*On a Non-Linear Integral Equation**

KOMARATH PADMAVALLY

Communicated by A. ZYGMUND

1. In this paper a non-linear integral equation of the form

$$y(t) = \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau$$

is studied. Such an equation arises in connection with the problem of determining the temperature of a semi-infinite rod whose end is heated by a source of (known) variable temperature $\phi(t)$. The rod is assumed to be initially at temperature 0 throughout. The particular case of this problem where $\phi(t) \equiv 1$ has been studied by W. R. MANN & F. WOLF [3] and by J. H. ROBERTS & W. R. MANN [4]. The temperature $U(x, t)$ of the point x of the rod at time t satisfies the equations

$$(1) \quad U_{xx}(x, t) = U_t(x, t), \quad x > 0, \quad t > 0,$$

$$(2) \quad U(x, 0) = 0,$$

$$(3) \quad -U_x(0, t) = g[U(0, t), \phi(t)]$$

where $g(u, v)$ denotes the rate of flow of heat from the source at temperature v to the end of the rod at temperature u .

In the special case where $\phi(t)$ is bounded in $t \geq 0$, a probable hypothesis is the following:

$$(4) \quad |U(x, t)| \leq \text{some constant } M \text{ for } x \geq 0, t \geq 0.$$

As in [3] and [4], the rate of heat diffusion from the source to the end of the rod is assumed to follow a general non-linear law, so that the function $g(u, v)$ satisfies the following conditions, which are clearly direct generalizations of the conditions on g assumed in [3] and [4]:

$$(5) \quad g(u, v) \text{ is continuous in both the variables } u \text{ and } v.$$

*This work was sponsored by the U. S. Air Force, Contract Number AF 18(600)-1139. This paper was submitted to the University of North Carolina as a thesis for the Ph.D. degree. The author wishes to thank Professor W. R. MANN for his help and encouragement during the completion of the work. Her thanks are also due to Professor C. T. RAJAGOPAL and Dr. M. VENKETARAMAN for their help during the earlier stage of this work.

- (6) $g(u, v)$ is a strictly decreasing function of u as well as a strictly increasing function of v .
- (7) $g(u, u) = 0$ for all values of u .

Further, $\phi(t)$ is assumed to satisfy the following condition, which can be easily seen to be satisfied under all physically possible variations of the source temperature:

- (8) The discontinuities of $\phi(t)$ in $t \geq 0$ form a discrete set, and $\phi(t)$ is bounded in any finite interval of t .

By the same method as in [3] it can be shown that any function $U(x, t)$ satisfying (1), (2), (3) and (4) satisfies the integral equation

$$(e) \quad U(x, t) = \int_0^t \frac{g[U(0, \tau), \phi(\tau)]}{\sqrt{\pi(t - \tau)}} e^{-x^2/4(t - \tau)} d\tau;$$

if, further, $U(x, t)$ is continuous in $x \geq 0, t \geq 0$, then

$$(f) \quad U(0, t) = \int_0^t \frac{g[U(0, \tau), \phi(\tau)]}{\sqrt{\pi(t - \tau)}} d\tau;$$

and if $U(0, t)$ satisfies equation (f), the corresponding function $U(x, t)$ given by (e) satisfies (1), (2) and (3). Also in Theorem H, §3 of this paper it will be shown that if $\phi(t)$ is bounded in $t \geq 0$, then the solution $U(0, t)$ of equation (f), as well as the corresponding solution $U(x, t)$ of equation (e), satisfies (4), thereby showing that for any given function $\phi(t)$ which is bounded in $t \geq 0$, the set of equations (1), (2), (3) and (4) is equivalent to equations (e) and (f).

In [3] it is shown that for $\phi(t) \equiv 1$, equation (f), and therefore (e) also, has a solution lying between 0 and 1 throughout; further, under the additional hypothesis that $g(u, 1) = g(u)$ satisfies a Lipschitz condition in $0 \leq u \leq 1$ it is shown that the solution $U(0, t)$ of (f) is unique, non-decreasing, and approaches 1 as $t \rightarrow \infty$. In [4] the solution $U(0, t)$ of (f) is shown to be unique and strictly increasing and to approach 1 as $t \rightarrow \infty$ without the assumption of any Lipschitz condition on $g(u)$ (again for the special case $\phi(t) \equiv 1$).

The present paper gives generalizations of some of these results to the case where $\phi(t)$ is variable, subject only to (8). In §3 the existence and uniqueness of a solution of equation (f), and hence of (e), is proved. In §4 it is shown that if $\phi(t)$ is non-negative and non-decreasing, then the solution $U(0, t)$ of equation (f) is non-decreasing and lies between 0 and $\phi(t)$ for all $t \geq 0$. Also if $\lim_{t \rightarrow \infty} \phi(t) = l$, then $\lim_{t \rightarrow \infty} U(0, t)$ also exists and equals l . It is further shown that $U(x, t)$ is a non-decreasing functional of $\phi(t)$, i.e. $\phi_1(t) \leq \phi_2(t)$ for all $t \geq 0$ implies that the corresponding solutions $U_1(x, t), U_2(x, t)$ of equation (e) (for $\phi(t) \equiv \phi_1(t), \phi(t) \equiv \phi_2(t)$ respectively) satisfy the relation $U_1(x, t) \leq U_2(x, t)$ for $x \geq 0, t \geq 0$.

2. This section deals with certain lemmas which are made use of in §3.

Lemma 1. If $\theta(t) \geq 0$, bounded and Riemann-integrable in $0 \leq t \leq T$, if $L(t) \leq 0$, bounded and L -integrable in $0 \leq t \leq T$, and if

$$z(t) = \int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t-\tau)}} d\tau + \int_0^t \frac{L(\tau)}{\sqrt{\pi(t-\tau)}} z(\tau) d\tau$$

then for $0 \leq t \leq T$ we have

$$0 \leq z(t) \leq \int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t-\tau)}} d\tau.$$

Proof. Consider any equation of the form

$$z(t) = f(t) + \int_0^t \frac{L(\tau)z(\tau)}{\sqrt{\pi(t-\tau)}} d\tau$$

where $L(t)$ satisfies the same conditions as mentioned above, and $f(t)$ is such that

$$\int_0^t \frac{|f(\tau)|}{\sqrt{t-\tau}} d\tau$$

exists for all $t \geq 0$.

It can be easily verified that a unique solution $z(t)$ exists for $0 \leq t \leq T$, and that this can be obtained by iteration, *i.e.*

$$\begin{aligned} z(t) = f(t) + \int_0^t \frac{L(\tau)f(\tau)}{\sqrt{\pi(t-\tau)}} d\tau \\ + \int_0^t \frac{L(\tau_1)}{\sqrt{\pi(t-\tau_1)}} d\tau_1 \int_0^{\tau_1} \frac{L(\tau_2)}{\sqrt{\pi(\tau_1-\tau_2)}} f(\tau_2) d\tau_2 + \cdots \end{aligned}$$

The following results, which can be easily verified, are made use of in proving Lemma 1:

(c) If $f_n(t)$, $n = 1, 2, \dots$ are functions which converge to $f(t)$ uniformly in $0 \leq t \leq T$,

$$\int_0^t \frac{|f_n(\tau)|}{\sqrt{t-\tau}} d\tau \quad \text{and} \quad \int_0^t \frac{|f(\tau)|}{\sqrt{t-\tau}} d\tau$$

exist for $0 \leq t \leq T$ and $n = 1, 2, \dots$, $L(\tau)$ satisfies the same conditions as above, and

$$Z_n(t) = f_n(t) + \int_0^t \frac{L(\tau)Z_n(\tau)}{\sqrt{\pi(t-\tau)}} d\tau,$$

then $Z_n(t) \rightarrow z(t)$ uniformly in $0 \leq t \leq T$.

(d) If $L_n(t)$, $n = 1, 2, \dots$, is a sequence of functions which converges in measure [2, p. 91] to $L(t)$ in $0 \leq t \leq T$, $L_n(t)$, $n = 1, 2, \dots$, is uniformly bounded

in $0 \leq t \leq T$, $L(t)$, as well as each $L_n(t)$, $n \geq 1$, is L -integrable in $(0, T)$, $f(t)$ satisfies the same conditions as above, and we have

$$Z_n(t) = f(t) + \int_0^t \frac{L_n(\tau)Z_n(\tau)}{\sqrt{\pi(t-\tau)}} d\tau, \quad n \geq 1,$$

then $Z_n(t) \rightarrow z(t)$ uniformly in $0 \leq t \leq T$.

Using (c) the proof of Lemma 1 can be reduced to proving the following result:

Lemma 1a. *If*

$$(F) \quad Z(t) = \frac{1}{\sqrt{\pi(t+a)}} + \int_0^t \frac{L(\tau)Z(\tau)}{\sqrt{\pi(t-\tau)}} d\tau$$

where a is a constant which is ≥ 0 and $L(t)$ satisfies the hypothesis of Lemma 1, then

$$(g) \quad 0 \leq Z(t) \leq \frac{1}{\sqrt{\pi(t+a)}}, \quad 0 \leq t \leq T.$$

For let

$$f(t) = \int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t-\tau)}} d\tau.$$

For each $n \geq 1$ let $0 = \tau_0 < \tau_1 < \dots < \tau_n = T$ be a subdivision of $(0, T)$ whose norm $\rightarrow 0$ as $n \rightarrow \infty$. For $t \in (0, T)$ let $n(t)$ denote the largest value of i for which $\tau_i < t$, and let

$$f_n(t) = \sum_{i=0}^{n(t)-1} \frac{\theta(\tau_i)}{\sqrt{\pi(t-\tau_i)}} (\tau_{i+1} - \tau_i), \quad 0 \leq t \leq T.$$

Since $\theta(t)$ is bounded and R -integrable in $(0, T)$, $f(t)$ is a Riemann integral and we have $\lim_{n \rightarrow \infty} f_n(t) = f(t)$, $0 \leq t \leq T$. This convergence of $f_n(t)$ to $f(t)$ in $(0, T)$ can be shown to be uniform (by using the fact that $f(t)$ is continuous in $(0, T)$, while, for any given t in $(0, T)$, $|f_n(t + \Delta t) - f_n(t)|$ can be made arbitrarily small by choosing Δt sufficiently small and n sufficiently large).

Let

$$z_n(t) = f_n(t) + \int_0^t \frac{L(\tau)z_n(\tau)}{\sqrt{\pi(t-\tau)}} d\tau.$$

By (c), $\lim_{n \rightarrow \infty} z_n(t) = z(t)$, $0 \leq t \leq T$, so that it is sufficient to show that $0 \leq z_n(t) \leq f_n(t)$, $0 \leq t \leq T$, $n = 1, 2, \dots$.

This latter result is a direct consequence of Lemma 1a. For consider any given $n \geq 1$. Let

$$\mu_i(t) = \begin{cases} \frac{1}{\sqrt{\pi(t-\tau_i)}}, & t \geq \tau_{i+1}, \\ 0, & t < \tau_{i+1}, \end{cases} \quad i = 0, 1, \dots, n-1.$$

Then

$$f_n(t) = \sum_{i=0}^{n(t)-1} \mu_i(t) \theta(\tau_i) (\tau_{i+1} - \tau_i)$$

and it can be easily verified that if

$$Z_i(t) = \mu_i(t) + \int_0^t \frac{L(\tau) Z_i(\tau)}{\sqrt{\pi(t-\tau)}} d\tau, \quad i = 0, 1, \dots,$$

then

$$z_n(t) = \sum_{i=0}^{n(t)-1} Z_i(t) \theta(\tau_i) (\tau_{i+1} - \tau_i).$$

But by Lemma 1a (setting $a = \tau_{i+1} - \tau_i$ and taking $t^1 = t - \tau_{i+1}$ as the variable, and noting that

$$Z_i(t^1) = \frac{1}{\sqrt{\pi(t^1 + a)}} + \int_0^{t^1} \frac{L(\tau + \tau_{i+1}) Z_i(\tau)}{\sqrt{\pi(t^1 - \tau)}} d\tau,$$

$Z_i(t) = 0$ for $t \leq \tau_{i+1}$) we have $0 \leq Z_i(t) \leq \mu_i(t)$, $0 \leq t \leq T$, $i = 0, 1, \dots$. Therefore, since by hypothesis $\theta(\tau_i) \geq 0$ in $0 \leq \tau_i \leq T$, it follows that $0 \leq z_n(t) \leq f_n(t)$, $0 \leq t \leq T$, $n \geq 1$. This proves Lemma 1.

It remains to establish Lemma 1a. Using (d), the proof of Lemma 1a can be reduced to proving the following result:

Lemma 1b. *Lemma 1a holds under the following additional restriction:*

$$(d^1) \quad L(t) \text{ is analytic in } 0 \leq t \leq T.$$

For suppose that Lemma 1b is true. To prove Lemma 1a it is sufficient, in view of (d), to show that the given function $L(t)$ can be approximated by a sequence of polynomials $L_n(t)$, $n = 1, 2, \dots$, which are uniformly bounded in $(0, T)$, converge in measure to $L(t)$ in $(0, T)$, and are such that each $L_n(t)$ satisfies the hypothesis of Lemma 1. Such a sequence can be constructed with the help of Weierstrass's approximation theorem [1, p. 65] (i.e. by first constructing a sequence of uniformly bounded continuous functions which converges in measure to $L(t)$ in $(0, T)$, each member of the sequence having an upper bound < 0 , and then uniformly approximating each of these continuous functions in $(0, T)$ by polynomials).

Proof of Lemma 1b: It suffices to prove the first half of (9) since the second half follows from the first when we use the hypothesis $L(t) \leq 0$, $0 \leq t \leq T$, in equation (F). From equation (F) it is clear that $Z(t)$ has the same sign as $1/\sqrt{t+a}$, i.e. is positive, for all sufficiently small $t > 0$. Also since $L(t)$ is analytic in $0 \leq t \leq T$ it can be verified that $Z(t)$ is continuous and has no singularities in $0 < t \leq T$. Hence we have either

$$(10) \quad Z(t) \geq 0 \text{ for } 0 < t \leq T,$$

or

$$(11) \quad \begin{aligned} Z(t) &\geq 0 \text{ for } 0 < t < t_0, Z(t_0) = 0, \text{ and} \\ Z(t) &< 0 \text{ for } t_0 < t \leq t_1, \text{ for some } t_0, t_1 \text{ such that } 0 < t_0 < t_1 < T. \end{aligned}$$

We prove (9) by showing that (11) leads to the following contradiction and so has to be ruled out:

$$\begin{aligned} 0 > Z(t_1) &= \frac{1}{\sqrt{\pi(t_1 + a)}} + \int_0^{t_0} \frac{L(\tau)Z(\tau)}{\sqrt{\pi(t_1 - \tau)}} d\tau + \int_{t_0}^{t_1} \frac{L(\tau)Z(\tau)}{\sqrt{\pi(t_1 - \tau)}} d\tau \\ &\geq \frac{1}{\sqrt{\pi(t_1 + a)}} + \int_0^{t_0} \frac{L(\tau)Z(\tau)}{\sqrt{\pi(t_1 - \tau)}} d\tau \end{aligned}$$

since $L(\tau)Z(\tau) \geq 0$ for $t_0 < \tau < t_1$. Hence

$$\begin{aligned} 0 > Z(t_1) &\geq \left(\frac{t_0 + a}{t_1 + a}\right)^{\frac{1}{2}} \left\{ \frac{1}{\sqrt{\pi(t_0 + a)}} + \int_0^{t_0} \sqrt{\frac{t_1 + a}{t_0 + a}} \cdot \frac{L(\tau)Z(\tau)}{\sqrt{\pi(t_1 - \tau)}} d\tau \right\} \\ &\geq \left(\frac{t_0 + a}{t_1 + a}\right)^{\frac{1}{2}} \left\{ \frac{1}{\sqrt{\pi(t_0 + a)}} + \int_0^{t_0} \frac{L(\tau)Z(\tau)}{\sqrt{\pi(t_0 - \tau)}} d\tau \right\} \equiv \left(\frac{t_0 + a}{t_1 + a}\right)^{\frac{1}{2}} Z(t_0) = 0, \end{aligned}$$

giving $0 > 0$.

The inequality in the last step is justified on account of the fact that, for $0 < \tau < t_0$ and $a \geq 0$,

$$\left(\frac{t_1 + a}{t_0 + a}\right)^{\frac{1}{2}} < \left(\frac{t_1 - \tau}{t_0 - \tau}\right)^{\frac{1}{2}},$$

and $L(\tau)Z(\tau) \leq 0$ for $0 < \tau < t_0$. This completes the proof of Lemma 1b and hence of Lemma 1.

Lemma 2. *In Lemma 1 the hypothesis $\theta(t) \geq 0$, bounded and R-integrable in $0 \leq t \leq T$ can be replaced by the weaker condition $\theta(t) \geq 0$, bounded and L-integrable in $0 \leq t \leq T$ (other conditions remaining unchanged).*

Proof. This is a direct consequence of (c). For the given function $\theta(t)$, satisfying the weaker hypothesis given above, can evidently be approximated by a sequence of continuous non-negative functions $\theta_n(t)$, $n = 1, 2, \dots$, which converges in measure to $\theta(t)$ in $0 \leq t \leq T$. The corresponding sequence

$$\int_0^t \frac{\theta_n(\tau)}{\sqrt{\pi(t - \tau)}} d\tau, \quad n = 1, 2, \dots,$$

converges uniformly in $(0, T)$ to

$$\int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t - \tau)}} d\tau.$$

Hence if

$$Z_n(t) = \int_0^t \frac{\theta_n(\tau)}{\sqrt{\pi(t-\tau)}} d\tau + \int_0^t \frac{L(\tau)Z_n(\tau)}{\sqrt{\pi(t-\tau)}} d\tau,$$

then $\lim_{n \rightarrow \infty} Z_n(t) = z(t)$ in $(0, T)$, while by Lemma 1

$$0 \leq Z_n(t) \leq \int_0^t \frac{\theta_n(\tau)}{\sqrt{\pi(t-\tau)}} d\tau, \quad n = 1, 2, \dots$$

Corollary 1. *If in Lemma 2 the hypothesis $\theta(t) \geq 0$ in $(0, T)$ is omitted (other conditions remaining unchanged) and*

$$z(t) = \int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t-\tau)}} d\tau + \int_0^t \frac{L(\tau)z(\tau)}{\sqrt{\pi(t-\tau)}} d\tau$$

then we have

$$|z(t)| \leq \int_0^t \frac{|\theta(\tau)|}{\sqrt{\pi(t-\tau)}} d\tau, \quad 0 \leq t \leq T.$$

Proof. It is clearly sufficient, in view of (c), to consider the case where $\theta(t)$ is continuous in $(0, T)$ (since otherwise

$$\int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t-\tau)}} d\tau$$

can be uniformly approximated in $(0, T)$ by a sequence

$$\int_0^t \frac{\theta_n(\tau)}{\sqrt{\pi(t-\tau)}} d\tau, \quad n \geq 1,$$

where $\theta_n(t)$ is continuous in $(0, T)$).

For the special case where $\theta(t)$ is continuous in $(0, T)$, the result can be deduced from Lemma 1 by writing $\theta(t) = \theta_1(t) + \theta_2(t)$ where

$$\theta_1(t) = \begin{cases} \theta(t) & \text{whenever } \theta(t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\theta_2(t) = \begin{cases} \theta(t) & \text{whenever } \theta(t) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then if

$$Z_1(t) = \int_0^t \frac{\theta_1(\tau)}{\sqrt{\pi(t-\tau)}} d\tau + \int_0^t \frac{L(\tau)Z_1(\tau)}{\sqrt{\pi(t-\tau)}} d\tau$$

and

$$Z_2(t) = \int_0^t \frac{\theta_2(\tau)}{\sqrt{\pi(t-\tau)}} d\tau + \int_0^t \frac{L(\tau)Z_2(\tau)}{\sqrt{\pi(t-\tau)}} d\tau,$$

$Z(t)$ is evidently equal to $Z_1(t) + Z_2(t)$, while, by Lemma 1, for $0 \leq t \leq T$ we have

$$|Z_1(t)| < \int_0^t \frac{\theta_1(\tau)}{\sqrt{\pi(t-\tau)}} d\tau$$

and

$$|Z_2(t)| < - \int_0^t \frac{\theta_2(\tau)}{\sqrt{\pi(t-\tau)}} d\tau.$$

Hence

$$\begin{aligned} |Z(t)| &\leq |Z_1(t)| + |Z_2(t)| \leq \int_0^t \frac{\theta_1(\tau) - \theta_2(\tau)}{\sqrt{\pi(t-\tau)}} d\tau \\ &= \int_0^t \frac{|\theta(\tau)|}{\sqrt{\pi(t-\tau)}} d\tau, \quad 0 \leq t \leq T. \end{aligned}$$

Corollary 2. *If*

$$Z(t) = \int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t+a-\tau)}} d\tau + \int_0^t \frac{L(\tau)Z(\tau)}{\sqrt{\pi(t-\tau)}} d\tau$$

where $L(t)$, $\theta(t)$ satisfy the same conditions as in Lemma 2 and $a \geq 0$, then

$$0 \leq Z(t) \leq \int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t+a-\tau)}} d\tau, \quad 0 \leq t \leq T.$$

The proof proceeds on the same lines as that of Lemma 1 and Lemma 2.

3. In this section an equation similar to (f) is investigated. Writing $g[u, \phi(t)] = G(u, t)$ and $U(0, t) = y(t)$, equation (f) becomes

$$(E) \quad y(t) = \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau.$$

Theorems I–IV below relate to equation (E) under conditions (A), (B), (C) and (D) given below. It can be easily verified that these conditions are implied by, but do not imply, the relation $G(u, t) = g[u, \phi(t)]$ where $g(u, v)$ satisfies (5), (6) and (7) and $\phi(t)$ satisfies (8). Thus equation (E) is slightly more general than equation (f).

Let $G(u, t)$ be any function defined in $-\infty < u < \infty$, $0 \leq t < \infty$, which satisfies the following conditions:

(A) $G(u, t)$ is L -integrable w.r.t. t in any finite interval of the latter variable.

This, together with condition (C) below, implies that if $Z(t)$ is any continuous function of t , then $G[Z(t), t]$ is integrable in any finite interval of t .

(B) $G(u, t)$ is non-increasing w.r.t. u .

- (C) $G(u, t)$ is bounded and uniformly continuous w.r.t. u in any finite rectangle of the form $u_1 \leq u \leq u_2, 0 \leq t \leq t_0$.
- (D) For each $t \geq 0$ there exists a finite value $u(t)$ of u such that $G[u(t), t] = 0$, $G(u, t) > 0$ for $u < u(t)$, and $u(t)$ is bounded in any finite interval of t .

Theorem I. *The equation*

$$(E) \quad y(t) = \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau$$

has a unique solution $y(t)$ for all $t \geq 0$.

Theorem II. *The solution $y(t)$ of equation (E) is a non-decreasing functional of $G(u, t)$, i.e. if $G_1(u, t), G_2(u, t)$ both satisfy (A), (B), (C) and (D), if*

$$G_1(u, t) \leq G_2(u, t) \quad \text{for} \quad -\infty < u < \infty, \quad 0 \leq t < \infty,$$

and if

$$y_1(t) = \int_0^t \frac{G_1[y_1(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau, \quad y_2(t) = \int_0^t \frac{G_2[y_2(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau,$$

then $y_1(t) \leq y_2(t)$ for $t \geq 0$.

Theorem III. *If in any interval of the form $0 \leq t \leq T$, we have $m \leq u(t) \leq M$ where $m \leq 0$ and $M \geq 0$, then the solution $y(t)$ of equation (E) also lies between m and M for $0 \leq t \leq T$.*

Theorem IV. *If $G(u, t)$ is a non-decreasing function of t , and $G(0, 0) = \lim_{t \rightarrow 0} G(u, t)|_{u=0}$ exists and is > 0 , the solution $y(t)$ of equation (E) is also non-decreasing w.r.t. t .*

These results are proved below first for the special case where the function $G(u, t)$ satisfies the following additional condition:

$$(D^1) \quad \left| \frac{G(u_1, t) - G(u_2, t)}{u_1 - u_2} \right| \leq \text{a constant } L$$

$$\text{for} \quad -\infty < u_1 < u_2 < +\infty, \quad 0 \leq t < +\infty.$$

Later they are extended to the class $\{\gamma\}$ of all functions $G(u, t)$ which can be uniformly approximated in any finite rectangle of the u - t plane by functions of the above-mentioned type, i.e. satisfying (A), (B), (C), (D) and (D¹). Then using condition (D) it is shown that the class $\{\gamma\}$ contains all functions $G(u, t)$ which satisfy (A), (B), (C) and (D).

Let $G(u, t)$ satisfy (A), (B), (C), (D) and (D¹). Then Theorem I can be proved by iteration as follows: Let

$$Y_0(t) = \int_0^t \frac{G(0, \tau)}{\sqrt{\pi(t-\tau)}} d\tau,$$

and let

$$\begin{aligned}
 Y_n(t) &= \int_0^t \frac{G[Y_{n-1}(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau, \quad n = 1, 2, \dots \\
 |Y_{n+1}(t) - Y_n(t)| &= \left| \int_0^t \frac{G[Y_n(\tau), \tau] - G[Y_{n-1}(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau \right| \\
 &\leq \int_0^t \frac{|G[Y_n(\tau), \tau] - G[Y_{n-1}(\tau), \tau]|}{|Y_n(\tau) - Y_{n-1}(\tau)|} \cdot \frac{|Y_n(\tau) - Y_{n-1}(\tau)|}{\sqrt{\pi(t-\tau)}} d\tau \\
 &\leq L \int_0^t \frac{|Y_n(\tau) - Y_{n-1}(\tau)|}{\sqrt{\pi(t-\tau)}} d\tau.
 \end{aligned}$$

Replacing n successively by $n-1, n-2, \dots$ in the above, it follows that

$$\begin{aligned}
 |Y_{n+1}(t) - Y_n(t)| &\leq L^2 \int_0^t \frac{d\tau_1}{\sqrt{\pi(t-\tau_1)}} \int_0^{\tau_1} \frac{d\tau_2}{\sqrt{\pi(\tau_1-\tau_2)}} \\
 &\quad \cdot |Y_{n-1}(\tau_2) - Y_{n-2}(\tau_2)| \leq \dots \\
 &\leq L^n \int_0^t \frac{d\tau_1}{\sqrt{\pi(t-\tau_1)}} \int_0^{\tau_1} \frac{d\tau_2}{\sqrt{\pi(\tau_1-\tau_2)}} \dots \int_0^{\tau_{n-1}} \frac{d\tau_n}{\sqrt{\pi(\tau_{n-1}-\tau_n)}} \\
 &\quad \cdot |Y_1(\tau_n) - Y_0(\tau_n)|.
 \end{aligned}$$

If $M(t)$ denotes $\text{Max}_{0 \leq \tau \leq t} |Y_1(\tau) - Y_0(\tau)|$, it follows that

$$|Y_{n+1}(t) - Y_n(t)| \leq \frac{L^n}{\sqrt{\pi}} M(t) \frac{t^{\frac{1}{2}m} \Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2}m + 1)}.$$

Hence

$$\begin{aligned}
 |Y_{m+n}(t) - Y_m(t)| &\leq \sum_{i=0}^{n-1} |Y_{m+i+1}(t) - Y_{m+i}(t)| \\
 &\leq \frac{M(t) \Gamma(\frac{3}{2})}{\sqrt{\pi}} \cdot \sum_{i=0}^{n-1} \frac{L^{m+i} t^{\frac{1}{2}(m+i)}}{\Gamma(\frac{1}{2}(m+i) + 1)}
 \end{aligned}$$

The functions $Y_0(t)$, $Y_1(t)$, and hence $M(t)$, are clearly bounded in any finite interval J of t of the form $0 \leq t \leq T$. It follows that the series $\{Y_n(t)\}$, $n = 1, 2, \dots$, is uniformly bounded and converges uniformly in J . Let $Y(t) = \lim_{n \rightarrow \infty} Y_n(t)$, $t \in J$. Then by (H), in the finite rectangle $|u| \leq \text{Max}_{t \in J, n \geq 0} |Y_n(t)|$, $0 \leq t \leq T$, the sequence $G[Y_n(t), t]$, $n = 0, 1, \dots$, converges uniformly to $G[Y(t), t]$. Hence, for $t \in J$,

$$\lim_{n \rightarrow \infty} \int_0^t \frac{G[Y_n(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau = \int_0^t \frac{G[Y(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau,$$

i.e.

$$\lim_{n \rightarrow \infty} Y_{n+1}(t) = Y(t) = \int_0^t \frac{G[Y(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau.$$

This proves the existence of a solution $Y(t)$ in any finite interval $0 \leq t \leq T$ of t .

The uniqueness of the solution follows by the same arguments as given in the proof of Theorem 1 of [4].

Theorem II follows by applying Lemma 2.

Let $G_1(u, t)$, $G_2(u, t)$ be any two functions, each satisfying (A), (B), (C), (D) and (D¹), such that $G_2(u, t) \leq G_1(u, t)$ in $-\infty < u < \infty$, $0 \leq t < \infty$, and let

$$y_1(t) = \int_0^t \frac{G_1[y_1(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau, \quad y_2(t) = \int_0^t \frac{G_2[y_2(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau$$

for all $t \geq 0$. Then

$$\begin{aligned} y_1(t) - y_2(t) &= \int_0^t \frac{G_1[y_1(\tau), \tau] - G_2[y_2(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau \\ &= \int_0^t \frac{G_1[y_1(\tau), \tau] - G_2[y_1(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau \\ &\quad + \int_0^t \frac{G_2[y_1(\tau), \tau] - G_2[y_2(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau \\ &= \int_0^t \frac{G_1[y_1(\tau), \tau] - G_2[y_1(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau \\ &\quad + \int_0^t \frac{G_2[y_1(\tau), \tau] - G_2[y_2(\tau), \tau]}{y_1(\tau) - y_2(\tau)} \cdot \frac{y_1(\tau) - y_2(\tau)}{\sqrt{\pi(t-\tau)}} d\tau. \end{aligned}$$

It follows from Lemma 2 that $y_1(t) - y_2(t) \geq 0$ for all $t \geq 0$. For, by hypothesis, $G_1[y_1(\tau), \tau] - G_2[y_1(\tau), \tau] \geq 0$, for all $\tau \geq 0$, and is bounded and L -integrable w.r.t. τ in $(0, t)$ for all $t > 0$. The second integral on the right can evidently be written as

$$\int_0^t \frac{M(\tau)}{\sqrt{\pi(t-\tau)}} [y_1(\tau) - y_2(\tau)] d\tau$$

where

$$M(\tau) = \begin{cases} \frac{G_2[y_1(\tau), \tau] - G_2[y_2(\tau), \tau]}{y_1(\tau) - y_2(\tau)} & \text{whenever } y_1(\tau) \neq y_2(\tau) \\ 0 & \text{whenever } y_1(\tau) = y_2(\tau). \end{cases}$$

By conditions (B) and (D¹) on $G_2(u, t)$, $M(\tau)$ is bounded and ≤ 0 in $\tau \geq 0$. Also $M(\tau)$ is L -integrable in $0 \leq \tau \leq t$. For since the function $y_1(\tau) - y_2(\tau)$ is continuous in $\tau \geq 0$, the set of points in $0 \leq \tau \leq t$ at which $y_1(\tau) = y_2(\tau)$ is a closed set, therefore its complement S in $(0, t)$ is an open set. Consider any open interval I contained in S . $[y_1(\tau) - y_2(\tau)]^{-1}$ is continuous in I . Therefore, from condition (A) on $G_2(u, t)$ and the fact that $M(\tau)$ is bounded in I , it follows

that

$$M(\tau) = \frac{G_2[y_1(\tau), \tau] - G_2[y_2(\tau), \tau]}{y_1(\tau) - y_2(\tau)}$$

is L -integrable in I . Therefore, since $M(\tau)$ is bounded in S , it follows that $M(\tau)$ is L -integrable in the open set S (S being the union of open intervals I). Hence $M(\tau)$ is L -integrable in $(0, t)$ ($M(\tau)$ being equal to 0 outside S). Hence all the conditions of Lemma 1 are fulfilled, and we have

$$y_1(t) - y_2(t) = \int_0^t \frac{G_1[y_1(\tau), \tau] - G_2[y_1(\tau), \tau]}{\sqrt{\pi(t - \tau)}} d\tau + \int_0^t \frac{M(\tau)}{\sqrt{\pi(t - \tau)}} [y_1(\tau) - y_2(\tau)] d\tau$$

is ≥ 0 for all $t \geq 0$.

The following result is also an immediate consequence of the above arguments:

Corollary 3. *If $G_1(u, t)$, $G_2(u, t)$ both satisfy (A), (B), (C), (D) and (D¹), $G_2(u, t) \leq G_1(u, t)$ for $0 \leq t \leq T$, and*

$$y_1(t) = \int_0^t \frac{G_1[y_1(\tau), \tau]}{\sqrt{\pi(t - \tau)}} d\tau, \quad y_2(t) = \int_0^t \frac{G_2[y_2(\tau), \tau]}{\sqrt{\pi(t - \tau)}} d\tau,$$

then $y_2(t) \leq y_1(t)$ for $0 \leq t \leq T$.

Theorem III follows by an application of Theorem II and of Theorem 4 of [3].

Consider any function $G(u, t)$ which satisfies (A), (B), (C), (D) and (D¹) and for which $u(t) \leq M$ for $0 \leq t \leq T$ where $M \geq 0$ and T is finite. Let

$$y(t) = \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t - \tau)}} d\tau, \quad t \geq 0.$$

Then $y(t) \leq M$ for $0 \leq t \leq T$. For let $H(u) = \text{l.u.b.}_{0 \leq t \leq T} G(u, t)$. $H(u)$ exists for all u since, for any given u , $G(u, t)$ is bounded in the finite interval $0 \leq t \leq T$. $H(u)$ is continuous and non-increasing w.r.t. u and satisfies condition (D¹). For let $u > u^1$. By (B) we have, for $0 \leq t \leq T$, $G(u^1, t) \geq G(u, t)$. Hence $\text{l.u.b.}_{0 \leq t \leq T} G(u^1, t) \geq \text{l.u.b.}_{0 \leq t \leq T} G(u, t)$, i.e. $H(u^1) \geq H(u)$.

By condition (D¹) on $G(u, t)$, $|G(u^1 + \Delta u, t) - G(u^1, t)| / |\Delta u|$ is bounded in $-\infty < u < +\infty$, $0 \leq t \leq T$, $\Delta u \geq 0$. Hence

$$\begin{aligned} \text{l.u.b.}_{0 \leq t \leq T} \left| \frac{G(u^1 + \Delta u, t) - G(u^1, t)}{\Delta u} \right| &\geq \left| \frac{\text{l.u.b.}_{0 \leq t \leq T} G(u^1 + \Delta u, t) - \text{l.u.b.}_{0 \leq t \leq T} G(u^1, t)}{\Delta u} \right| \\ &= \left| \frac{H(u^1 + \Delta u) - H(u^1)}{\Delta u} \right|. \end{aligned}$$

Hence it follows that $H(u)$ satisfies a Lipschitz condition w.r.t. u in $-\infty < u < +\infty$ and is therefore continuous in u .

Let $h = 1.u.b._{0 \leq t \leq T} u(t)$. Then $G(h, t) \leq G(u(t), t) = 0$ for $0 \leq t \leq T$, so that $H(h) = 1.u.b._{0 \leq t \leq T} G(h, t) \leq 0$. Also if $u^1 < h$, then there exists t^1 in $(0, T)$ such that $u(t^1) > u^1$ and hence $G(u^1, t^1) > 0$. Hence $1.u.b._{0 \leq t \leq T} G(u^1, t) = H(u^1) \geq G(u^1, t^1) > 0$, i.e. $H(u^1) > 0$ for all $u^1 < h$. Therefore, since $H(u)$ is continuous in u , $H(h) = 0$. Also h is clearly finite. Thus the function $H(u) = H(u, t)$ satisfies (A), (B), (C), (D) and (D¹), with the corresponding function $u(t) = h$. Hence by the special case of Theorem I proved above, the equation

$$z(t) = \int_0^t \frac{H[z(\tau)]}{\sqrt{\pi(t-\tau)}} d\tau$$

has a unique solution $z(t)$ for $t \geq 0$.

Also since $H(u) = 1.u.b._{0 \leq t \leq T} G(u, t) \geq G(u, t)$ for $0 \leq t \leq T$, therefore, by Corollary 2 above, $z(t) \geq y(t)$ for $0 \leq t \leq T$.

By the same arguments as used in the proof of Theorem 4 of [3] (where the hypothesis $G(u)$ is a strictly decreasing function of u is not made use of) it will follow that, for $0 \leq t \leq T$, we have either $0 \leq z(t) \leq h$ or $0 \geq z(t) \geq h$ according as $h \geq 0$ or ≤ 0 . In either case, $y(t) \leq z(t) \leq \text{Max}(0, h)$ for $0 \leq t \leq T$. But by hypothesis, $M \geq 0$ and $M \geq u(t)$, $0 \leq t \leq T$, so that $M \geq h = 1.u.b._{0 \leq t \leq T} u(t)$. Hence $M \geq \text{Max}(0, h) \geq y(t)$ for $0 \leq t \leq T$.

An exactly similar proof holds for the result $m \leq y(t)$ for $0 \leq t \leq T$. This proves Theorem III under the additional condition (D¹).

Remark. It may be noted that Theorem III implies the following: $|y(t)| \leq \text{Max}_{0 \leq t \leq T} |u(t)|$ in $0 \leq t \leq T$.

Theorem IV (under the additional condition (D¹)) is a direct consequence of Lemma 2 and Corollary 2.

Let $G(u, t)$ satisfy (A), (B), (C), (D) and (D¹) and let

$$\begin{aligned} y(t) &= \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau, \quad t \geq 0. \\ y(t + \Delta t) - y(t) &= \int_0^{t+\Delta t} \frac{G[y(\tau), \tau]}{\sqrt{\pi(t+\Delta t-\tau)}} d\tau - \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau \\ &= \int_0^{\Delta t} \frac{G[y(\tau), \tau]}{\sqrt{\pi(t+\Delta t-\tau)}} d\tau + \int_{\Delta t}^{t+\Delta t} \frac{G[y(\tau), \tau]}{\sqrt{\pi(t+\Delta t-\tau)}} d\tau \\ &\quad - \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau \\ &= \int_0^{\Delta t} \frac{G[y(\tau), \tau]}{\sqrt{\pi(t+\Delta t-\tau)}} d\tau + \int_0^t \frac{G[y(\tau + \Delta t), \tau + \Delta t]}{\sqrt{\pi(t-\tau)}} d\tau \\ &\quad - \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t-\tau)}} d\tau. \end{aligned}$$

Let

$$\theta(\tau) = \begin{cases} G[y(\tau), \tau] & \text{for } 0 < \tau < \Delta t \\ 0 & \text{for } \tau \geq \Delta t \end{cases}$$

and

$$M^1(\tau) = \begin{cases} \frac{G[y(\tau + \Delta t), \tau] - G[y(\tau), \tau]}{y(\tau + \Delta t) - y(\tau)} & \text{whenever } y(\tau + \Delta t) \neq y(\tau) \\ 0 & \text{whenever } y(\tau + \Delta t) = y(\tau). \end{cases}$$

Then, as in the proof of the special case of Theorem II,

$$\begin{aligned} y(t + \Delta t) - y(t) &= \int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t + \Delta t - \tau)}} d\tau \\ &+ \int_0^t \frac{G[y(\tau + \Delta t), \tau + \Delta t] - G[y(\tau + \Delta t), \tau]}{\sqrt{\pi(t - \tau)}} d\tau \\ &+ \int_0^t \frac{G[y(\tau + \Delta t), \tau] - G[y(\tau), \tau]}{\sqrt{\pi(t - \tau)}} d\tau = \int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t + \Delta t - \tau)}} d\tau \\ &+ \int_0^t \frac{G[y(\tau + \Delta t), \tau + \Delta t] - G[y(\tau + \Delta t), \tau]}{\sqrt{\pi(t - \tau)}} d\tau \\ &+ \int_0^t \frac{M^1(\tau)[y(\tau + \Delta t) - y(\tau)]}{\sqrt{\pi(t - \tau)}} d\tau. \end{aligned}$$

Since $G(u, t)$ is a non-decreasing function of t , $G[y(\tau + \Delta t), \tau + \Delta t] - G[y(\tau + \Delta t), \tau] \geq 0$ for $\tau \geq 0$, $\Delta t \geq 0$.

Also this function is bounded and L -integrable (by (A)) in any finite interval of τ .

Since by hypothesis $\lim_{t \rightarrow 0, u \rightarrow 0} G(u, t) = G(0, 0)$ exists and is > 0 , therefore, by choosing Δt sufficiently small, we can make $\theta(\tau) \geq 0$ for all $\tau \geq 0$, and $\theta(\tau)$ is L -integrable and bounded in any finite interval of τ .

Hence, writing $y(t + \Delta t) - y(t) = Z_1(t) + Z_2(t)$ where

$$\begin{aligned} Z_1(t) &= \int_0^t \frac{\theta(\tau)}{\sqrt{\pi(t + \Delta t - \tau)}} d\tau + \int_0^t \frac{M^1(\tau)Z_1(\tau)}{\sqrt{\pi(t - \tau)}} d\tau, \\ Z_2(t) &= \int_0^t \frac{G[y(\tau + \Delta t), \tau + \Delta t] - G[y(\tau + \Delta t), \tau]}{\sqrt{\pi(t - \tau)}} d\tau + \int_0^t \frac{M^1(\tau)Z_2(\tau)}{\sqrt{\pi(t - \tau)}} d\tau \end{aligned}$$

and arguing as in the proof of Theorem II above, by Lemma 2 and Corollary 2, we have, for $t \geq 0$, $Z_1(t) \geq 0$, $Z_2(t) \geq 0$ and hence $y(t + \Delta t) - y(t) \geq 0$ for all sufficiently small $\Delta t > 0$. This proves Theorem IV (under the additional restriction (D¹)).

Now consider the general case where the function $G(u, t)$ satisfies (A), (B), (C) and (D) but not necessarily (D¹).

Lemma 3. Given finite numbers $\epsilon > 0$, $U_0 > 0$ and $T > 0$, a function $\bar{G}(u, t)$ can be defined which satisfies (A), (B), (C), (D), (D¹) and the following:

- (a) $|G(u, t) - \bar{G}(u, t)| < \epsilon$ for $|u| \leq U_0$, $0 \leq t \leq T$.
 (b) The function $\bar{u}(t)$ corresponding to $\bar{G}(u, t)$ is such that $|u(t) - \bar{u}(t)| < \epsilon$ for $0 \leq t \leq T$.

Proof. There is evidently no loss of generality in assuming that $U_0 > |u(t)|$ for $0 \leq t \leq T$ since, by (D), $u(t)$ is bounded in this finite interval. Since, by (C), $G(u, t)$ is uniformly continuous w.r.t. u in the finite rectangle $|u| \leq U_0$, $0 \leq t \leq T$, it is possible to find a finite set of points $u_1 = -U_0 < u_2 < u_3 < \dots < u_{n-1} < u_n = U_0$ such that $u_{i+1} - u_i < \epsilon$ for $i = 1, 2, \dots, n-1$, and $|G(u, t) - G(u_i, t)| < \epsilon$ for $u_i \leq u \leq u_{i+1}$, $0 \leq t \leq T$.

In the interval $0 \leq t \leq T$, let

$$\bar{G}(u, t) = \begin{cases} G(u_i, t) + \frac{u - u_i}{u_{i+1} - u_i} [G(u_{i+1}, t) - G(u_i, t)] \\ \quad \text{for } u_i \leq u < u_{i+1}, \quad i = 1, 2, \dots, n-1; \\ G(-U_0, t) \quad \text{for } u \leq -U_0 = u_1; \\ G(U_0, t) \quad \text{for } u \geq U_0 = u_n; \end{cases}$$

for $t \geq T$ let $\bar{G}(u, t) = \bar{G}(u, T)$ (which is already defined).

(a) follows by noting that for $0 \leq t \leq T$, $u_i \leq u < u_{i+1}$ ($1 \leq i < n$)

$$\begin{aligned} |\bar{G}(u, t) - G(u, t)| &= \left| \left(1 - \frac{u - u_i}{u_{i+1} - u_i}\right) G(u_i, t) + \frac{u - u_i}{u_{i+1} - u_i} G(u_{i+1}, t) \right. \\ &\quad \left. - G(u, t) \right| = |k_1[G(u_i, t) - G(u, t)] + k_2[G(u_{i+1}, t) - G(u, t)]| \end{aligned}$$

where $k_1 + k_2 = 1$ and $k_1 \geq 0$, $k_2 \geq 0$.

The last expression on the right side is clearly $\leq (k_1 + k_2) \epsilon = \epsilon$. Also, for $u = u_n = U_0$, $0 \leq t \leq T$, $\bar{G}(u, t) = G(u, t)$.

Conditions (A), (B) and (C) can be easily verified for the function $\bar{G}(u, t)$. Also it can be verified that $\partial \bar{G}(u, t) / \partial u$ exists for $u \neq$ any u_i , $1 \leq i \leq n$, and that the only values it assumes are 0 and $[G(u_{i+1}, t) - G(u_i, t)] / (u_{i+1} - u_i)$, $i = 1, 2, \dots, n-1$, $0 \leq t \leq T$.

Since the function $G(u, t)$ is bounded in $0 \leq t \leq T$, $|u| \leq U_0$, and the u_i 's are finite in number, it is clear that

$$\left| \frac{G(u_{i+1}, t) - G(u_i, t)}{u_{i+1} - u_i} \right|, \quad 1 \leq i < n, \quad 0 \leq t \leq T,$$

is bounded. This proves (D¹) for $\bar{G}(u, t)$.

It remains to prove (D) and (b). By hypothesis, $-U_0 < u(t) < U_0$ for $0 \leq t \leq T$. Hence, corresponding to any given t^1 in $(0, T)$, there exists $i (= i(t^1))$ such that $1 \leq i < n$ and $u_i < u(t^1) \leq u_{i+1}$. Then $G(u_i, t^1) > 0$ and $G(u_{i+1}, t^1)$

≤ 0 . But $\tilde{G}(u_i, t) = G(u_i, t)$ for $j = 1, 2, \dots, n, 0 \leq t \leq T$. Hence $\tilde{G}(u_i, t^1) > 0$ and $\tilde{G}(u_{i+1}, t) \leq 0$. Therefore from the continuity of $\tilde{G}(u, t)$ w.r.t. u , it follows that $\bar{u}(t^1)$ exists and $u_i < \bar{u}(t^1) \leq u_{i+1}$. Therefore $|\bar{u}(t^1) - u(t^1)| \leq u_{i+1} - u_i < \epsilon$. But t^1 is any number in $(0, T)$. This proves (b).

Corollary 3. *From the fact that*

$$\tilde{G}(u, t) = k_1(u) G(u_i, t) + k_2(u) G(u_{i+1}, t)$$

where k_1, k_2 are independent of t , it follows that if $G(u, t)$ is a non-decreasing function of t in $-\infty < u < \infty, 0 \leq t < +\infty$, then $\tilde{G}(u, t)$ is also a non-decreasing function of t in $-\infty < u < +\infty, 0 \leq t < +\infty$.

Proof of Theorem I: In Lemma 3 above let $U_0 = \text{Max}_{0 \leq t \leq T} |u(t)| + 1$ and let ϵ be any given positive number ≤ 1 . By Lemma 3, a function $\tilde{G}(u, t) = G_\epsilon(u, t)$ can be defined satisfying (A), (B), (C), (D), (D'), (a) and (b). Let $u(t, \epsilon)$ denote the corresponding function $\bar{u}(t)$.

By the special case of Theorem I proved above, the integral equation

$$y(t, \epsilon) = \int_0^t \frac{G_\epsilon[y(\tau, \epsilon), \tau]}{\sqrt{\pi(t - \tau)}} d\tau$$

has a unique solution $y(t, \epsilon)$ for $t \geq 0$.

Also by the special case of Theorem III proved above, we have, for $0 \leq t \leq T$, $|y(t, \epsilon)| \leq \text{Max}_{0 \leq t \leq T} |u(t, \epsilon)|$. But, by (b), $|u(t, \epsilon)| \leq |u(t)| + |\epsilon|$, so that $\text{Max}_{0 \leq t \leq T} |u(t, \epsilon)| \leq \text{Max}_{0 \leq t \leq T} |u(t)| + \epsilon \leq \text{Max}_{0 \leq t \leq T} |u(t)| + 1 = U_0$. It follows that for any given ϵ in $(0, 1)$, $|y(t, \epsilon)| \leq U_0$ for $0 \leq t \leq T$, i.e. the function $y(t, \epsilon)$ is uniformly bounded in $(0, T)$.

It follows that if, as $\epsilon \rightarrow 0$, $y(t, \epsilon)$ converges uniformly to a limit $y(t)$ in $0 \leq t \leq T$, then, for $0 \leq t \leq T$,

$$y(t) = \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t - \tau)}} d\tau.$$

For from condition (a) it is clear that in the rectangle $|u| \leq U_0, 0 \leq t \leq T$, $G_\epsilon(u, t)$ converges uniformly to $G(u, t)$ as $\epsilon \rightarrow 0$. Therefore, since $|y(t, \epsilon)| \leq U_0$ for $0 \leq t \leq T$ and $G(u, t)$ is continuous in u , $G_\epsilon[y(\tau, \epsilon), \tau] \rightarrow G[y(\tau), \tau]$ as $\epsilon \rightarrow 0$, uniformly in $0 \leq \tau \leq T$. Hence, taking the limit as $\epsilon \rightarrow 0$ of both sides of the equation

$$y(t, \epsilon) = \int_0^t \frac{G_\epsilon[y(\tau, \epsilon), \tau]}{\sqrt{\pi(t - \tau)}} d\tau$$

(for $0 \leq t \leq T$), we have

$$y(t) = \int_0^t \frac{G[y(\tau), \tau]}{\sqrt{\pi(t - \tau)}} d\tau$$

(for $0 \leq t \leq T$).

Since $T > 0$ in Lemma 3 is arbitrary, therefore to prove the first part of Theorem I it is sufficient, in view of the above result, to show that, as $\epsilon \rightarrow 0$, $y(t, \epsilon)$ converges uniformly in $0 \leq t \leq T$ to a limit $y(t)$. This can be done as follows: Let ϵ_1, ϵ_2 both lie in $(0, 1)$. By hypothesis

$$\begin{aligned} y(t, \epsilon_1) - y(t, \epsilon_2) &= \int_0^t \frac{G_{\epsilon_1}[y(\tau, \epsilon_1), \tau] - G_{\epsilon_2}[y(\tau, \epsilon_2), \tau]}{\sqrt{\pi(t - \tau)}} d\tau \\ &= \int_0^t \frac{G_{\epsilon_1}[y(\tau, \epsilon_1), \tau] - G_{\epsilon_2}[y(\tau, \epsilon_1), \tau]}{\sqrt{\pi(t - \tau)}} d\tau \\ &\quad + \int_0^t \frac{G_{\epsilon_2}[y(\tau, \epsilon_1), \tau] - G_{\epsilon_2}[y(\tau, \epsilon_2), \tau]}{\sqrt{\pi(t - \tau)}} d\tau \\ &= \int_0^t \frac{G_{\epsilon_1}[y(\tau, \epsilon_1), \tau] - G_{\epsilon_2}[y(\tau, \epsilon_1), \tau]}{\sqrt{\pi(t - \tau)}} d\tau \\ &\quad + \int_0^t \frac{G_{\epsilon_2}[y(\tau, \epsilon_1), \tau] - G_{\epsilon_2}[y(\tau, \epsilon_2), \tau]}{y(\tau, \epsilon_1) - y(\tau, \epsilon_2)} \cdot \frac{y(\tau, \epsilon_1) - y(\tau, \epsilon_2)}{\sqrt{\pi(t - \tau)}} d\tau. \end{aligned}$$

As in the proof of the special case of Theorem II above, let

$$M_1(\tau) = \begin{cases} \frac{G_{\epsilon_1}[y(\tau, \epsilon_1), \tau] - G_{\epsilon_2}[y(\tau, \epsilon_2), \tau]}{y(\tau, \epsilon_1) - y(\tau, \epsilon_2)} & \text{whenever } y(\tau, \epsilon_1) \neq y(\tau, \epsilon_2) \\ 0 & \text{whenever } y(\tau, \epsilon_1) = y(\tau, \epsilon_2). \end{cases}$$

Then using conditions (A), (B) and (D¹) on $G_\epsilon(u, t)$ it can be shown that $M_1(\tau) \leq 0$, bounded and L -integrable in $0 \leq \tau \leq t \leq T$. Therefore, since

$$\begin{aligned} y(t, \epsilon_1) - y(t, \epsilon_2) &= \int_0^t \frac{G_{\epsilon_1}[y(\tau, \epsilon_1), \tau] - G_{\epsilon_2}[y(\tau, \epsilon_1), \tau]}{\sqrt{\pi(t - \tau)}} d\tau \\ &\quad + \int_0^t \frac{M_1(\tau)[y(\tau, \epsilon_1) - y(\tau, \epsilon_2)]}{\sqrt{\pi(t - \tau)}} d\tau, \end{aligned}$$

by Corollary 1,

$$|y(t, \epsilon_1) - y(t, \epsilon_2)| \leq \int_0^t \frac{|G_{\epsilon_1}[y(\tau, \epsilon_1), \tau] - G_{\epsilon_2}[y(\tau, \epsilon_1), \tau]|}{\sqrt{\pi(t - \tau)}} d\tau.$$

Since, as $\epsilon \rightarrow 0$, $G_\epsilon(u, t) \rightarrow G(u, t)$ uniformly in $|u| \leq U_0$, $0 \leq t \leq T$, and $|y(\tau, \epsilon_1)| \leq U_0$ for $0 \leq \tau \leq t \leq T$, therefore $|G_{\epsilon_1}[y(\tau, \epsilon_1), \tau] - G_{\epsilon_2}[y(\tau, \epsilon_1), \tau]| \rightarrow 0$ as $\epsilon_1, \epsilon_2 \rightarrow 0$ independently, uniformly in $0 \leq \tau \leq t$, $0 \leq t \leq T$. It follows that, in $0 \leq t \leq T$, $|y(t, \epsilon_1) - y(t, \epsilon_2)| \rightarrow 0$ uniformly as $\epsilon_1, \epsilon_2 \rightarrow 0$ independently. Hence $\lim_{\epsilon \rightarrow 0} y(t, \epsilon) = y(t)$ exists for $0 \leq t \leq T$, and the convergence of $y(t, \epsilon)$ to $y(t)$ is uniform in $(0, T)$. This proves the existence of a solution of equation (E) in any arbitrary interval $0 \leq t \leq T$. The uniqueness of the solution follows by the same arguments as given in the proof of Theorem I of [4].

Remark 1: With the above solution, for $0 \leq t \leq T$ we have

$$|y(t) - y(t, \epsilon)| \leq \int_0^t \frac{|G[y(\tau), \tau] - G_\epsilon[y(\tau), \tau]|}{\sqrt{\pi(t - \tau)}} d\tau \leq \frac{\epsilon \cdot 2\sqrt{t}}{\sqrt{\pi}} \leq \frac{\epsilon \cdot 2\sqrt{T}}{\sqrt{\pi}}.$$

This follows by taking $\epsilon_1 = 0$ in the foregoing.

Proof of Theorem II: Consider any finite interval of t of the form $0 \leq t \leq T$. In Lemma 3 let

$$U_0 = \max_{0 \leq t \leq T} |u_1(t)| + \max_{0 \leq t \leq T} |u_2(t)| + 1$$

where $u_1(t)$, $u_2(t)$ respectively are the smallest values of u for which $G_1(u, t) = 0$, $G_2(u, t) = 0$. Let ϵ be any given positive number ≤ 1 . Consider a subdivision $-U_0 = u_1 < u_2 < u_3 < \dots < u_n = U_0$ of the interval $(-U_0, U_0)$ such that, for $i = 1, 2, \dots, n-1$, $u_{i+1} - u_i < \epsilon$, and we have $|G_1(u, t) - G_1(u_i, t)| < \epsilon$, and $|G_2(u, t) - G_2(u_i, t)| < \epsilon$ for $u_i \leq u \leq u_{i+1}$, $0 \leq t \leq T$. (Such a subdivision can be obtained by superposing the subdivisions $\{u_i^1\}$, $\{u_i^2\}$ of $(-U_0, U_0)$ where, for $i = 1, 2, \dots, n-1$, $u_{i+1}^1 - u_i^1 < \epsilon$, $u_{i+1}^2 - u_i^2 < \epsilon$ and $|G_1(u, t) - G_1(u_i^1, t)| < \epsilon$ for $0 \leq t \leq T$, $u_i^1 \leq u \leq u_{i+1}^1$, while $|G_2(u, t) - G_2(u_i^2, t)| < \epsilon$ for $0 \leq t \leq T$, $u_i^2 \leq u \leq u_{i+1}^2$.) Let the functions $G_\epsilon^1(u, t)$, $G_\epsilon^2(u, t)$ be defined in the same manner as $\tilde{G}(u, t)$ in Lemma 3, by taking $G(u, t)$ successively equal to $G_1(u, t)$ and $G_2(u, t)$.

Since $G_1(u_i, t) \leq G_2(u_i, t)$, $t \geq 0$, $i = 1, 2, \dots, n$, it can be easily seen from the definition of $\tilde{G}(u, t) = G_\epsilon(u, t)$ that $G_\epsilon^1(u, t) \leq G_\epsilon^2(u, t)$ for all u and for $t \geq 0$.

Now consider the equations

$$y_1(t, \epsilon) = \int_0^t \frac{G_\epsilon^1[y_1(\tau, \epsilon), \tau]}{\sqrt{\pi(t - \tau)}} d\tau, \quad y_2(t, \epsilon) = \int_0^t \frac{G_\epsilon^2[y_2(\tau, \epsilon), \tau]}{\sqrt{\pi(t - \tau)}} d\tau.$$

As in the proof of Theorem I, using the fact that $U_0 \geq \max_{0 \leq t \leq T} |u_1(t)| + 1$ and $U_0 \geq \max_{0 \leq t \leq T} |u_2(t)| + 1$, it can be shown that, for $0 \leq t \leq T$, $\lim_{\epsilon \rightarrow 0} y_1(t, \epsilon) = y_1(t)$ and $\lim_{\epsilon \rightarrow 0} y_2(t, \epsilon) = y_2(t)$. Also, since $G_\epsilon^1(u, t) \leq G_\epsilon^2(u, t)$, $t \geq 0$ and $G_\epsilon^1(u, t)$, $G_\epsilon^2(u, t)$ both satisfy (A), (B), (C), (D) and (D¹), therefore, by the special case of Theorem II proved above, $y_1(t, \epsilon) \leq y_2(t, \epsilon)$, $t \geq 0$, $0 < \epsilon < 1$. Taking the limits of both sides as $\epsilon \rightarrow 0$ we have $y_1(t) \leq y_2(t)$ for $0 \leq t \leq T$. Since $T > 0$ is arbitrarily chosen, this proves Theorem II.

Proof of Theorem III: For every ϵ in $(0, 1)$ let $G_\epsilon(u, t)$, $u(t, \epsilon)$ and $y(t, \epsilon)$ be defined as in the proof of Theorem I. Since, in $0 \leq t \leq T$, $m \leq u(t) \leq M$ and $|u(t, \epsilon) - u(t)| < \epsilon$, therefore, for $0 \leq t \leq T$, $m - \epsilon \leq u(t, \epsilon) \leq M + \epsilon$. Also since $\epsilon > 0$ and $m \leq 0$, $M \geq 0$, $m - \epsilon < 0$, $M + \epsilon > 0$. Hence by the special case of Theorem III proved above (which holds since $G_\epsilon(u, t)$ satisfies (A), (B), (C), (D) and (D¹)) $m - \epsilon \leq y(t, \epsilon) \leq M + \epsilon$ for $0 \leq t \leq T$. The required result follows by taking limits as $\epsilon \rightarrow 0$.

Proof of Theorem IV: Let $G_\epsilon(u, t)$ and $y(t, \epsilon)$ be defined as in the proof of Theorem I. Since, by hypothesis, $G(u, t)$ is a non-decreasing function of t , therefore, by Corollary 3, $G_\epsilon(u, t)$ is also a non-decreasing function of t .

In the definition of $\tilde{G}(u, t)$ in Lemma 3 it is clear that it is always possible to choose 0 as one of the u_i 's, so that $\tilde{G}(0, t) = G(0, t)$. Hence, in the proof of Theorem I, $G_\epsilon(u, t)$ can be chosen so that $G_\epsilon(0, 0) = G(0, 0)$. But by hypothesis $G(0, 0) > 0$. Hence $G_\epsilon(0, 0) > 0$, i.e. $G_\epsilon(u, t)$ satisfies the hypothesis of Theorem IV as well as (D¹). Hence, by the special case of Theorem IV proved above, $y(t, \epsilon)$ is a non-decreasing function of t . Hence $y(t) = \lim_{\epsilon \rightarrow 0} y(t, \epsilon)$ is also a non-decreasing function of t .

4. Consider the special case where $G(u, t) = g[u, \phi(t)]$ and conditions (5)–(8) are satisfied. The function $u(t)$ defined in (D) is evidently $\equiv \phi(t)$. Theorem I proves the existence and uniqueness of a solution of equation (f) and hence of (e) also. Theorem II leads to the following result:

Corollary 4. *If $\phi_1(t) \leq \phi_2(t)$ for $0 \leq t \leq T$ and $U_1(x, t)$, $U_2(x, t)$ are the solutions of equation (e) corresponding to $\phi(t) \equiv \phi_1(t)$, $\phi(t) \equiv \phi_2(t)$ respectively, then $U_1(x, t) \leq U_2(x, t)$ for $x \geq 0$, $t \geq 0$.*

Proof. In Theorem II let $G_1(u, t) = g[u, \phi_1(t)]$, $G_2(u, t) = g[u, \phi_2(t)]$. The hypothesis of Theorem II is evidently satisfied since, on account of (6), $g[u, \phi_1(t)] \leq g[u, \phi_2(t)]$ for all u and t . Let the functions $G_\epsilon^1(u, t)$, $G_\epsilon^2(u, t)$, $y_1(t, \epsilon)$ and $y_2(t, \epsilon)$ be defined as in the proof of Theorem II. By Theorem I the equations

$$y_1(t, \epsilon, x) = \int_0^t \frac{G_\epsilon^1[y_1(\tau, \epsilon, 0), \tau]}{\sqrt{\pi(t-\tau)}} e^{-x^2/4(t-\tau)} d\tau,$$

$$y_2(t, \epsilon, x) = \int_0^t \frac{G_\epsilon^2[y_2(\tau, \epsilon, 0), \tau]}{\sqrt{\pi(t-\tau)}} e^{-x^2/4(t-\tau)} d\tau$$

have unique solutions $y_1(t, \epsilon, x)$, $y_2(t, \epsilon, x)$. Also $y_1(t, \epsilon, 0) = y_1(t, \epsilon)$, $y_2(t, \epsilon, 0) = y_2(t, \epsilon)$. Therefore since, as shown in the proof of Theorem I, $y_1(t, \epsilon) \rightarrow y_1(t)$ and $y_2(t, \epsilon) \rightarrow y_2(t)$ as $\epsilon \rightarrow 0$, uniformly in $0 \leq t \leq T$,

$$y_1(t, \epsilon, x) \rightarrow \int_0^t \frac{G_1[y_1(\tau), \tau]}{\sqrt{\pi(t-\tau)}} e^{-x^2/4(t-\tau)} d\tau$$

$$= \int_0^t \frac{g[y_1(\tau), \phi_1(\tau)]}{\sqrt{\pi(t-\tau)}} e^{-x^2/4(t-\tau)} d\tau = U_1(x, t)$$

and similarly $y_2(t, \epsilon, x) \rightarrow U_2(x, t)$ as $\epsilon \rightarrow 0$.

Writing

$$y_1(t, \epsilon, x) - y_2(t, \epsilon, x) = \int_0^t \frac{G_\epsilon^1[y_1(\tau, \epsilon), \tau] - G_\epsilon^2[y_1(\tau, \epsilon), \tau]}{\sqrt{\pi(t-\tau)}} e^{-x^2/4(t-\tau)} d\tau$$

$$+ \int_0^t \frac{M(\tau) e^{-x^2/4(t-\tau)}}{\sqrt{\pi(t-\tau)}} d\tau$$

($M(\tau)$ being defined as in §3) and using Lemma 1, it can be shown that $y_1(t, \epsilon, x) \leq y_2(t, \epsilon, x)$ for $x \geq 0$, $t \geq 0$, $0 < \epsilon < 1$. The required result follows by taking

limits as $\epsilon \rightarrow 0$ of both sides of this inequality. Similarly Theorem III has the following extension:

Corollary 5. *If $m \leq 0$, $M \geq 0$ are constants such that $m \leq \phi(t) \leq M$ for $0 \leq t \leq T$, then $m \leq U(x, t) \leq M$, $0 \leq t \leq T$.*

This result can be seen to follow from Corollary 4 above and from Conclusion 1, p. 183, of [4].

Theorem V. *If $\phi(0) > 0$ and $\phi(t)$ is a non-decreasing function, then*

(12) *the solution $U(0, t)$ of equation (f) is also a non-decreasing function;*

(13) $0 \leq U(0, t) \leq \phi(t)$ for all $t \geq 0$;

(14) $U(0, t) < \phi(t)$ whenever $\phi(t) > 0$.

Proof. The hypothesis clearly implies that if $G(u, t) = g[u, \phi(t)]$, then $G(0, 0) = g[0, \phi(0)] > 0$ and that $G(u, t)$ is a non-decreasing function of t . Hence (12) follows from Theorem IV.

Since $\phi(t)$ is non-decreasing and $\phi(0) > 0$, therefore, for any interval of t of the form $0 \leq t \leq T$, we have $0 \leq \phi(t) \leq \phi(T)$. Hence, by Corollary 5 (taking $x = 0$), $0 \leq U(0, t) \leq \phi(T)$ for $0 \leq t \leq T$. In particular, $0 \leq U(0, T) \leq \phi(T)$. But $T \geq 0$ is arbitrary. This proves (13).

To prove (14), suppose if possible that $t_0 \geq 0$ is such that $\phi(t_0) = U(0, t_0) > 0$. $t_0 > 0$ since $U(0, 0) = 0$. Consider the equation

$$\bar{U}(0, t) = \int_0^t \frac{g[\bar{U}(0, \tau), \bar{\phi}(\tau)]}{\sqrt{\pi(t - \tau)}} d\tau$$

where

$$\bar{\phi}(t) = \begin{cases} \phi(t), & t < t_0 \\ \phi(t_0), & t \geq t_0. \end{cases}$$

Conditions (5)–(8) are clearly satisfied and hence a unique solution $\bar{U}(0, t)$ exists by Theorem I. Since $\phi(t)$ is non-decreasing, $\bar{\phi}(t)$ is also clearly non-decreasing, and $\bar{\phi}(0) = \phi(0) > 0$. Hence, by (12) proved above, $\bar{U}(0, t)$ is a non-decreasing function of t . Also it is clear that $\bar{U}(0, t) = U(0, t)$ for $t \leq t_0$, so that $\bar{U}(0, t_0) = \phi(t_0)$. Hence for $t \geq t_0$ we have $\bar{U}(0, t) \geq \bar{U}(0, t_0) = \phi(t_0) = \bar{\phi}(t)$, and $\phi(t_0) > 0$. But for $t > t_0$

$$\bar{U}(0, t) = \int_0^{t_0} \frac{g[\bar{U}(0, \tau), \bar{\phi}(\tau)]}{\sqrt{\pi(t - \tau)}} d\tau + \int_{t_0}^t \frac{g[\bar{U}(0, \tau), \bar{\phi}(\tau)]}{\sqrt{\pi(t - \tau)}} d\tau.$$

The first term on the right clearly $\rightarrow 0$ as $t \rightarrow \infty$, while the second is ≤ 0 since, for $\tau > t_0$, $\bar{U}(0, \tau) \geq \bar{\phi}(\tau)$ and hence $g[\bar{U}(0, \tau), \bar{\phi}(\tau)] \leq 0$. Hence if $\epsilon > 0$ is any given number, then, for all sufficiently large t , $\bar{U}(0, t) < \epsilon$. This contradicts the result proved above that $\bar{U}(0, t) \geq \phi(t_0) > 0$ for all $t > t_0$. Hence the hypothesis $U(0, t_0) = \phi(t_0) > 0$ is wrong, and (14) follows.

Theorem VI. If $\lim_{t \rightarrow \infty} \phi(t) = l$, a finite number, then $\lim_{t \rightarrow \infty} U(0, t)$ also exists and equals l .

Proof. Given $\eta > 0$, t_0 can be found such that $l - \eta \leq \phi(t) \leq l + \eta$ for $t \geq t_0$. Let

$$\phi_1(t) = \begin{cases} \phi(t), & t < t_0 \\ l - \eta, & t \geq t_0, \end{cases} \quad \phi_2(t) = \begin{cases} \phi(t), & t < t_0 \\ l + \eta, & t \geq t_0. \end{cases}$$

Consider the equations

$$y_1(t) = \int_0^t \frac{g[y_1(\tau), \phi_1(\tau)]}{\sqrt{\pi(t-\tau)}} d\tau, \quad y_2(t) = \int_0^t \frac{g[y_2(\tau), \phi_2(\tau)]}{\sqrt{\pi(t-\tau)}} d\tau.$$

By Theorem I, both these equations have unique solutions $y_1(t), y_2(t)$ respectively. From the definition it is clear that $\phi_1(t) \leq \phi(t) \leq \phi_2(t)$, $t \geq 0$. Hence by Corollary 4, $y_1(t) \leq U(0, t) \leq y_2(t)$, $t \geq 0$. Therefore, since $\eta > 0$ is arbitrary, it is sufficient to prove that $\lim_{t \rightarrow \infty} y_1(t) = l - \eta$ and $\lim_{t \rightarrow \infty} y_2(t) = l + \eta$.

For this purpose, consider the equation

$$z_1(t) = \int_{t_0}^t \frac{g[z_1(\tau), l - \eta]}{\sqrt{\pi(t-\tau)}} d\tau, \quad t \geq t_0.$$

$\lim_{t \rightarrow \infty} z_1(t) = l - \eta$. For, writing $w(t^1) = z(t^1 + t_0)$ and $t = t^1 + t_0$, we have

$$w(t^1) = \int_0^{t^1} \frac{g[w(\tau), l - \eta]}{\sqrt{\pi(t-\tau)}} d\tau$$

and the required result follows by a straightforward extension of Theorem 2 of [4]. Hence, given $\zeta > 0$, $t_1 > t_0$ can be found such that $|z_1(t) - (l - \eta)| < \zeta$ for $t \geq t_1$.

Consider any $T \geq t_1$. In Lemma 3 let $G(u, t) = g[u, \phi(t)]$. Then given $\epsilon > 0$ and $U_0 > 0$ a function $G_\epsilon(u, t)$ can be constructed such that $|G_\epsilon(u, t) - G(u, t)| < \epsilon$ for $|u| \leq U_0$, $0 \leq t \leq T$. Let U_0 be chosen = $\text{Max}_{t \geq 0} |\phi(t)| + |l| + \eta + 1$ (which is evidently finite by hypothesis). Using Corollary 5 it can be shown that $|y_1(t)|$, as well as $|z_1(t)|$, is $\leq U_0 - \zeta$ for $t \geq 0$ and $0 < \zeta < 1$. As in the proof of Theorem I, let

$$y_1(t, \epsilon) = \int_0^t \frac{G_\epsilon[y_1(\tau, \epsilon), \tau]}{\sqrt{\pi(t-\tau)}} d\tau, \quad z_1(t, \epsilon) = \int_{t_0}^t \frac{G_\epsilon[z_1(\tau, \epsilon), \tau]}{\sqrt{\pi(t-\tau)}} d\tau.$$

Using Corollary 1 it can be shown as before in Remark 1 that, for $0 \leq t \leq T$,

$$|y_1(t) - y_1(t, \epsilon)| \leq \int_0^t \frac{|G[y_1(\tau), \tau] - G_\epsilon[y_1(\tau), \tau]|}{\sqrt{\pi(t-\tau)}} d\tau \leq \frac{\epsilon \cdot 2\sqrt{t}}{\sqrt{\pi}} \leq \frac{2\epsilon\sqrt{T}}{\sqrt{\pi}}$$

and similarly

$$|z_1(t) - z_1(t, \epsilon)| \leq \int_{t_0}^t \frac{|G[z_1(\tau), \tau] - G_\epsilon[z_1(\tau), \tau]|}{\sqrt{\pi(t-\tau)}} d\tau \leq \frac{\epsilon \cdot 2\sqrt{t-t_0}}{\sqrt{\pi}} \leq \frac{2\epsilon\sqrt{T}}{\sqrt{\pi}}.$$

Let $\epsilon = \sqrt{\pi} \zeta / 2 \sqrt{T}$. Then for $0 \leq t \leq T$ we have

$$|y_1(t) - z_1(t)| \leq |y_1(t) - y_1(t, \epsilon)| + |y_1(t, \epsilon) - z_1(t, \epsilon)| + |z_1(t, \epsilon) - z_1(t)| \\ \leq 2\zeta + |y_1(t, \epsilon) - z_1(t, \epsilon)|.$$

Now

$$y_1(t, \epsilon) - z_1(t, \epsilon) = \int_0^t \frac{G_\epsilon[y_1(\tau, \epsilon), \tau]}{\sqrt{\pi(t-\tau)}} d\tau - \int_{t_0}^t \frac{G_\epsilon[z_1(\tau, \epsilon), \tau]}{\sqrt{\pi(t-\tau)}} d\tau \\ = \int_0^{t_0} \frac{G_\epsilon[y_1(\tau, \epsilon), \tau]}{\sqrt{\pi(t-\tau)}} d\tau + \int_{t_0}^t \frac{G_\epsilon[y_1(\tau, \epsilon), \tau] - G_\epsilon[z_1(\tau, \epsilon), \tau]}{\sqrt{\pi(t-\tau)}} d\tau.$$

As in the proof of Theorem II, defining $m(\tau)$ as

$$m(\tau) = \begin{cases} \frac{G_\epsilon[y_1(\tau, \epsilon), \tau] - G_\epsilon[z_1(\tau, \epsilon), \tau]}{y_1(\tau, \epsilon) - z_1(\tau, \epsilon)} & \text{whenever } y_1(\tau, \epsilon) \neq z_1(\tau, \epsilon) \\ 0 & \text{whenever } y_1(\tau, \epsilon) = z_1(\tau, \epsilon) \end{cases}$$

and applying Corollary 1 to the equation

$$y_1(t, \epsilon) - z_1(t, \epsilon) = \int_0^{t_0} \frac{G_\epsilon[y_1(\tau, \epsilon), \tau]}{\sqrt{\pi(t-\tau)}} d\tau + \int_{t_0}^t \frac{m(\tau)[y_1(\tau, \epsilon) - z_1(\tau, \epsilon)]}{\sqrt{\pi(t-\tau)}} d\tau,$$

it can be shown that

$$|y_1(t, \epsilon) - z_1(t, \epsilon)| \leq \int_0^{t_0} \frac{|G_\epsilon[y_1(\tau, \epsilon), \tau]|}{\sqrt{\pi(t-\tau)}} d\tau.$$

The integral on the right $\rightarrow 0$ as $t \rightarrow \infty$ uniformly w.r.t. ϵ . For, in $0 \leq t \leq T$, $|y_1(t, \epsilon) - y_1(t)| \leq \zeta$ and $|y_1(t)| \leq U_0 - \zeta$ so that $|y_1(t, \epsilon)| \leq U_0$. Hence

$$|G_\epsilon[y_1(t, \epsilon), t]| \leq |G[y_1(t, \epsilon), t]| + \epsilon \leq \max_{\substack{|u| \leq U_0 \\ 0 \leq t \leq T}} |G(u, t)| + 1 = m$$

where m is independent of ϵ (assuming that $\epsilon < 1$). Hence

$$\int_0^{t_0} \frac{|G_\epsilon[y_1(\tau, \epsilon), \tau]|}{\sqrt{\pi(t-\tau)}} d\tau \rightarrow 0$$

as $t \rightarrow \infty$, for fixed t_0 . Let $t_2 \geq t_1$ be chosen so that for $t \geq t_2$

$$\int_0^{t_0} \frac{|G_\epsilon[y_1(\tau, \epsilon), \tau]|}{\sqrt{\pi(t-\tau)}} d\tau \leq m \cdot \frac{1}{\sqrt{\pi}} \int_0^{t_0} \frac{d\tau}{\sqrt{t-\tau}} < \zeta.$$

Then consider any $T \geq t_2$. Let $\epsilon = \zeta \sqrt{\pi} / 2 \sqrt{T}$. Then, as shown above,

$$|y_1(T) - z_1(T)| \leq |y_1(T) - y_1(T, \epsilon)| + |y_1(T, \epsilon) - z_1(T, \epsilon)| \\ + |z_1(T, \epsilon) - z_1(T)| \leq 3\zeta.$$

Also since $T \geq t_1$, $|z_1(T) - (l - \eta)| < \zeta$. Hence

$$|y_1(T) - (l - \eta)| < |y_1(T) - z_1(T)| + |z_1(T) - (l - \eta)| < 4\zeta.$$

Now $\zeta > 0$, as well as $T \geq t_2$, is arbitrary. Hence $\lim_{T \rightarrow \infty} y_1(T) = l - \eta$. An exactly similar proof holds to show that $\lim_{T \rightarrow \infty} y_2(T) = l + \eta$. This completes the proof of Theorem VI.

BIBLIOGRAPHY

- [1] R. COURANT & D. HILBERT: *Methods of Mathematical Physics*, Vol. 1, New York: Interscience Publishers, Inc.
- [2] P. R. HALMOS: *Measure Theory*, New York: Van Nostrand, 1950.
- [3] W. R. MANN & F. WOLF: Heat Transfer between solids and gases under non-linear boundary conditions, *Quarterly of Applied Mathematics*, 9 (1951), pp. 163-184.
- [4] J. H. ROBERTS & W. R. MANN: A nonlinear integral equation of Volterra type, *Pacific Journal of Mathematics*, 1 (1951), pp. 431-445.

Ramanujan Institute of Mathematics
and
University of North Carolina
Chapel Hill, North Carolina