*Fractional Integrals on n -dimensional Euclidean Space**

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1. Introduction. In their study of fractional integrals, HARDY & LITTLEWOOD, [3], considered the operator U_λ , acting on functions in the interval $(0, \infty)$, given by

$$U_\lambda f = \int_0^\infty \frac{f(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda < 1.$$

They proved the following theorem:

Theorem A. *If f is a function in $L^p(0, \infty)$, $1 < p < \infty$, and $p < q < \infty$, with $1/q = (1/p) + \lambda - 1$, then $U_\lambda f$ exists almost everywhere and*

$$\left(\int_0^\infty |U_\lambda f|^q dx \right)^{1/q} \leq A \left(\int_0^\infty |f(x)|^p dx \right)^{1/p},$$

where A is independent of f , but may depend on p and q .

This theorem has been extended to n dimensions by SOBOLEFF, [5], and THORIN, [6]. Before stating their generalization, we recall some standard notation.

We let $x = (x_1, x_2, \dots, x_n)$ denote a general point in Euclidean n -space E^n , $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$, and $dx = dx_1 \dots dx_n$ the standard Euclidean Lebesgue measure.

We let

$$T_\lambda f = \int_{E^n} \frac{f(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda < n.$$

The statement of the Soboleff-Thorin theorem is as follows:¹

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¹The case $n = 1$ of Theorem A* is stated for the interval $(-\infty, \infty)$, while Theorem A is stated for the interval $(0, \infty)$. A simple argument, however, shows that these two cases can be reduced to each other.

Theorem A*. If f is a function in $L^p(E^n)$, $\infty > q > p > 1$, and $1/q = (1/p) + (\lambda/n) - 1$, then $T_\lambda f$ exists almost everywhere and

$$\left(\int |T_\lambda f|^q dx \right)^{1/q} \leq A \left(\int |f(x)|^p dx \right)^{1/p},$$

where A is independent of f , but may depend on p and q .²

For some purposes, however, it is of interest to study the modifications of Theorems A and A* when instead of the norms $\|f\|_p = (\int_{E^n} |f(x)|^p dx)^{1/p}$, we consider norms given by

$$\|f\|_{p,\alpha} = \left(\int_{E^n} |f(x)|^p |x|^{p\alpha} dx \right)^{1/p},$$

for some suitable α . One such extension of Theorem A was given by HARDY & LITTLEWOOD, [3], themselves. Their theorem is as follows:

Theorem B. Let $0 < \lambda < 1$, $1 < p < \infty$, $\alpha < 1/p'$,³ $\beta < 1/q$, $\alpha + \beta \geq 0$, and $1/q = (1/p) + \lambda + \alpha + \beta - 1$. If $p \leq q < \infty$, then

$$\left(\int_0^\infty \{|U_\lambda f| x^{-\beta}\}^q dx \right)^{1/q} \leq A \left(\int_0^\infty \{|f(x)| x^\alpha\}^p dx \right)^{1/p},$$

where A is independent of f , but may depend on p , q , α and β .

Our purpose is to prove the following generalization of Theorem B—and also of Theorem A*:

Theorem B*. Let $0 < \lambda < n$, $1 < p < \infty$, $\alpha < n/p'$, $\beta < n/q$, $\alpha + \beta \geq 0$, and $1/q = (1/p) + [(\lambda + \alpha + \beta)/n] - 1$. If $p \leq q < \infty$, then

$$\left(\int_{E^n} \{|T_\lambda f| |x|^{-\beta}\}^q dx \right)^{1/q} \leq A \left(\int_{E^n} \{|f(x)| |x|^\alpha\}^p dx \right)^{1/p}.$$

Theorem B* and Theorem B are composed of two cases: the case $p < q$ and the case $p = q$. The method of proof in each case is different and deserves comment. In their proof of Theorem B, HARDY & LITTLEWOOD reduced the case $p < q$ to Theorem A. Their proof of this point, however, is incorrect.⁴ We remove the gap in their proof when we prove Theorem B* for the case $p < q$. The proof of Theorem B* when $p = q$ is more difficult than the special case when $n = 1$, and takes up a large share of this paper.

An important tool for the proof of Theorem B* is a generalization to n dimensions of a very useful theorem on integral operators whose kernels are homogeneous of degree -1 . (For this theorem see HARDY, LITTLEWOOD & POLYA, [4], Theorem 319; our generalization is Lemma (2.1) below.)

²An elegant reduction of Theorem A* was given by N. DU PLESSIS, [2]. In this paper we assume theorem A* as known.

³For any number $p > 1$ we let p' denote its conjugate index. That is, $(1/p) + (1/p') = 1$.

⁴See HARDY & LITTLEWOOD, [3], page 579, lines 11 and 12.

We conclude this introductory section by stating two alternative rewordings of Theorem B*. We leave it to the reader to verify that, for each p, q, λ, α and β , these three theorems are immediately equivalent.

Theorem B₁*. Let p, q, λ, α and β be as in Theorem B*. Define the operator $S_{\lambda, \alpha, \beta}(f)$ by

$$Sf = S_{\lambda, \alpha, \beta}(f) = \int_{E^n} \frac{f(y)}{|x|^\beta |x - y|^\lambda |y|^\alpha} dy.$$

Then

$$\left(\int_{E^n} |Sf|^q dx \right)^{1/q} \leq A \left(\int_{E^n} |f(x)|^p dx \right)^{1/p}.$$

Theorem B₂*. Again, let p, q, λ, α and β be as in Theorem B*. If $f \in L^p(E^n)$, $g \in L^{q'}(E^n)$, then

$$\left| \int_{E^n} \int_{E^n} \frac{f(y)g(x)}{|x|^\beta |x - y|^\lambda |y|^\alpha} dy dx \right| \leq A \|f\|_p \|g\|_{q'}.$$

2. Proof of Theorem B* when $p = q$. Throughout this paper we shall adhere to the following notation. We shall often write $R = |x|$ and $r = |y|$. ξ and η will represent unit vectors having the same directions as x and y ; thus $x = R\xi$ and $y = r\eta$. The surface of the unit sphere in E^n will be denoted by Σ^n , or, if there is no danger of confusion, simply by Σ . If the domain of integration is not specified we shall mean the integration to be extended over all of E^n . We write $d\sigma$ to represent the element of area of Σ ; sometimes we shall use the symbols $d\sigma_\xi$ or $d\sigma_\eta$ to indicate that the variable of integration is ξ or η .

We have noted that Theorem B* is immediately equivalent to Theorem B₁*. In this section we prove the latter theorem when $p = q$. In this case $\lambda + \alpha + \beta = n$. Therefore the kernel of the integral operator S is a function in x and y which is homogeneous of degree $-n$. The following lemma will enable us to exploit this property of S .

Lemma (2.1). Let $K(u, v) \geq 0$ be a function defined on the quadrant $\{(u, v): u \geq 0 \text{ and } v \geq 0\}$ which is homogeneous of degree $-n$, satisfying, for $p \geq 1$,

$$J = \int_0^\infty K(1, t) t^{n/p'-1} dt < \infty.$$

Let $[Uf](x) = \int K(|x|, |y|) f(y) dy$; then

$$\left(\int |Uf|^p dx \right)^{1/p} \leq A_J \left(\int |f|^p dx \right)^{1/p}$$

where A_J is a number depending only on J (and, hence, on p) and the dimension n , but not on f .

We write

$$(2.2) \quad [Uf](x) = [Uf](R) = \int_\Sigma \left\{ \int_0^\infty K(R, r) f(r\eta) r^{n-1} dr \right\} d\sigma$$

and we define

$$(2.3) \quad [U_{\eta}f](R) = \int_0^{\infty} K(R, r)f(r\eta)r^{n-1} dr = \int_0^{\infty} K(1, t)f(tR\eta)t^{n-1} dt$$

where the last expression is derived from the second by making the change of variable $tR = r$ and using the homogeneity of K . Then, clearly,

$$(2.4) \quad |[Uf](R)| \leq \int_{\Sigma} |U_{\eta}f| d\sigma.$$

On the other hand, if g is a measurable function on E^n and $t > 0$,

$$(2.5) \quad \left(\int_0^{\infty} |g(tR\eta)|^p R^{n-1} dR \right)^{1/p} = t^{-n/p} \left(\int_0^{\infty} |g(R\eta)|^p R^{n-1} dR \right)^{1/p}.$$

Let h be a function satisfying $(\int_0^{\infty} |h(r)|^p R^{n-1} dR) = 1$ such that

$$\left(\int_0^{\infty} |[U_{\eta}f](R)|^p R^{n-1} dR \right)^{1/p} = \int_0^{\infty} [U_{\eta}f](R) R^{n-1} h(R) dR.$$

Then, using (2.3), (2.5), Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned} \left(\int_0^{\infty} |[U_{\eta}f](R)|^p R^{n-1} dR \right)^{1/p} &= \int_0^{\infty} [U_{\eta}f](R) h(R) R^{n-1} dR \\ &= \int_0^{\infty} \left\{ \int_0^{\infty} K(1, t)f(tR\eta)t^{n-1} dt \right\} R^{n-1} h(R) dR \\ &= \int_0^{\infty} K(1, t)t^{n-1} \left\{ \int_0^{\infty} f(tR\eta)h(R)R^{n-1} dR \right\} dt \\ &\leq \int_0^{\infty} K(1, t)t^{n-1} \left\{ \int_0^{\infty} |f(tR\eta)|^p R^{n-1} dR \right\}^{1/p} dt \\ &= \int_0^{\infty} K(1, t)t^{n-1}t^{-n/p} \left\{ \int_0^{\infty} |f(R\eta)|^p R^{n-1} dR \right\}^{1/p} dt \\ &= J \left\{ \int_0^{\infty} |f(R_{\eta})|^p R^{n-1} dR \right\}^{1/p}. \end{aligned}$$

We have shown

$$(2.6) \quad \left(\int_0^{\infty} |[U_{\eta}f](R)|^p R^{n-1} dR \right)^{1/p} \leq J \left(\int_0^{\infty} |f(R_{\eta})|^p R^{n-1} dR \right)^{1/p}.$$

Applying Jensen's inequality to (2.4), we obtain

$$|[Uf](R)|^p \leq \{\sigma(\Sigma)\}^{p-1} \int_{\Sigma} |U_{\eta}f|^p d\sigma.$$

Using this inequality and (2.6), we thus have

$$\begin{aligned}
\left(\int_0^\infty |[Uf](R)|^p R^{n-1} dR \right) &\leq \{\sigma(\Sigma)\}^{p-1} \left(\int_0^\infty \left\{ \int_\Sigma |[U_\eta f](R)|^p d\sigma \right\} R^{n-1} dR \right) \\
&= \{\sigma(\Sigma)\}^{p-1} \left(\int_\Sigma \left\{ \int_0^\infty |[U_\eta f](R)|^p R^{n-1} dR \right\} d\sigma \right) \\
&\leq J^p \{\sigma(\Sigma)\}^{p-1} \left(\int_\Sigma \left\{ \int_0^\infty |f(R\eta)|^p R^{n-1} dR \right\} d\sigma \right) \\
&= J^p \{\sigma(\Sigma)\}^{p-1} \int |f(x)|^p dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\int |[Uf](x)|^p dx &= \int |[Uf](R)|^p dx = \int_\Sigma \left\{ \int_0^\infty |[Uf](R)|^p R^{n-1} dR \right\} d\sigma \\
&= \sigma(\Sigma) \left(\int_0^\infty |[Uf](R)|^p R^{n-1} dR \right) \leq \sigma(\Sigma) J^p \{\sigma(\Sigma)\}^{p-1} \int |f|^p dx.
\end{aligned}$$

Thus

$$\left(\int |[Uf](x)|^p dx \right)^{1/p} \leq J \sigma(\Sigma) \left(\int |f(x)|^p dx \right)^{1/p}$$

and the lemma is proved with $A_J = J\sigma(\Sigma)$.

Lemma (2.7). Let $\Delta = \Delta(t, \xi, \eta) = |1 - 2t \cos(\xi, \eta) + t^2|^{1/2}$ where $0 < m \leq t \leq M < \infty$ and $\cos(\xi, \eta)$ is the inner product of the unit vectors ξ and η . If g is a function in $L^p(\Sigma)$, $p \geq 1$, we define

$$\mathcal{G}_t(\xi) = \int_\Sigma \frac{g(\eta)}{\Delta^\lambda} d\sigma_\eta.$$

Then

$$\left(\int_\Sigma |\mathcal{G}_t(\xi)|^p d\sigma_\xi \right)^{1/p} \leq A |1 - t|^{-\lambda/n} \left(\int_\Sigma |g(\eta)|^p d\sigma_\eta \right)^{1/p}$$

where A is independent of g and t but may depend on λ , n , p and the numbers m and M .

Let us write

$$\mathcal{G}(\xi) = \mathcal{G}_t(\xi) = \int_\Sigma \frac{g(\eta)}{\Delta^{\lambda/p}} \cdot \frac{1}{\Delta^{\lambda/p'}} d\sigma_\eta.$$

Thus, applying Hölder's inequality, we see that

$$|\mathcal{G}(\xi)|^p \leq \left(\int_\Sigma \frac{|g(\eta)|^p}{\Delta^\lambda} d\sigma_\eta \right) \left(\int_\Sigma \frac{d\sigma_\eta}{\Delta^\lambda} \right)^{p/p'}.$$

Making use of the fact⁵

$$(2.8) \quad \int_{\Sigma} \frac{d\sigma_{\eta}}{\Delta^{\lambda}} \leq A |1 - t|^{-\lambda/n},$$

where A depends only on λ , n , m and M but not on t or ξ , we have

$$(2.9) \quad |\hat{g}(\xi)|^p \leq A^{p/p'} |1 - t|^{-\lambda p/n p'} \int_{\Sigma} \frac{|g(\eta)|^p}{\Delta^{\lambda}} d\sigma_{\eta}.$$

If we now integrate both sides of (2.9) with respect to ξ and apply (2.8) again we obtain

$$\int_{\Sigma} |\hat{g}(\xi)|^p d\sigma_{\xi} \leq A^p |1 - t|^{-\lambda p/n} \int_{\Sigma} |g(\eta)|^p d\sigma_{\eta},$$

and Lemma (2.7) is proved.

We now proceed to prove Theorem B₁^{*} (when $p = q$). We write

$$\begin{aligned} Sf &= \int K_1(x, y)f(y) dy + \int K_2(x, y)f(y) dy + \int K_3(x, y)f(y) dy \\ &= U_1f + U_2f + U_3f, \end{aligned}$$

where

$$\begin{aligned} K_1(x, y) &= \begin{cases} |x|^{-\beta} |x - y|^{-\lambda} |y|^{-\alpha} & \text{if } 0 \leq \frac{|y|}{|x|} \leq \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases} \\ K_2(x, y) &= \begin{cases} |x|^{-\beta} |x - y|^{-\lambda} |y|^{-\alpha} & \text{if } 2 \leq \frac{|y|}{|x|} \\ 0 & \text{otherwise,} \end{cases} \\ K_3(x, y) &= \begin{cases} |x|^{-\beta} |x - y|^{-\lambda} |y|^{-\alpha} & \text{if } \frac{1}{2} < \frac{|y|}{|x|} < 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We will show that for $i = 1, 2, 3$

$$\left(\int |U_i f|^p dx \right)^{1/p} \leq A_i \left(\int |f|^p dx \right)^{1/p}$$

where the numbers A_i do not depend on f in $L^p(E^n)$.

If $|y|/|x| \leq \frac{1}{2}$ then $|x - y| \geq \frac{1}{2}|x|$; thus

$$K_1(x, y) \leq 2^{\lambda} |x|^{-\beta} |x|^{-\lambda} |y|^{-\alpha} \quad \text{if } |y|/|x| \leq \frac{1}{2},$$

⁵We prove (2.8) in an appendix.

while $K_1(x, y) = 0$ otherwise. Therefore

$$J_1 = \int_0^\infty K_1(1, t) t^{n/p'-1} dt \leq 2^\lambda \int_0^{\frac{1}{2}} t^{-\alpha} t^{n/p'-1} dt < \infty$$

since $\alpha < n/p'$. Thus by Lemma (2.1)

$$\left(\int |U_1 f|^p dx \right)^{1/p} \leq A_{J_1} \left(\int |f|^p dx \right)^{1/p}.$$

Similarly, $|y|/|x| \geq 2$ implies $|x - y| \geq \frac{1}{2}|y|$, which leads to the estimate

$$J_2 = \int_0^\infty K_2(1, t) t^{n/p'-1} dt \leq 2^\lambda \int_2^\infty t^{-\lambda} t^{-\alpha} t^{n/p'-1} dt < \infty$$

since $n/p' - (\lambda + \alpha) = n/p' - (n - \beta) = n[(1/p') - 1] + \beta = \beta - n/p < 0$. Applying Lemma (2.1) again, we obtain

$$\left(\int |U_2 f|^p dx \right)^{1/p} \leq A_{J_2} \left(\int |f|^p dx \right)^{1/p}.$$

We shall now consider $U_3 f = \int K_3(x, y) f(y) dy$. Using the notation introduced at the beginning of this section, we have

$$[U_3 f](R\xi) = \int_{\mathbb{R}^n} \left\{ \int_0^\infty K_3(R\xi, r\eta) f(r\eta) r^{n-1} dr \right\} d\sigma.$$

Making the change of variable $r = tR$, this reduces to

$$\begin{aligned} [U_3 f](R\xi) &= \int_{\mathbb{R}^n} \left\{ \int_{\frac{1}{2}}^2 \frac{f(tR\eta) t^{n-\alpha-1}}{|\xi - t\eta|^\lambda} dt \right\} d\sigma \\ &\leq A \int_{\mathbb{R}^n} \left\{ \int_{\frac{1}{2}}^2 \left| 1 - \frac{f(tR\eta)}{2t \cos(\xi, \eta) + t^2} \right| dt \right\} d\sigma, \end{aligned}$$

where $A = \max \{t^{n-\alpha-1}; \frac{1}{2} < t < 2\}$.

Using the notation introduced in Lemma (2.7), we write, for $\frac{1}{2} < t < 2$,

$$F_t(R\xi) = \int_{\mathbb{R}^n} \frac{|f(tR\eta)|}{\Delta^\lambda} d\sigma, \quad F(x) = F(R\xi) = \int_{\frac{1}{2}}^2 F_t dt.$$

Because of the last inequality, it suffices to show that there exists a number B , independent of f , such that

$$\left(\int |F(x)|^p dx \right)^{1/p} \leq B \left(\int |f(x)|^p dx \right)^{1/p}.$$

Let $h(R)$ be a function satisfying $\int_0^\infty |h(R)|^{p'} R^{n-1} dR = 1$ such that

$$\left(\int_0^\infty |F_t(R\xi)|^p R^{n-1} dR \right)^{1/p} = \int_0^\infty h(R) F_t(R\xi) R^{n-1} dR.$$

We then have

$$\begin{aligned}
 \left(\int_0^\infty |F_t(R\xi)|^p R^{n-1} dR \right)^{1/p} &= \int_0^\infty h(R) \left\{ \int_\Sigma \frac{|f(tR\eta)|}{\Delta^\lambda} d\sigma_\eta \right\} R^{n-1} dR \\
 &= \int_\Sigma \frac{1}{\Delta^\lambda} \left\{ \int_0^\infty h(R) |f(tR\eta)| R^{n-1} dR \right\} d\sigma_\eta \\
 &\leq \int_\Sigma \frac{1}{\Delta^\lambda} \left\{ \int_0^\infty |f(tR\eta)|^p R^{n-1} dR \right\}^{1/p} d\sigma_\eta \\
 &\leq 2^n \int_\Sigma \frac{1}{\Delta^\lambda} \left\{ \int_0^\infty |f(R\eta)|^p R^{n-1} dR \right\}^{1/p} d\sigma_\eta,
 \end{aligned}$$

the first inequality being a consequence of Hölder's inequality, while the second follows from the change of variable $p = tR$ using the fact that $t > \frac{1}{2}$.

Let us now write $g(\eta) = (\int_0^\infty |f(R\eta)|^p R^{n-1} dR)^{1/p}$ and apply Lemma (2.7) as well as the last chain of inequalities. We obtain

$$\left(\int_\Sigma \left\{ \int_0^\infty |F_t(R\xi)|^p R^{n-1} dR \right\} d\sigma_\xi \right)^{1/p} \leq C |1 - t|^{-\lambda/n} \left(\int_\Sigma \left\{ \int_0^\infty |f(R\eta)|^p R^{n-1} dR \right\}^{1/p} d\sigma_\eta \right)^{1/p}.$$

Hence, using Minkowski's inequality for integrals, keeping in mind that $0 < \lambda < n$ and letting $Q = \{\int_{\frac{1}{2}}^2 |F_t(R\xi)|^p dt\}^{1/p}$, we have by standard arguments

$$\begin{aligned}
 \left(\int |F(x)|^p dx \right)^{1/p} &= \left(\int \left| \int_{\frac{1}{2}}^2 F_t(R\xi) dt \right|^p dx \right)^{1/p} \leq \int \left\{ \int_{\frac{1}{2}}^2 |F_t(R\xi)|^p dt \right\}^{1/p} dx \\
 &= \int \left\{ \int_{\frac{1}{2}}^2 |F_t(R\xi)| \cdot \frac{|F_t(R\xi)|^{p-1}}{Q^{p-1}} dt \right\} dx \\
 &= \int_{\frac{1}{2}}^2 \left\{ \int |F_t(R\xi)| \left[\frac{|F_t(R\xi)|}{Q} \right]^{p-1} dx \right\} dt \\
 &\leq \int_{\frac{1}{2}}^2 \left[\int |F_t(R\xi)|^p dx \right]^{1/p} dt \\
 &\leq \int_{\frac{1}{2}}^2 C |1 - t|^{-\lambda/n} \left(\int |f(x)|^p dx \right)^{1/p} dt \leq B \left(\int |f(x)|^p dx \right)^{1/p}
 \end{aligned}$$

and Theorem B₁^{*} (when $p = q$) is proved.

3. Proof of Theorem B^{*} when $p < q < \infty$. For the consideration of this case it is more convenient to use the form of the statement given in Theorem B₂^{*}. We wish, therefore, to prove that when $f \in L^p(E^m)$ and $g \in L^{q'}(E^n)$, $p < q < \infty$, we have

$$(3.1) \quad \left| \int_{E^m} \int_{E^n} \frac{f(y)g(x)}{|x|^\beta |x - y|^\lambda |y|^\alpha} dy dx \right| \leq A \|f\|_p \|g\|_{q'}.$$

We may limit ourselves to non-negative f and g , without loss of generality. We now divide the product space $E^n \times E^n$ into three disjoint regions R_1 , R_2 , R_3 given as follows:

$$R_1 = \{(x, y); \tfrac{1}{2} |y| < |x| < 2 |y|\},$$

$$R_2 = \{(x, y); |x| \leq \tfrac{1}{2} |y|\},$$

$$R_3 = \{(x, y); |y| \leq \tfrac{1}{2} |x|\}.$$

We set

$$I = \int_{E^n} \int_{E^n} \frac{f(y)g(x)}{|x|^\beta |x-y|^\lambda |y|^\alpha} dy dx$$

and

$$I_i = \iint_{R_i} \frac{f(y)g(x)}{|x|^\beta |x-y|^\lambda |y|^\alpha} dy dx, \quad i = 1, 2, 3.$$

Thus $I = I_1 + I_2 + I_3$. It is therefore sufficient to show that

$$(3.2) \quad I_i \leq A \|f\|_p \|g\|_{q'}, \quad i = 1, 2, 3.$$

We consider I_1 first.

Since I_1 is taken over R_1 , $\tfrac{1}{2} |y| < |x| < 2 |y|$ in this case. Therefore $|x-y| < 3 |x|$. Now, by the hypothesis of the theorem, $\alpha + \beta \geq 0$. Thus we obtain

$$|x-y|^{\alpha+\beta} < 3^{\alpha+\beta} \cdot |x|^{\alpha+\beta} < 3^{\alpha+\beta} \cdot 2^{|\beta|} \cdot |x|^\alpha \cdot |y|^\beta.$$

From this it follows that

$$\begin{aligned} \iint_{R_1} \frac{f(y)g(x)}{|x|^\beta |x-y|^\lambda |y|^\alpha} dy dx &\leq A \iint_{R_1} \frac{f(y)g(x)}{|x-y|^{\lambda+\alpha+\beta}} dy dx \\ &\leq A \int_{E^n} \int_{E^n} \frac{f(y)g(x)}{|x-y|^{\lambda+\alpha+\beta}} dy dx. \end{aligned}$$

Stating this another way,

$$(3.3) \quad I_1 \leq A \int_{E^n} \int_{E^n} \frac{f(y)g(x)}{|x-y|^\sigma} dy dx, \quad \text{where } \sigma = \alpha + \beta + \lambda.$$

Since $0 < \lambda$, and $\alpha + \beta \geq 0$, then $0 < \sigma$. Furthermore, since $1/q = (1/p) + [(\lambda + \alpha + \beta)/n]^{-1}$ and $p < q$, $\sigma = \lambda + \alpha + \beta < n$. Thus $0 < \sigma < n$. We may therefore apply Theorem A* to T_σ , and obtain, from (3.3), (3.4), $I_1 \leq A \|f\|_p \cdot \|g\|_{q'}$, and the desired estimate for I_1 is obtained.

For the estimates of I_2 and I_3 we shall need the following lemma.

Lemma (3.5). *Let*

$$V_\delta f = |x|^{-n+\delta} \int_{|y| < |x|} |y|^{-\delta} f(y) dy$$

and $\delta < n/p'$. Then

$$(a) \quad \|V_\delta f\|_p \leq A \|f\|_p$$

and

$$(b) \quad |V_\delta f| \leq A |x|^{-n/p} \|f\|_p .^6$$

We notice that $V_\delta(f) = \int_{E^n} K(|x|, |y|) f(y) dy$ where

$$K(u, v) = \begin{cases} u^{-n+\delta} \cdot v^{-\delta} & \text{if } v < u \\ 0 & \text{otherwise} \end{cases}$$

for all (u, v) with $u, v \geq 0$. Since K is homogeneous of degree $-n$, Lemma (2.1) is applicable. Thus

$$\|V_\delta f\|_p \leq A \|f\|_p$$

whenever $\int_0^1 t^{-\delta} t^{(n/p')-1} dt < \infty$, which holds provided $\delta < n/p'$. Thus part (a) is proved. To prove part (b), we observe that

$$|V_\delta f| \leq |x|^{-n+\delta} \int_{|y| < |x|} |y|^{-\delta} |f(y)| dy \leq |x|^{-n+\delta} \left\{ \int_{|y| < |x|} |y|^{-\delta p'} dy \right\}^{1/p'} \|f\|_p ,$$

by Hölder's inequality.

Since

$$\left\{ \int_{|y| < |x|} |y|^{-\delta p'} dy \right\}^{1/p'} \leq A |x|^{n-\delta-(n/p')} ,$$

part (b) follows and the proof of the lemma is complete.

Now let us examine I_2 .

$$I_2 = \iint_{R_2} \frac{f(y)g(x)}{|x|^\beta |x-y|^\lambda |y|^\alpha} dy dx .$$

In R_2 , $|x| < \frac{1}{2} |y|$; thus $|x-y|^{-\lambda} \leq 2^\lambda |y|^{-\lambda}$. Therefore

$$\begin{aligned} I_2 &\leq A \iint_{R_2} \frac{f(y)g(x)}{|x|^\beta |y|^{\alpha+\lambda}} dy dx = A \iint_{\{|x| < |y|/2\}} \frac{f(y)g(x)}{|x|^\beta |y|^{\alpha+\lambda}} dy dx \\ &\leq A \int_{E^n} |y|^{-\alpha-\lambda} f(y) \left\{ \int_{|x| < |y|} g(x) |x|^{-\beta} dx \right\} dy . \end{aligned}$$

Thus

$$\begin{aligned} I_2 &\leq A \int_{E^n} f(y) \left\{ |y|^{-\alpha-\lambda} \int_{|x| < |y|} g(x) |x|^{-\beta} dx \right\} dy \\ &= A \int_{E^n} f(y) |y|^{n-\alpha-\beta-\lambda} V_\beta(g) dy . \end{aligned}$$

⁶This theorem generalizes an inequality of HARDY, when $n = 1$, which may be found in A. ZYGMUND, [7], on page 72.

Hölder's inequality therefore yields

$$(3.6) \quad I_2 \leq A \|f\|_p \|(V_\beta g) |y|^{n-\beta-\alpha-\lambda}\|_p.$$

We wish now to estimate

$$\|(V_\beta g) |y|^{n-\alpha-\beta-\lambda}\|_p.$$

We therefore write

$$\int_{E^n} \{[V_\beta g] |y|^{n-\alpha-\beta-\lambda}\}^{p'} dy = \int_{E^n} [V_\beta g]^{q'} [V_\beta g]^{p'-q'} \{ |y|^{n-\alpha-\beta-\lambda} \}^{p'} dy.$$

Since $1 < p < q < \infty$, then $1 < q' < p' < \infty$, and $p' - q' > 0$. Using Lemma (3.5), part (b) we obtain

$$V_\beta g \leq A |y|^{-n/q'} \|g\|_{q'},$$

since $\beta < n/q$. Therefore

$$[V_\beta g]^{p'-q'} \{ |y|^{n-\alpha-\beta-\lambda} \}^{p'} \leq A \|g\|_{q'}^{p'-q'} |y|^{-n/p(p'-q') + (n-\alpha-\beta-\lambda)p'}.$$

But it is easy to check that $-(n/p)(p' - q') + (n - \alpha - \beta - \lambda)p' = 0$. We thus obtain

$$\int \{[V_\beta g] |y|^{n-\alpha-\beta-\lambda}\}^{p'} dy \leq A \left\{ \int [V_\beta g]^{q'} dy \right\} \|g\|_{q'}^{p'-q'}.$$

If we use part (a) of Lemma (3.5) we then have

$$(3.7) \quad \int \{[V_\beta g] |y|^{n-\alpha-\beta-\lambda}\}^{p'} dy \leq A \|g\|_{q'}^{p'}.$$

When we combine (3.7) and (3.6) we obtain

$$(3.8) \quad I_2 \leq A \|f\|_p \|g\|_{q'}.$$

A similar argument proves

$$(3.9) \quad I_3 \leq A \|f\|_p \|g\|_{q'},$$

and the proof of the theorem is complete.

APPENDIX

It remains to verify inequality (2.8). We do this by observing, [1], p. 242, that for $t \geq 0$

$$\frac{1-t^2}{\sigma(\Sigma)} \int_{\Sigma} \frac{d\sigma}{\Delta^n} = \text{signum}(1-t) \quad (t \neq 1),$$

since $(1-t^2)/\sigma(\Sigma)\Delta^n$ is the Poisson kernel of the unit sphere Σ in E^n .

Now an application of Jensen's inequality yields the desired result.

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