follow from Lemma 3.

These lemmas are used in the proofs of the main existence and uniqueness theorems which are now presented.

Theorem 1. Let

$$(38) \quad q = \sum_{n=0}^{\infty} \Lambda^n K_3,$$

$$(39) \quad p = x + q.$$

Then q and p are continuous and

$$(40) \quad 0 < q \leq C,$$

$$(41) \quad q - \Lambda q = K_3,$$

$$(42) \quad p = \Lambda p, \quad x < p \leq x + C.$$

Furthermore, the unique bounded solution of (41) is given by (38).

Proof. By (26) and (28), each term in the series of (38) is continuous. From (10) and (24), $0 \leq \Lambda^n K_3 \leq C \Lambda^n K_2$ for $n \geq 0$. So, by Lemma 3, the series converges uniformly on each finite interval, q and p are continuous, and $0 \leq q \leq C$. It follows by induction from (17), (31) and (38) that $(\Lambda^n K_3)(x) > 0$ and $q(x) > 0$ for $0 \leq x < (n+1)\alpha$, where $\alpha > 0$. Thus (40) holds. Lemma 1 and (38) imply (41). These results, plus $\Lambda x = x + K_3$ and Lemma 4, imply (42) and the uniqueness of q .

Theorem 2. A necessary and sufficient condition that $f = \Lambda f$, $f \geq 0$ and $f \equiv 0$ is that $f = bp$ for some $b > 0$.

Proof. The sufficiency is obvious. For the necessity let $f = \Lambda f$, $f \geq 0$, $f \equiv 0$ and $g = f - bp$, where $b = \text{g.l.b. } (f/p)$. Then $g = \Lambda g$ and $g \geq 0$. Choose x_n , $n \geq 1$, such that $\lim_{n \rightarrow \infty} f(x_n)/p(x_n) = b$ and $\lim_{n \rightarrow \infty} x_n$ exists, finite or infinite. Suppose first that $x_n \rightarrow y < \infty$. Since p is continuous, $p(x_n) \rightarrow p(y)$. Then $f(x_n) = [f(x_n)/p(x_n)]p(x_n) \rightarrow bp(y)$ and $g(x_n) = f(x_n) - bp(x_n) \rightarrow 0$. If $g \equiv 0$, then (37) implies that $\text{g.l.b. } g(x) > 0$, a contradiction. Thus $g \equiv 0$ if $x_n \rightarrow y < \infty$. Suppose that $x_n \rightarrow \infty$. By (42), $p(x_n)/x_n \rightarrow 1$, so that $f(x_n)/x_n \rightarrow b$ and $g(x_n)/x_n \rightarrow 0$. If $g \equiv 0$, then (37) implies that $\text{g.l.b. } [g(x)/x] > 0$, a contradiction. Hence $g \equiv 0$ in either case and $f = bp$. It follows from $f \geq 0$, $f \equiv 0$ and $p > 0$ that $b > 0$.

6. A normalization condition. In the radiative transfer case the quantity $B_2 = B_2(f)$ appearing in (32) is one-fourth the outward normal net flux associated with the intensity f . It is customary to specify this constant in order to add a normalization condition to the problem and, thus, to have a unique solution. The following is a derivation of $B_2(p)$ in the present more general case. Substitute $f(y) = p(y) = y + q(y)$ into (36), replace y by $x - t$, divide through by x and let $x \rightarrow \infty$ to obtain

$$(43) \quad B_2(p) = \int_{-\infty}^{\infty} K_3(t) dt = K_4(0+) + K_4(0-) = \frac{1}{2}M_2,$$

where M_2 is given by (13) or (14).

If $J = bp$, then $B_2(J) = \frac{1}{2}bM_2$. By analogy with the radiative transfer case, let J denote henceforth the unique function such that $J = \Lambda J$, $J \geq 0$, $J \equiv 0$ and $B_2(J) = \frac{1}{4}F$, where F is an arbitrary positive constant. Thus

$$(44) \quad J = \frac{F}{2M_2} p = \frac{F}{2M_2} (x + q).$$

In the radiative transfer case, $M_2 = \frac{2}{3}$, so that $J = \frac{3}{4}Fp = \frac{3}{4}F(x + q)$.

7. The adjoint operator. Corresponding to K_1 and Λ , define $\bar{K}_1$ and $\bar{\Lambda}$ such that $\bar{K}_1(t) = K_1(-t)$ and

$$(45) \quad (\bar{\Lambda}f)(x) = \int_0^\infty \bar{K}_1(x-y)f(y) dy = \int_0^\infty K(y-x)f(y) dy, \quad x \geq 0.$$

Since $\bar{K}_1$ satisfies (3)–(6), all results obtained above hold with appropriate notational changes. It suffices to replace K_n , M_n , q , p , J , F , $\dots$ by $\bar{K}_n$, $\bar{M}_n$, $\bar{q}$, $\bar{p}$, $\bar{J}$, $\bar{F}$, $\dots$ respectively. In particular, one has $\bar{M}_n = (-1)^n M_n$ and

$$(46) \quad \begin{aligned} \bar{q} &= \sum_{n=0}^{\infty} \bar{\Lambda}^n \bar{K}_3, & \bar{q} - \bar{\Lambda} \bar{q} &= \bar{K}_3, \\ \bar{p} &= x + \bar{q}, & \bar{p} &= \bar{\Lambda} \bar{p}, & \bar{J} &= \frac{\bar{F} \bar{p}}{2M_2}. \end{aligned}$$

Of course, if K_1 is even then $\bar{\Lambda} = \Lambda$ and these results reduce to the earlier ones.

For Φ and Θ non-negative and measurable, let

$$(47) \quad (\Phi, \Theta) = \int_0^\infty \Phi(t)\Theta(t) dt,$$

finite or infinite. If, also, $\Phi \in D(\Lambda)$ and $\Theta \in D(\bar{\Lambda})$, then

$$(\Lambda \Phi, \Theta) = \int_0^\infty \int_0^\infty K_1(x-y)\Phi(y)\Theta(x) dy dx = (\Phi, \bar{\Lambda} \Theta),$$

so that $(\Lambda^n \Phi, \Theta) = (\Phi, \bar{\Lambda}^n \Theta)$ for $n \geq 0$. Suppose that ϕ and θ exist, where

$$(48) \quad \phi = \sum_{n=0}^{\infty} \Lambda^n \Phi, \quad \theta = \sum_{n=0}^{\infty} \bar{\Lambda}^n \Theta.$$

Then, by Lemma 1 and the results just obtained,

$$(49) \quad (\phi, \Theta) = (\Phi, \theta),$$

$$(50) \quad \phi - \Lambda \phi = \Phi, \quad \theta - \bar{\Lambda} \theta = \Theta,$$

$$(51) \quad (\phi, \theta - \bar{\Lambda} \theta) = (\phi - \Lambda \phi, \theta).$$

8. Analogue of the Hopf-Bronstein relation. Let

$$(52) \quad p_s(x) = \frac{1}{s} [p(x+s) - p(x)]$$

for $x \geq 0$ and $s > 0$. Then, by (20) and $p = \Lambda p$,

$$(53) \quad p_s - \Lambda p_s = Q_s,$$

where

$$(54) \quad Q_s(x) = \frac{1}{s} \int_x^{x+s} K_1(t)p(x+s-t) dt.$$

Since $p_*(x)$ is bounded in x for each fixed s , (53) and Lemma 4 imply that

$$(55) \quad p_s = \sum_{n=0}^{\infty} \Lambda^n Q_s.$$

Incidental consequences of (55) are that p_s is non-negative and that p is non-decreasing. It follows from (46), (53) and (51) with $\phi = p_*$ and $\theta = \bar{q}$ that

$$(56) \quad (p_s, \bar{K}_s) = (Q_s, \bar{q}).$$

The desired result will be obtained by letting $s \rightarrow 0$ in (56). From

$$(47), (52) \quad \text{and} \quad \bar{K}_s(x) = K_s(-x),$$

$$(p_s, \bar{K}_s) = -\frac{1}{s} \int_0^s K_s(-x) p(x) dx + \int_s^{\infty} \frac{K_s(-x+s) - K_s(-x)}{s} p(x) dx.$$

Then, by (9), (35) and (43),

$$(57) \quad \lim_{s \rightarrow 0} (p_s, \bar{K}_s) = -K_3(0-)p(0) + \frac{1}{2}M_2.$$

Refer again to (54). Since p is continuous, $\lim_{s \rightarrow 0} Q_s(x) = p(0)K_1(x)$ almost everywhere and, by (47),

$$(58) \quad \lim_{s \rightarrow 0} (Q_s, \bar{q}) = p(0)(K_1, \bar{q}).$$

Since $\bar{q} = \bar{p} - x$, it follows from (45), (46) and (12) that

$$\begin{aligned} (K_1, \bar{q}) &= \int_0^{\infty} K_1(x) \bar{p}(x) dx - \int_0^{\infty} K_1(x) x dx \\ &= (\bar{\Lambda} \bar{p})(0) - K_3(0+) = \bar{p}(0) - K_3(0+) \end{aligned}$$

and

$$(59) \quad \lim_{s \rightarrow 0} (Q_s, \bar{q}) = p(0) \bar{p}(0) - p(0) K_3(0+).$$

Therefore, by (56), (57), (59) and (16),

$$(60) \quad p(0) \bar{p}(0) = \frac{1}{2} M_2.$$

It follows from (60), (46) and (44) that

$$(61) \quad J(0) \bar{J}(0) = \frac{F \bar{F}}{8M_2}.$$

If K_1 is even and $\bar{F} = F$ then (61) reduces to

$$(62) \quad J^2(0) = \frac{F^2}{8M_2}.$$

In the astrophysical case $M_2 = \frac{2}{3}$ and (62) yields the Hopf-Bronstein relation

$$(63) \quad J(0) = \frac{\sqrt{3}F}{4}.$$

9. A convergent sequence of operators. Let K_1 be a non-negative function such that $\int_{-\infty}^{\infty} K_1(t) dt = 1$, and define K_2 and Λ as above. Let S denote the set of bounded measurable functions $f(x)$, $x \geq 0$, with $\|f\| = \text{l.u.b. } |f|$. It follows as in Section 3 that $S \subset D(\Lambda)$, $\Lambda 1 = 1 - K_2$, $\|\Lambda 1\| = \|1\| = 1$ and $\|\Lambda\| = 1$ on S . For each $m \geq 1$, define K_{m1} , K_{m2} and Λ_m analogously. The principal theorem of this section gives sufficient conditions in order that $\|\Lambda f - \Lambda_m f\| \rightarrow 0$ on S . One of the conditions, $K_{m2} \rightarrow K_2$ pointwise, is also necessary, as the following theorem indicates.

Theorem 3. $(\Lambda_m f)(x) \rightarrow (\Lambda f)(x)$ for each $f \in S$ and each x iff $K_{m2} \rightarrow K_2$ pointwise.

Proof. Assume first that $(\Lambda_m f)(x) \rightarrow (\Lambda f)(x)$. Let $f_n(y) = 0$ for $0 \leq y < n$ and $f_n(y) = 1$ for $y \geq n$. Then, by (20) and (9), $(\Lambda f_n)(x + n) = K_2(x)$ for $-n \leq x < 0$ and $(\Lambda f_n)(x + n) = 1 - K_2(x)$ for $x \geq 0$. Similar statements hold for Λ_m . Hence $K_{m2}(x) \rightarrow K_2(x)$ for each x . The converse is a direct consequence of the Helly-Bray theorem on completely convergent sequences of distribution functions of random variables (see LOÈVE [2], p. 182). Further details are omitted because no application is to be made of this half of the theorem.

It can be proved that the hypothesis $K_{m2} \rightarrow K_2$ pointwise is not sufficient to imply that $\|\Lambda f - \Lambda_m f\| \rightarrow 0$ on S . It suffices, for example, to require also that $K_{m1} \rightarrow K_1$ uniformly on each compact subset of the complement of a finite set $\{x_i \mid i = 1, \dots, j\}$. This condition is weakened somewhat in the following theorem so as to admit certain singular kernels.

Theorem 4. Suppose that $K_{m2} \rightarrow K_2$ pointwise and that $K_{m1} \rightarrow K_1$ uniformly on each compact subset of the complement of a finite set $\{x_i \mid i = 1, \dots, j\}$. Then $\|\Lambda f - \Lambda_m f\| \rightarrow 0$ on S .

Proof. Choose y_k , $0 \leq k \leq 2j + 3$, arbitrarily such that

$$-\infty = y_0 < y_1 < \dots < y_{2j} < x_i < y_{2j+1} < \dots < y_{2j+2} < y_{2j+3} = \infty.$$

Then, for each $f \in S$,

$$\begin{aligned} |(\Lambda f)(x) - (\Lambda_m f)(x)| &= \left| \int_{-\infty}^x [K_1(t) - K_{m1}(t)] f(x-t) dt \right| \\ &\leq \|f\| \int_{-\infty}^{\infty} |K_1(t) - K_{m1}(t)| dt \leq \|f\| \sum_{i=0}^{j+1} \int_{y_{2i}}^{y_{2i+1}} [K_1(t) + K_{m1}(t)] dt \\ &\quad + \|f\| \sum_{i=1}^{j+1} \int_{y_{2i-1}}^{y_{2i}} |K_1(t) - K_{m1}(t)| dt. \end{aligned}$$

By the hypothesis on K_{m1} , each term in the second sum tends to zero as $m \rightarrow \infty$. The hypothesis on K_{m2} implies that $\int_x^y K_{m1}(t) dt \rightarrow \int_x^y K_1(t) dt$ for $-\infty \leq x < y \leq \infty$. Therefore

$$\limsup_{m \rightarrow \infty} \|\Lambda f - \Lambda_m f\| \leq 2 \|f\| \sum_{i=0}^{j+1} \int_{y_{2i}}^{y_{2i+1}} K_1(t) dt.$$

Let $y_1 \rightarrow -\infty$, $y_{2i} \rightarrow x_i$ and $y_{2i+1} \rightarrow x_i$ for $1 \leq i \leq j$, and $y_{2i+2} \rightarrow \infty$ to obtain the desired result.

For each positive number N let S_N denote the subset of S such that $f \in S_N$ iff $\|f\| \leq N$. Then the conclusion of Theorem 4 can be replaced by $\|\Delta f - \Delta_m f\| \rightarrow 0$ uniformly on each S_N without altering the proof. As a matter of fact, it can be shown that whenever $\|\Delta f - \Delta_m f\| \rightarrow 0$ on S the convergence is uniform on each S_N .

The usual argument involving the triangle inequality yields, for each $f \in S$, $\|\Delta f - \Delta_m f\| \rightarrow 0 \Rightarrow \|\Delta^n f - \Delta_m^n f\| \rightarrow 0$ for $n \geq 0$. A more general but hardly more novel result is that, for each $f \in S$,

$$(64) \quad \|f - f_m\| \rightarrow 0, \|\Delta f - \Delta_m f\| \rightarrow 0 \Rightarrow \|\Delta^n f - \Delta_m^n f_m\| \rightarrow 0, \quad n \geq 0.$$

The case with $n = 0$ is obvious. Since

$$\|\Delta f - \Delta_m f_m\| \leq \|\Delta f - \Delta_m f\| + \|\Delta_m(f - f_m)\| \leq \|\Delta f - \Delta_m f\| + \|f - f_m\| \rightarrow 0,$$

(64) holds with $n = 1$. Since $\|\Delta^{n+1} f - \Delta_m^{n+1} f_m\| = \|\Delta^n(\Delta f) - \Delta_m^n(\Delta_m f_m)\|$, induction completes the proof.

10. Convergence theorems for the q_m and J_m . Suppose that K_1 and K_{m_1} , $m \geq 1$, satisfy (3)–(6). Let Λ, K_n, q, J, M_n and F be associated with K_1 as above and define $\Lambda_m, K_{mn}, q_m, J_m, M_{mn}$ and F_m analogously in terms of K_{m_1} . Assume that K_1 and K_{m_1} satisfy (6) with some single constant C . Equivalently, there exists $C > 0$ such that

$$(65) \quad K_2 \leq CK_1, \quad K_{m_2} \leq CK_{m_1} \quad \text{for } m \geq 1.$$

It is also assumed that

$$(66) \quad \|\Delta f - \Delta_m f\| \rightarrow 0 \quad \text{on } S.$$

These assumptions will be repeated in the statements of theorems.

One of the principal theorems of this paper asserts that $q_m \rightarrow q$ uniformly on each finite interval whenever (65) and (66) are satisfied. First, some preliminary results are derived.

Theorem 5. *If $\|\Delta f - \Delta_m f\| \rightarrow 0$ on S , then $K_{m_2} \rightarrow K_2$ pointwise. If $K_2 \leq CK_1$, $K_{m_2} \leq CK_{m_1}$ and $K_{m_2} \rightarrow K_2$ pointwise, then $\|K_n - K_{mn}\| \rightarrow 0$ and $K_{mn}(0-) \rightarrow K_n(0-)$ for $n \geq 2$.*

Proof. The first statement is implied by Theorem 3. Since $K_2(0) = K_2(0+) = 1 - K_2(0-)$ and similarly for K_{m_2} , one has $K_{m_2}(0-) \rightarrow K_2(0-)$. Let $g(x) = \int_{-\infty}^x K_1(t)dt$ for $-\infty < x < \infty$, $g(-\infty) = 0$ and $g(\infty) = 1$. Define g_m similarly in terms of K_{m_1} . Since g and g_m are simply related to K_2 and K_{m_2} , $g_m(x) \rightarrow g(x)$ for $-\infty \leq x \leq \infty$. It is not difficult to prove that any sequence of monotone functions which converges to a finite valued continuous function on a finite or infinite closed interval is uniformly convergent. Therefore $\|g - g_m\| \rightarrow 0$ and

the theorem is valid for $n = 2$. Assume the validity of the theorem for some fixed n . It follows from (9) that

$$K_{m,n+1}(x) - K_{n+1}(x) = \int_x^y [K_{mn}(t) - K_n(t)] dt + K_{m,n+1}(y) + K_{n+1}(y)$$

for $y \geq x \geq 0$. Therefore, by (10),

$$|K_{m,n+1}(x) - K_{n+1}(x)| \leq \int_x^y |K_{mn}(t) - K_n(t)| dt + C[K_{mn}(y) + K_n(y)].$$

First let $m \rightarrow \infty$ and then let $y \rightarrow \infty$ to obtain $K_{m,n+1}(x) \rightarrow K_{n+1}(x)$ for $x \geq 0$. In a similar fashion it is proved that $K_{m,n+1}(x) \rightarrow K_{n+1}(x)$ for $x < 0$ and that $K_{m,n+1}(0-) \rightarrow K_{n+1}(0-)$. The uniformity of the convergence is obtained by the same argument as in the case with $n = 2$.

From (41), $q - \Lambda q = K_3$ and $\sum_{i=0}^{n-1} \Lambda^i K_3 = q - \Lambda^n q$ for $n \geq 1$. Similar statements involve q_m , Λ_m and K_{m3} . Therefore

$$(67) \quad q = q_{0n} + \Lambda^n q, \quad q_m = q_{mn} + \Lambda_m^n q_m,$$

for $n \geq 1$ and $m \geq 1$, where

$$(68) \quad q_{0n} = \sum_{i=0}^{n-1} \Lambda^i K_3, \quad q_{mn} = \sum_{i=0}^{n-1} \Lambda_m^i K_{m3}.$$

Theorem 6. *If $K_2 \leq CK_1$, $K_{m2} \leq CK_{m1}$ and $\| \Lambda f - \Lambda_m f \| \rightarrow 0$ on S , then $q_m \rightarrow q$ uniformly on each finite interval.*

Proof. According to (40), $0 < q \leq C$. Similarly, $0 < q_m \leq C$. Therefore, by (67) and (24),

$$\begin{aligned} |q - q_m| &\leq |q_{0n} - q_{mn}| + C\Lambda^n 1 + C\Lambda_m^n 1 \\ &\leq |q_{0n} - q_{mn}| + C|\Lambda_m^n 1 - \Lambda^n 1| + 2C\Lambda^n 1 \end{aligned}$$

for $n \geq 1$. By Lemma 3, $\Lambda^n 1 \rightarrow 0$ uniformly on each finite interval. Choose $\epsilon > 0$ and $y > 0$ arbitrarily. Fix n arbitrarily such that $(\Lambda^n 1)(x) < \epsilon$ for $0 \leq x \leq y$. Then

$$|q(x) - q_m(x)| \leq \|q_{0n} - q_{mn}\| + C\|\Lambda_m^n 1 - \Lambda^n 1\| + 2C\epsilon, \quad 0 \leq x \leq y.$$

Since it follows from (64) and Theorem 5 that $\|q_{0n} - q_{mn}\| \rightarrow 0$ and $\|\Lambda_m^n 1 - \Lambda^n 1\| \rightarrow 0$, the theorem is proved.

According to (44), $J = (F/2M_2)(x + q)$, where F is an arbitrary positive constant and M_2 is the second moment of K_1 . Correspondingly, $J_m = (F_m/2M_{m2})(x + q_m)$.

Theorem 7. *Suppose that $K_2 \leq CK_1$, $K_{m2} \leq CK_{m1}$, $\| \Lambda f - \Lambda_m f \| \rightarrow 0$ on S , and $F_m \rightarrow F$. Then $J_m \rightarrow J$ uniformly on each finite interval.*

Proof. By (14) and $K_n(0+) = K_n(0)$, one has $M_2 = 2[K_4(0) + K_4(0-)]$ and similarly for M_{m2} . The desired result now follows from Theorems 5 and 6.

It follows from Theorem 4 that the condition $\| \Lambda f - \Lambda_m f \| \rightarrow 0$ on S can be replaced in Theorems 6 and 7 by the conditions that $K_{m_2} \rightarrow K_2$ pointwise and that $K_{m_1} \rightarrow K_1$ uniformly on each compact subset of the complement of a particular finite set. This condition on K_{m_1} is weaker than uniform convergence on each finite interval and stronger than almost uniform convergence on each finite interval. It was designed principally so that the theory would apply with $K_1(t) = \frac{1}{2}E_1(|t|)$, which has a logarithmic singularity at $t = 0$. This is the kernel of the Schwarzschild-Milne equation.

I should like to express my appreciation to Dr. MICHAEL I. AISSSEN of The Johns Hopkins University Radiation Laboratory for many helpful suggestions.

REFERENCES

1. EBERHARD HOPF, *Mathematical Problems of Radiative Equilibrium*, Cambridge University Press, 1934.
2. MICHEL LOÈVE, *Probability Theory*, D. Van Nostrand Company, Inc., 1955.
3. GIAN CARLO WICK, Über ebene diffusionsprobleme, *Zeitschrift für Physik* 121, 702-718 (1943).
4. S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, 1950.
5. MAX KROOK, On the solution of equations of transfer, *Journal of Astrophysics* 122, 488-497 (1955).
6. J. B. SYKES, Approximate integration of the equation of transfer, *Monthly Notices of the Royal Astronomical Society* 111, 337-386 (1951).
7. PHILIP M. ANSELONE, *Convergence of the Wick-Chandrasekhar approximation technique in radiative transfer*, Oregon State College doctoral thesis, 1957.
8. FRANK SPITZER, The Wiener-Hopf equation whose kernel is a probability density, *Duke Mathematical Journal* 24, 327-343 (1957).
9. HAROLD WIDOM, Equations of Wiener-Hopf type (abstract), *Bulletin of the American Mathematical Society* 63, 386 (1957).

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