*The Method of Characteristics for a Perfect Compressible Fluid in General Relativity and Non-steady Newtonian Mechanics**

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Communicated by B. HOFFMANN

1. Introduction. In the case of the non-steady flow of a compressible fluid in Newtonian mechanics, the known expression for the characteristic partial differential equation (the first of the characteristic relations) does not lead to any simple geometric relation between the three-dimensional velocity vector and the vectors of the normal and bicharacteristic cones. This is so whether we consider these cones as lying in a three-dimensional Euclidean space or as lying in a four-dimensional Euclidean space-time manifold [1]. It should be noted that in the steady case the velocity vector lies along the axis of these cones which are right circular cones [2]. The difficulty in the non-steady case is that the velocity vector lies in a three-dimensional space but the generators of the above cones lie in a four-dimensional space. Another way of stating this result is that Newtonian mechanics does not possess space-time invariance. Thus it appears desirable to determine the first characteristic relation and then the complete characteristic system in the theory of general relativity.

In addition to the value of the characteristic system of general relativity in studying the motions of compressible fluids in astrophysical problems, such a characteristic system can also be used to study motions of compressible fluids in Newtonian mechanics. This can be done in two ways by use of the usual approximation method for going from relativity theory to Newtonian mechanics. First, we can determine the characteristic system for the Newtonian case. Second, the reduction of solutions of the characteristic system in the relativistic case to the solutions in the Newtonian case can be studied. We shall develop the first method in this paper.

In the present paper we shall proceed as follows in order to obtain the character-

* This work was done under a grant of the National Science Foundation, administered by the Engineering Research Institute of the University of Michigan.

istic system in the relativistic case. First, by use of a previously developed technique [1], we determine the discontinuity manifolds for the discontinuities in the first derivatives of the density and the unit four-vector of a world line which enter the general relativistic form of the equations of motion and continuity of an isentropic perfect fluid [3]. We show that the above unit four-vector of a world line lies along the common axis of the normal and bicharacteristic three-dimensional cones and that these cones are right "circular" cones.

Then we study the geometric relations among the above unit four-vector, two distinct vectors of the normal cone, their corresponding bicharacteristics, and two vectors each of which is orthogonal to both of the above normal vectors. Since the unit four-vector of a world line is time-like, it follows that all vectors of the normal cone are space-like and all vectors of the bicharacteristic cone are time-like. Further, it is shown that at each point any two normal vectors and their corresponding bicharacteristic vectors lie in a local three-dimensional subspace of the four-dimensional space-time manifold of general relativity. The first of the above vectors, which are orthogonal to the two normal vectors, is chosen to be orthogonal to this local three-dimensional subspace. The additional vector, which is orthogonal to two definite vectors of the normal cone, lies in the above local three-space. The relations between these vectors furnish *the first set of characteristic relations*. By expressing the equations of motion and continuity in terms of directional derivatives along the above vectors, we obtain *the second set of characteristic relations*. Finally, we show by use of a method similar to that used by E. W. TIRT [4], for the general three-dimensional hyperbolic partial differential equation, that a class of coordinate systems exists in space-time such that the coordinate curve directions at any point coincide with two bicharacteristic directions and two directions both of which are orthogonal to the corresponding normals. In other words, the coordinate curves consist of two families of bicharacteristic curves and two families of curves which lie in the intersection of two families of characteristic surfaces. The latter are coordinate surfaces. Thus directional derivatives can be transformed into partial derivatives and the complete characteristic system for the relativistic case can be determined.

Finally, we shall reduce the characteristic system obtained for the relativistic case to that for the non-steady flow of a compressible fluid in Newtonian mechanics by use of the customary first approximation technique. First, we show that the relativistic jumps and characteristic partial differential equation reduce to the well known jumps and characteristic equation for the non-steady flow of a compressible fluid in the Newtonian case. Second, we obtain *the complete characteristic system for the non-steady rotational isentropic flow of a compressible fluid*. An existence theorem for a special case is verified.

In concluding this introduction we list some recent related developments in the non-linear theory. The steady, rotational, three-dimensional case has recently been studied by M. HOLT [5], who expanded the theory to take account of rotational effects and entropy changes in the method of characteristics. HOLT

also proposed a finite difference scheme whereby three-dimensional steady flows can be computed. In addition, G. C. McVITTIE obtained a class of solutions of the Newtonian equations of motion of a compressible, rotational, non-isentropic fluid by studying special solutions of the relativistic equations [6].

In the present paper the fluid does not create the gravitational field. That is, the metric tensor does not depend upon the velocity and density of the fluid. S. O'BRIEN & J. L. SYNGE [7], A. LICHTNEROWICZ [8], and A. H. TAUB [9] have recently studied the discontinuity manifolds of a perfect compressible fluid which creates the gravitational field. Further, TAUB [10] has discussed some interesting methods of solution of the field equations when the fluid creates a plane symmetric space-time.

2. The Discontinuity Relations and the Characteristic Manifolds in the Relativistic Case. Let x^i ($j = 0, 1, 2, 3$) denote a curvilinear coordinate system in the space-time of general relativity. If g_{ij} is the metric tensor, then the indefinite form

$$(2.1) \quad \sigma ds^2 = g_{ij} dx^i dx^j$$

has the signature $(+, -, -, -)$. This means that at any point (2.1) can be reduced to

$$(2.2) \quad \sigma ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

The differential ds is called the element of arc. The scalar σ is plus one when the differentials dx^i determine a so-called time-like direction; σ is minus one when dx^i determine a space-like direction. The sign of σ is such that $(ds)^2$ is always positive.

Next we introduce the customary quantities associated with a perfect compressible fluid in general relativity [3]. These are

$$(2.3) \quad \begin{aligned} \lambda^i &= \frac{dx^i}{ds} = \text{the unit four-vector along a world line,} \\ c &= \text{the speed of light,} \\ a &= \text{the local speed of sound,} \\ \rho &= \text{the specific density,} \\ \epsilon &= c^2 \rho = \text{the specific energy,} \\ p &= \text{the pressure,} \\ S &= \text{the specific entropy.} \end{aligned}$$

From (2.1) we see that the unit four-vector λ^i of a world line satisfies the relation

$$(2.4) \quad g_{ij} \lambda^i \lambda^j = 1.$$

This follows from the fact that λ^i is time-like and $\lambda^i = dx^i/ds$.

Finally, we introduce the symmetric energy tensor

$$(2.5) \quad T^{ik} = \epsilon \lambda^i \lambda^k - p g^{ik}.$$

If ∇_i denotes covariant differentiation, then the *equations of motion and continuity* in relativistic form are [3]

$$(2.6) \quad \nabla_i T^{ik} = 0.$$

When the fluid is a polytropic gas, then p, ρ are related by

$$(2.7) \quad p = K\gamma^{-1}\rho^\gamma,$$

where γ is the ratio of the specific heats of the gas (a constant) and K is a known function of the entropy (a constant). The four equations (2.6) plus the two equations (2.4), (2.7) are six equations for the six unknowns λ^i, ρ, p . Note $a^2 = dp/d\rho$.

Now we shall study the manifolds

$$(2.8) \quad \phi(x^i) = \text{constant},$$

along which the first derivatives of λ^i and ρ are discontinuous but λ^i, ρ are continuous. Let us denote the jump of the quantity α by

$$(2.9) \quad [\alpha] = \alpha_2 - \alpha_1$$

where α_2, α_1 are the values of α on the two sides of the surface $\phi(x^i) = \text{constant}$. By use of our previously developed theory [11] for determining the discontinuity manifolds (2.8), we find from (2.4), (2.6)

$$(2.10) \quad [\nabla_k T^{ki}] = 0,$$

$$(2.11) \quad \lambda_i [\nabla_k \lambda^i] = 0,$$

$$(2.12) \quad [\nabla_i T^{ik}] \frac{\partial \phi}{\partial x^k} = 0.$$

We note that the jump in the derivative of the product $\alpha\beta$ can be expanded as follows:

$$(2.13) \quad [\nabla_k (\alpha\beta)] = \alpha[\nabla_k \beta] + \beta[\nabla_k \alpha].$$

By substituting (2.5) into (2.12) and expanding by use of (2.13), we obtain, upon replacing ϵ by $c^2\rho$,

$$(2.14) \quad (Lc^2\lambda^i - a^2\phi^i)[\nabla_i \rho] + c^2\rho(\lambda^i\phi_k[\nabla_i \lambda^k] + L[\nabla_i \lambda^i]) = 0$$

where

$$(2.15) \quad \phi_k = \frac{\partial \phi}{\partial x^k}, \quad \phi^i = g^{ik}\phi_k, \quad L = \lambda^k\phi_k.$$

Forming the scalar product of (2.14) with λ_i and using (2.4), (2.11), we obtain

$$(2.16) \quad \rho c^2 \phi_k [\nabla_i \lambda^k] = (a^2 - c^2)L[\nabla_i \rho].$$

Eliminating $\phi_k[\nabla_i \lambda^k]$ in (2.14) by use of (2.16), we find that (2.14) furnishes

$$(2.17) \quad \rho c^2 L[\nabla_i \lambda^k] = a^2(\phi^k - L\lambda^k)[\nabla_i \rho].$$

Multiplying (2.17) by ϕ_k and (2.16) by L and equating the right hand sides of the resulting equations, we obtain *the characteristic partial differential equation (the first characteristic relation) in the relativistic case*,

$$(2.18) \quad (c^2 - 2a^2)L^2 + a^2 g^{ik} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^k} = 0,$$

where $[\nabla_i \rho] \neq 0$ and

$$(2.19) \quad L = \lambda^i \frac{\partial \phi}{\partial x^i}.$$

We shall now consider the significance of the relativistic characteristic relation (2.18). At each point of space-time, equation (2.18) defines a cone (more properly, a hypercone) of normals, the vectors $\partial \phi / \partial x^k$, to possible characteristic surfaces (more properly, hypersurfaces). In the present paper we consider the case

$$c^2 > 3a^2.$$

From (2.18) and the above, we find that $g^{ik} (\partial \phi / \partial x^i) (\partial \phi / \partial x^k) < 0$.

Thus the unit vector n_i along $\partial \phi / \partial x^i$ is

$$(2.20) \quad n_i = \frac{\frac{\partial \phi}{\partial x^i}}{\left(-g^{ik} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^k}\right)^{\frac{1}{2}}}.$$

Since (2.20) leads to $g^{ik} n_i n_k = -1$, *the vectors of the normal cone are space-like*. Further, (2.18) may be expressed by

$$(2.21) \quad \lambda^i n_i = \pm \frac{a}{(c^2 - 2a^2)^{\frac{1}{2}}},$$

or, as in the Newtonian case, we can say that *the unit time-like four-vector λ^i lies along the axis of the "right circular" normal cone*. It should be noted that this statement constitutes a geometric interpretation of (2.21). For the indefinite metric (2.1) angle does not exist. Thus $\lambda^i n_i$ is an invariant under those space-time coordinate transformations for which $(ds)^2$ of (2.1) is an invariant but $\lambda^i n_i$ is not necessarily the product of the magnitudes of λ_i , n_i and the cosine of some angle. No additional relation of the same type as (2.21), that is $\lambda^i n_i = R$, can exist when the characteristic surfaces are three-dimensional (as we assume to be the case). It is in this sense that λ^i is said to be the axis of the normal cone.

In order to obtain expressions for the jumps $[\nabla_i \rho]$, $[\nabla_i \lambda_k]$, we proceed as in the Newtonian case [7]. Let us write

$$(2.22) \quad [\nabla_i \rho] = \mu_i$$

where μ_i is an unknown vector. From (2.17) we find

$$(2.23) \quad [\nabla_i \lambda_k] = \frac{a^2}{\rho c^2 L} \left(\frac{\partial \phi}{\partial x^k} - L \lambda_k \right) \mu_i.$$

By use of (2.23), (2.19), (2.4), we see that (2.11) is identically satisfied. From (2.10) we find, by use of (2.22), (2.23), (2.18),

$$(2.24) \quad \left\{ L\mu^k \frac{\partial\phi}{\partial x^k} - (\lambda^i \mu_i) g^{pk} \frac{\partial\phi}{\partial x^p} \frac{\partial\phi}{\partial x^k} \right\} \lambda_i - L^2 \mu_i + L\lambda^k \mu_k \frac{\partial\phi}{\partial x^i} = 0.$$

If we assume that

$$(2.25) \quad \mu_k = A \frac{\partial\phi}{\partial x^k}$$

where A is an arbitrary scalar then (2.24) is *identically satisfied*. The most general expression for μ_k is

$$(2.26) \quad \mu_k = A \frac{\partial\phi}{\partial x^k} + B\tau_k$$

where B is a scalar and the unit vector τ_k is orthogonal to $\partial\phi/\partial x^k$. We shall prove that B is zero. If B is not zero and (2.26) is substituted into (2.24), we obtain

$$(2.27) \quad B \left(g^{pi} \frac{\partial\phi}{\partial x^p} \frac{\partial\phi}{\partial x^i} \right) (\lambda_k \tau^k) \lambda_i + L^2 \mu_i - L(\lambda^k \mu_k) \frac{\partial\phi}{\partial x^i} = 0.$$

Forming the scalar product of (2.27) with τ_i , we find

$$(2.28) \quad \left(g^{pi} \frac{\partial\phi}{\partial x^p} \frac{\partial\phi}{\partial x^i} \right) (\lambda_k \tau^k)^2 + L^2 \sigma = 0$$

where $\sigma = \tau^k \tau_k$. By use of (2.19), (2.20), the relation (2.28) becomes

$$(2.29) \quad (\lambda^k \tau_k)^2 = \sigma (\lambda^k n_k)^2.$$

From (2.29) we see that two conclusions follow. First, since σ must be positive, τ_k is time-like. Second, (2.29) shows that τ_k must belong to the normal cone. This is impossible. Hence B must vanish and (2.26) reduces to (2.25) or

$$(2.30) \quad [\nabla_k \rho] = A \frac{\partial\phi}{\partial x^k}.$$

The equation (2.23) leads to

$$(2.31) \quad [\nabla_i \lambda_k] = \frac{a^2 A}{\rho c^2 L} \frac{\partial\phi}{\partial x^i} \left(\frac{\partial\phi}{\partial x^k} - L\lambda_k \right).$$

3. The Basic Vector Fields of the Theory. We have introduced the unit time-like four-vector λ_i of a world line (see 2.4) and the unit space-like vectors n_i of the normal cone (see 2.20). If μ is a parameter along a bicharacteristic, then the differential equations of the bicharacteristics are [12]

$$(3.1) \quad \frac{dx^i}{d\mu} = (c^2 - 2a^2)L\lambda^i + a^2 g^{ik} \frac{\partial\phi}{\partial x^k}.$$

Forming the scalar product of (3.1) with $\partial\phi/\partial x^i$ and using (2.18), we see that $dx^i/d\mu$ is orthogonal to n_i . By use of (2.19), (2.20), and the positive nappe of the normal cone (see 2.21), the equation (3.1) becomes

$$(3.2) \quad \frac{dx^i}{d\mu} = a \left(-g^{ik} \frac{\partial\phi}{\partial x^i} \frac{\partial\phi}{\partial x^k} \right)^{\frac{1}{2}} [(c^2 - 2a^2)^{\frac{1}{2}} \lambda^i + an^i].$$

If we introduce the unit vector t^i along a bicharacteristic, then (3.2) may be written as

$$(3.3) \quad \beta t^i = (c^2 - 2a^2)^{\frac{1}{2}} \lambda^i + an^i$$

where β is a scalar, whose value is determined by forming the scalar product of (3.3) with itself. We find, by use of (2.21) and the fact that λ^i is time-like and n^i is space-like,

$$(3.4) \quad \sigma \beta^2 = c^2 - a^2$$

where

$$(3.5) \quad t^i t_i = \sigma.$$

Since $c^2 > a^2$, we see from (3.4) that $\sigma > 0$ or the vectors t^i are time-like. Further, (3.3) becomes, by use of (3.4),

$$(3.6) \quad \lambda^i = B t^i + C n^i$$

(which is true for *any* normal vector and its corresponding bicharacteristic) where

$$(3.7) \quad B = \left(\frac{c^2 - a^2}{c^2 - 2a^2} \right)^{\frac{1}{2}}, \quad C = \frac{a}{(c^2 - 2a^2)^{\frac{1}{2}}}.$$

Since t^i is orthogonal to n^i , we find, by forming the scalar product of (3.6) with t^i ,

$$(3.8) \quad \lambda^i t_i = B.$$

Hence the bicharacteristics form a cone of time-like vectors, with axis along λ^i . From (3.7) we find, since we are assuming $c^2 > 3a^2$, $a > 0$,

$$(3.9) \quad B > 1. \quad 0 < C < 1.$$

Since the normal cone consists of space-like vectors, the null cone separates these two cones at each point P . It should be noted that the vector λ^i is time-like and hence this vector and the vectors of the cone of bicharacteristics appear to lie "within" the null cone, whereas the vectors of the cone of normals lie "outside" of the null cone, in spite of (3.9).

In order to introduce the other vectors of the theory, we consider as given the following data: (1) a three-dimensional hypersurface S_3 ; (2) λ^i , ρ , and the sound speed a on S_3 . On this given hypersurface S_3 the normal and bicharacteristic cones are known at each point P and are given by (2.21) and (3.8). Further,

we assume that we can choose a family of ∞^1 two-dimensional surfaces S_2 lying in the given surface S_3 and such that the two conditions listed below are satisfied. It is understood that a surface S_2 passes through each point P of S_3 . Further, we shall consider two orthogonal families of curves $C_1, 'C_1$ with unit orthogonal tangent vectors $p^i, 'p^i$, respectively, lying in any surfaces S_2 . The two conditions on $C_1, 'C_1$ and S_2 are:

Condition 1. Let $H, 'H$ denote two three-dimensional hyperplanes which are orthogonal to the unit tangent vectors $p^i, 'p^i$ of $C_1, 'C_1$, respectively, at any point P of S_3 . We assume that these two hyperplanes intersect the normal cone at P in two distinct generators $n^i, 'n^i$. Thus both $p^i, 'p^i$ are orthogonal to both $n^i, 'n^i$ at P .

As an example we consider the case where $H, 'H$ are $x^0 = x^1 = 0$ in terms of the local orthogonal Cartesian coordinates x^i at P . The equation of the normal cone reduces to

$$\sum_{i,k=2}^3 \{(c^2 - 2a^2)\lambda_k\lambda_i + a^2 g_{ki}\} x^k x^i = 0.$$

This quadratic equation for x^2/x^3 has real distinct roots if the discriminant is positive.

Condition 2. Associated with $n^i, 'n^i$ are bicharacteristic directions $t^i, 't^i$, respectively. From (3.6) we see that if $n^i, 'n^i$ are real and distinct then $t^i, 't^i$ are real and distinct. We assume that by proper choice of $C_1, 'C_1, S_2$ of S_3 , the $t^i, 't^i$ will not lie on S_3 .

If these conditions are satisfied, then, along S_3 , two characteristic three-dimensional strips are determined. These strips can be extended into characteristic surfaces by proceeding along t^i (and also along $'t^i$) by the strip condition [8]

$$(3.10) \quad \frac{d}{d\mu} \left(\frac{\partial \phi}{\partial x^j} \right) = \frac{1}{2} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^k} \frac{\partial}{\partial x^i} \{(c^2 - 2a^2)\lambda^k \lambda^i + a^2 g^{ki}\}.$$

The two families of characteristic surfaces intersect in the two-dimensional surfaces S_2 . The surfaces S_2 are spanned by $p^i, 'p^i$. The intersections of these characteristic surfaces in *space-time* are two-dimensional surfaces L_2 . We define the orthogonal fields $p^i, 'p^i$ in *space-time* as any two orthogonal fields which span these L_2 and which reduce to the specified fields $p^i, 'p^i$ on S_2 . Both $p^i, 'p^i$ are assumed not to lie on the null cone, and both $p^i, 'p^i$ are orthogonal to both $n^i, 'n^i$ in *space-time*. It should be noted that at any point P of *space-time*, the plane of vectors determined by $p^i, 'p^i$ is orthogonal to the plane of vectors determined by $n^i, 'n^i$. Further, at all points of S_3 , $t^i, 't^i, p^i, 'p^i$ are four independent vectors which span *space-time*. We assume that this condition is satisfied in the neighborhood of S_3 .

Let us introduce the vectors

$$(3.11) \quad \frac{\alpha}{g^{\frac{1}{2}}} e^{ikqr} p_k p_q t_r ; \quad \frac{\beta}{g^{\frac{1}{2}}} e^{ikqr} 'p_k 'p_q t_r$$

where

$$\begin{aligned} e^{ikar} &= \text{the permutation tensor density,} \\ g &= \text{the absolute value of the determinant of the metric tensor,} \\ \alpha, \beta &= \text{two undetermined scalars.} \end{aligned}$$

In view of the skew-symmetry of e^{ikar} , we see that the first vector is orthogonal to $p_i, 'p_i, t_i$ and the second vector is orthogonal to $p_i, 'p_i, 't_i$. Since $p_i, 'p_i, t_i, 't_i$ are independent, the vector orthogonal to $p_i, 'p_i, t_i$ is unique except for magnitude and hence lies along n_i . Similarly, the second vector lies along $'n_i$. Hence we may write for proper choice of α, β

$$(3.12) \quad n^i = \frac{\alpha}{g^{\frac{1}{2}}} e^{ikar} p_k 'p_a t_r$$

$$(3.13) \quad 'n^i = \frac{\beta}{g^{\frac{1}{2}}} e^{ikar} 'p_k p_a 't_r$$

where $p_i, 'p_i$ are two unit orthogonal vector fields which lie in the intersections of the two families of characteristic surfaces.

In the future calculations we encounter the tensor

$$(3.14) \quad e^{ikar} e_{jab} = \delta_{ab}^{kar}.$$

The tensor δ_{ab}^{kar} is the generalized Kronecker delta tensor. This tensor is related to the ordinary Kronecker delta tensor by

$$(3.15) \quad \delta_{ab}^{kar} = \delta_{[a}^k \delta_b^a \delta_{c]}^r.$$

The brackets surrounding the indices a, b, c in the right hand side of (3.15) indicate the alternating sum. Thus for the case of two indices $\delta_{ab}^{ka} = \delta_{[a}^k \delta_{b]}^a = \delta_a^k \delta_b^a - \delta_b^k \delta_a^a$.

Now we introduce the scalar products

$$(3.16) \quad d = p^i t_i, \quad e = p^i 't_i, \quad f = 'p^i t_i, \quad h = 'p^i 't_i, \quad k = t^i t_i.$$

If we form the scalar product of (3.12) with $g^{\frac{1}{2}} e_{jab} t^a 't^b$, we find, by use of (3.14), (3.15), (3.16) and the fact that $t^i, 't^i$ are time-like,

$$(3.17) \quad (k - 1)\alpha(fp_e - d'p_e) = g^{\frac{1}{2}} e_{jab} n^i t^a 't^b.$$

Similarly, from (3.13) we find

$$(3.18) \quad (k - 1)\beta(-fp_e + d'p_e) = g^{\frac{1}{2}} e_{jab} 'n^i 't^a t^b.$$

Since $p_i, 'p_i$ are both orthogonal to $n_i, 'n_i$, we see that (3.17) (or 3.18) leads to

$$(3.19) \quad e_{jab} n^i t^a 't^b n^c = 0.$$

This means that every pair of normal vectors $n^i, 'n^i$ which are space-like and their corresponding bicharacteristics $t^i, 't^i$ (which are time-like vectors) lie in a three-dimensional local subspace of space-time at each point P . This result also follows from (3.6). We choose $'p^i$ to be orthogonal to this subspace at any point P .

This will determine (as we shall show) whether the unit vector $'p^i$ is time-like ($'\sigma = 1$) or space-like ($'\sigma = -1$). Hence, from (3.16), we write

$$(3.20) \quad 'p^i 'p_i = ' \sigma, \quad f = h = 0.$$

The vector p^i is orthogonal to $'p^i$ and hence p^i lies in the above subspace (formed by $n^i, 'n^i, t^i, 't^i$). In fact, p^i lies along the intersection of the two planes orthogonal to $n^i, 'n^i$. We have no criterion as yet for determining whether p^i is time-like ($\sigma = 1$) or space-like ($\sigma = -1$). Hence we write

$$(3.21) \quad p^i p_i = \sigma.$$

The vector t^i is time-like, and $'p^i$ is orthogonal to the local three-dimensional space containing t^i, n^i, p^i . Since the signature of the quadratic form (2.1) is three, *it can be shown that any 4-tuple of vectors (four orthogonal unit vectors) consists of three space-like vectors and one time-like vector.* From this it follows that $'p^i$ is space-like and $'\sigma = -1$.

The proof of the above underlined statement is as follows. Let $'g_{ii}$ ($i, j = 0, 1, 2, 3$) denote the metric coefficients of a quadratic form (2.2) or

$$(3.22) \quad 'g_{00} = 1, \quad 'g_{11} = 'g_{22} = 'g_{33} = -1, \quad 'g_{ii} = 0, \quad (i \neq j).$$

Thus the transformation from the holonomic x^i coordinates of (2.1) to the non-holonomic $'x^i$ coordinates of (2.2) is such that

$$(3.23) \quad 'g_{ii} = g_{kp} \frac{\partial x^k}{\partial 'x^i} \frac{\partial x^p}{\partial 'x^i}.$$

By use of (3.22), it follows that $d 'x^1$ is the arc length along the local coordinate $'x^1 = \text{variable}$, and similarly for the other $'x^i$. From this and the facts that $'g_{ii} = 0, i \neq j$, and (3.23) we see that $\partial x^i / \partial 'x^k$ are the components of *some* 4-tuple of vectors r^i_k . Three of these vectors are space-like; one is time-like.

Conversely, by starting with *any* 4-tuple of vectors r^k_i ,

$$(3.24) \quad \delta_{ii} = g_{kp} r^k_i r^p_i$$

where $\delta_{ii} = 0$ ($i \neq j$), $\delta_{ii} = +1$ or -1 , we can identify (1) r^k_i with $\partial x^k / \partial 'x^i$ where $'x^i$ are non-holonomic and (2) δ_{ii} with $'g_{ii}$. From the law of inertia of quadratic forms, it follows that *three of these vectors are space-like and one is time-like.* Since t^i is time-like, $'p^i$ is space-like or

$$(3.25) \quad 'p^i 'p_i = ' \sigma = -1.$$

4. The First and Second Sets of Characteristic Relations. In order to determine the characteristic system for the relativistic case, two problems must be solved. These same problems arose and were solved in the case of the steady supersonic flow of a compressible fluid in Newtonian mechanics [2]. The first

problem is to determine the geometrical relations among the vectors n^i , $'n^i$, t^i , $'t^i$, p^i , $'p^i$, and λ^i (the so-called *first set of characteristic relations*) and to determine the equations of motion and continuity in terms of directional derivatives of ρ , λ^i along t^i , $'t^i$, p^i , $'p^i$ (the so-called *second set of characteristic relations*).

To determine the relations among the vectors n^i , $'n^i$, t^i , $'t^i$, p^i , $'p^i$ and λ^i , we use (3.6) and find

$$(4.1) \quad Bt^i + Cn^i = B't^i + C'n^i.$$

Replacing n^i by (3.12) and $'n^i$ by (3.13), we obtain from relation (4.1)

$$(4.2) \quad Dt^i + \frac{\alpha}{g^{\frac{1}{2}}} e^{ikar} p_k 'p_a t_r = D't^i + \frac{\beta}{g^{\frac{1}{2}}} e^{ikar} 'p_k p_a 't_r,$$

where

$$(4.3) \quad D = \frac{B}{C}.$$

Forming the scalar product of (4.2) with t_i and then with $'t_i$, p_i , $'p_i$ and finally with n_i , $'n_i$, we obtain, through use of (3.16) and the fact that t_i , $'t_i$ are time-like,

$$(4.4) \quad k = 1 - \frac{\beta}{Dg^{\frac{1}{2}}} e^{ikar} t_i 'p_k p_a 't_r,$$

$$(4.5) \quad k = 1 - \frac{\alpha}{Dg^{\frac{1}{2}}} e^{ikar} 't_i p_k 'p_a t_r,$$

$$(4.6) \quad d = e, \quad f = h = 0,$$

$$(4.7) \quad 't^i n_i = \frac{e^{ikar}}{Dg^{\frac{1}{2}}} (\alpha n_i p_k 'p_a t_r - \beta n_i 'p_k p_a 't_r),$$

$$(4.8) \quad t^i 'n_i = \frac{e^{ikar}}{Dg^{\frac{1}{2}}} (\alpha 'n_i 'p_k p_a 't_r - \beta 'n_i p_k 'p_a t_r).$$

Now we consider some results of the equations (3.12), (3.13) and (4.4)–(4.8). From (4.4), (4.5) and the skew-symmetry of e^{ikar} , it follows that

$$(4.9) \quad \alpha = \beta.$$

Forming the scalar product of (3.12) with itself, we find, through use of the facts that n_i is space-like; t_i is time-like; $p^i p_i = \sigma$, $f = h = 0$, $'p^i 'p_i = ' \sigma = -1$,

$$(4.10) \quad \alpha^2 = \frac{' \sigma}{d^2 - \sigma} = \frac{1}{\sigma - d^2}.$$

From (4.10) we see that *since $'p^i$ is space-like ($' \sigma = -1$) then p^i is time-like ($\sigma = 1$) and*

$$(4.11) \quad p^i p_i = \sigma = 1, \quad |d| < 1.$$

The sign of α depends upon whether the ordered triad n^k, p^k, t^k forms a right or left handed system in the local space orthogonal to $'p^k$. If this system is assumed to be right handed, then, by forming the scalar product of (3.12) with n_i , we find

$$(4.12) \quad \alpha > 0.$$

By substituting (3.12), (3.13) into the right hand sides of (4.7), (4.8) and using (3.5), (3.16), (3.20), (3.25), (4.6), (4.9), (4.10), (4.11), we find

$$(4.13) \quad 't^i n_i = \frac{(1+k) - 2d^2}{(d^2 - 1)D},$$

$$(4.14) \quad 't^i n_i = t^i 'n_i.$$

Finally, if we let $f = 0$ in (3.17) and use this relation to eliminate $'p_i$ in the right hand side of (4.4), we obtain, upon noting that the pairs p_i, n_i and t_i, n_i are orthogonal,

$$(4.15) \quad 't^i n_i = -D(k - 1).$$

Now we determine $n^i, 'n^i, \lambda^i$ in terms of $t^i, 't^i, p^i$. From (4.13), (4.15), we find that D, k, d are related by

$$(4.16) \quad D^2 = \frac{(1+k) - 2d^2}{(1-d^2)(k-1)}.$$

In the steady Newtonian case, a similar relation to (4.16) is valid [13]. To determine n^i in terms of $t^i, 't^i, p^i$, we write (where a_1, a_2, a_3 are undetermined coefficients)

$$n^i = a_1 t^i + a_2 't^i + a_3 p^i.$$

Forming the scalar products of this equation with $t^i, 't^i, p^i$ and using (4.15) and

$$p^i n_i = t^i n_i = 0$$

we obtain, after solving for a_1, a_2, a_3 ,

$$(4.17) \quad \frac{1}{D} (1 + k - 2d^2) n^i = (d^2 - k) t^i + (1 - d^2) 't^i + d(k - 1) p^i.$$

In a similar manner, we find

$$(4.18) \quad \frac{1}{D} (1 + k - 2d^2) 'n^i = (d^2 - k) 't^i + (1 - d^2) t^i + d(k - 1) p^i.$$

Since λ^i, t^i, n^i and B, C, D are related by (see (4.1), (4.3))

$$(4.19) \quad \lambda^i = B t^i + C n^i = B 't^i + C 'n^i, \quad D = \frac{B}{C},$$

we find by use of (4.17) that

$$(4.20a) \quad \frac{1}{B} (1 + k - 2d^2) \lambda^i = (1 - d^2) (t^i + 't^i) + d(k - 1) p^i.$$

Further, we note that $'p^i$ is orthogonal to $p^i, t^i, 't^i$ and (4.6) is valid:

$$(4.20b) \quad 'p^i p_i = 'p^i t_i = 'p^i 't_i = 0,$$

$$(4.20c) \quad p^i t_i = p^i 't_i.$$

The relations (4.17), (4.18), (4.20) are often called *the first set of characteristic relations*. They determine the geometry of the vectors $t^i, 't^i, p^i, 'p^i, \lambda^i$ and are consequences of the characteristic partial differential equation (2.18).

Our next step will be to express the equations of motion (2.5), (2.6) in terms of directional derivatives along $t^i, 't^i, p^i$, and $'p^i$ of ρ , the density, and λ_k , the unit four-vector. *The resulting equations constitute a part of the second set of characteristic relations*. From (2.5), (2.6) we find

$$(4.21) \quad c^2 \rho (\lambda_k \nabla_i \lambda^i + \lambda^i \nabla_i \lambda_k) + c^2 \lambda_k \lambda^i \nabla_i \rho - a^2 \nabla_k \rho = 0.$$

Forming the scalar product of (4.21) with $t^k, 't^k, p^k, 'p^k$, we find, by use of (4.19),

$$(4.22) \quad c^2 \rho (B \nabla_i \lambda^i + t^k \lambda^i \nabla_i \lambda_k) + c^2 B \lambda^i \nabla_i \rho - a^2 t^k \nabla_k \rho = 0,$$

$$(4.23) \quad c^2 \rho (B \nabla_i \lambda^i + 't^k \lambda^i \nabla_i \lambda_k) + c^2 B \lambda^i \nabla_i \rho - a^2 't^k \nabla_k \rho = 0,$$

$$(4.24) \quad c^2 \rho p^k \lambda^i \nabla_i \lambda_k - a^2 p^k \nabla_k \rho = 0,$$

$$(4.25) \quad c^2 \rho 'p^k \lambda^i \nabla_i \lambda_k - a^2 'p^k \nabla_k \rho = 0.$$

The quantities $\lambda^i \nabla_i \lambda_k, \lambda^i \nabla_i \rho$ can be expressed in terms of the directional derivatives $t^i \nabla_i \lambda_k, 't^i \nabla_i \lambda_k, p^i \nabla_i \lambda_k, 'p^i \nabla_i \lambda_k, t^i \nabla_i \rho, 't^i \nabla_i \rho, p^i \nabla_i \rho, 'p^i \nabla_i \rho$ by use of (4.20a).

The term $\nabla_i \lambda^i = g^{ik} \nabla_i \lambda_k$ of (4.22), (4.23) remains to be analyzed. This means that the tensor g^{ik} must be decomposed with respect to the vectors $'p^i, p^i, t^i, n^i$ (or $'p^i, p^i, 't^i, 'n^i$) and then n^i (or $'n^i$) must be eliminated by use of (4.19). Now $'p^i$ is orthogonal to n^i, t^i, p^i and n^i is orthogonal to p^i, t^i at each point. Hence we may write (since $n^i n_i = -1, 'p^i 'p_i = -1$)

$$(4.26) \quad g^{ik} = -'p^i 'p^k - n^i n^k + F t^i t^k + G (t^i p^k + p^i t^k) + J p^i p^k$$

where F, G, J are unknown scalars. Forming the scalar product of (4.26) with $t_i t_k, t_i p_k, p_i p_k$ and using (3.16) and $p^i p_i = 1$, we find, after solving algebraically,

$$(4.27) \quad F = J = \frac{1}{1 - d^2}, \quad G = \frac{-d}{1 - d^2}.$$

By eliminating n^i from (4.26) by use of (4.19), we find

$$(4.28) \quad g^{ik} = -'p^i 'p^k - C^{-2} \lambda^i \lambda^k + (F - D^2) t^i t^k + J p^i p^k + DC^{-1} (t^i \lambda^k + \lambda^i t^k) + G (t^i p^k + p^i t^k).$$

Before computing $g^{ik} \nabla_i \lambda_k$ or $\nabla_i \lambda^i$, we notice one more result. Since λ^i is a unit time-like vector (see 2.4), we find

$$(4.29) \quad \lambda^k \nabla_i \lambda_k = 0.$$

With the aid of (4.28), (4.29), we obtain

$$(4.30) \quad g^{ik} \nabla_i \lambda_k = -'p^k 'p^i \nabla_i \lambda_k + (F - D^2) t^k t^i \nabla_i \lambda_k + J p^k p^i \nabla_i \lambda_k \\ + DC^{-1} t^k \lambda^i \nabla_i \lambda_k + G(p^k t^i \nabla_i \lambda_k + t^k p^i \nabla_i \lambda_k).$$

Evidently, a similar relation to (4.30) is valid with $'t^i$ replacing t^i . The term $\lambda^i \nabla_i \lambda_k$ of (4.30) can be expressed in terms of $t^i \nabla_i \lambda_k$, $'t^i \nabla_i \lambda_k$, $p^i \nabla_i \lambda_k$ by use of (4.20a). An alternative procedure would be to introduce the vorticity tensor

$$(4.31) \quad 2\omega_{ik} = \nabla_i \lambda_k - \nabla_k \lambda_i$$

and, by use of (4.20a) and (4.31), to find

$$(4.32) \quad \lambda^i \nabla_i \lambda_k = 2\lambda^i \omega_{ik} = \frac{2B\omega_{ik}}{1+k-2d^2} [(1-d^2)(t^i + 't^i) + d(k-1)p^i].$$

Of course, *then we must add the integrability conditions of (4.31) to complete the second set of characteristic relations.* These are [14]

$$(4.33) \quad \nabla_{[p} \omega_{ik]} = 0$$

where the brackets indicate alternation (see (3.15)). In the particular case of irrotational motion,

$$(4.34) \quad \omega_{ik} = 0, \quad \nabla_i \lambda_k = \nabla_k \lambda_i.$$

Then, by use of (4.31), (4.34), we find

$$(4.35) \quad \lambda^i \nabla_i \lambda_k = \lambda^i (2\omega_{ik} + \nabla_k \lambda_i) = 0.$$

Now we shall express (4.22), (4.23), (4.24), (4.25) in terms of the directional derivatives along t^i , $'t^i$, p^i , $'p^i$ and obtain *the first part of the second set of characteristic relations.* By substituting (4.30), (4.32) into (4.22)–(4.25), we find

$$(4.36) \quad -B 'p^k 'p^i \nabla_i \lambda_k + B[(F - D^2)t^k + Gp^k] t^i \nabla_i \lambda_k \\ + B(Gt^k + Jp^k) p^i \nabla_i \lambda_k + 2 \frac{(D^2 + 1)B}{1+k-2d^2} \omega_{ik} t^k [(1-d^2)'t^i \\ + d(k-1)p^i] + B^2 \frac{[(1-d^2)(t^i + 't^i) + d(k-1)p^i]}{1+k-2d^2} \nabla_i \ln \rho \\ - \frac{a^2}{c^2} t^i \nabla_i \ln \rho = 0,$$

$$(4.37) \quad -B 'p^k 'p^i \nabla_i \lambda_k + B[(F - D^2)'t^k + Gp^k] 't^i \nabla_i \lambda_k \\ + B(G't^k + Jp^k) p^i \nabla_i \lambda_k + \frac{2(D^2 + 1)B}{1+k-2d^2} \omega_{ik} 't^k [(1-d^2)t^i \\ + d(k-1)p^i] + B^2 \frac{[(1-d^2)(t^i + 't^i) + d(k-1)p^i]}{1+k-2d^2} \nabla_i \ln \rho \\ - \frac{a^2}{c^2} 't^i \nabla_i \ln \rho = 0,$$

$$(4.38) \quad \frac{2(1-d^2)B\omega_{jk}}{1+k-2d^2} p^k (t^i + {}'t^i) - \frac{a^2}{c^2} p^k \nabla_k \ln \rho = 0,$$

$$(4.39) \quad \frac{2B\omega_{jk} {}'p^k}{1+k-2d^2} [(1-d^2)(t^i + {}'t^i) + d(k-1)p^i] - \frac{a^2}{c^2} {}'p^k \nabla_k \ln \rho = 0.$$

The remaining equations of the second set of characteristic relations are the ten integrability equations (4.31), (4.33)

$$(4.40) \quad \begin{aligned} t^i {}'t^k (2\omega_{jk} - \nabla_j \lambda_k + \nabla_k \lambda_j) &= 0, & t^i p^k (2\omega_{jk} - \nabla_j \lambda_k + \nabla_k \lambda_j) &= 0, \\ t^i {}'p^k (2\omega_{jk} - \nabla_j \lambda_k + \nabla_k \lambda_j) &= 0, & {}'t^i p^k (2\omega_{jk} - \nabla_j \lambda_k + \nabla_k \lambda_j) &= 0, \\ {}'t^i {}'p^k (2\omega_{jk} - \nabla_j \lambda_k + \nabla_k \lambda_j) &= 0, & {}'p^i p^k (2\omega_{jk} - \nabla_j \lambda_k + \nabla_k \lambda_j) &= 0, \end{aligned}$$

$$(4.41) \quad \begin{aligned} p^i {}'p^k t^a \nabla_{[j} \omega_{kq]} &= 0, & p^i {}'p^k {}'t^a \nabla_{[j} \omega_{kq]} &= 0, \\ p^i t^k {}'t^a \nabla_{[j} \omega_{kq]} &= 0, & {}'p^i t^k {}'t^a \nabla_{[j} \omega_{kq]} &= 0, \end{aligned}$$

and the relation $\lambda^i \lambda_i = 1$ or the relations (4.29) which imply that λ_i is a vector of constant magnitude. By use of (4.20), (4.32), these are

$$(4.42) \quad [(1-d^2)(t^i + {}'t^i) + d(k-1)p^i](\nabla_i \lambda_k - 2\omega_{ik}) = 0.$$

Thus equations (4.36)–(4.42) are a system of eighteen equations in eleven unknowns ($\lambda_k, \rho, \omega_{jk}$).

Let us consider the range of permissible values of k . In (4.11) we found that $|d| < 1$. From (4.10), (4.16) we obtain

$$(4.43) \quad D^2 = \frac{(1+k) - 2d^2}{(1-d^2)(k-1)} = -\frac{[(1+k) - 2d^2]\alpha^2}{(1-k)}$$

where

$$d = p^i t_i = p^i {}'t_i, \quad k = t^i {}'t_i.$$

From (4.43) we see that if $k < 1$, then $(1+k) < 2d^2$; if $k > 1$, then $2d^2 < (1+k)$. Since these results are consistent with $|d| < 1$, all values of k except $k = 1$ are permissible.

Finally, it should be noted that the relation (4.16) (or the equation (4.43)) does not furnish a new relation between d, k, a, c . This can be seen as follows. In (3.7) we defined B^2, C^2 , and since $D^2 = B^2/C^2$ (see (4.3)),

$$(4.44) \quad \frac{1}{B^2} = 1 - \frac{1}{D^2}.$$

If $1/D^2$ is obtained from (4.43), and if $1/B^2$ is obtained by forming the magnitude of equation (4.20a), we find from (4.44)

$$(4.45) \quad \begin{aligned} 2(1-d^2)[(1-d^2)(k+1) + 2d^2(k-1) + d^2(k-1)^2] \\ = [(1+k) - 2d^2][(1+k) - 2d^2 - (1-d^2)(k-1)]. \end{aligned}$$

Equation (4.45) is an identity in d^2 . Thus (4.20a) implies (4.16).

The theory of sections 2, 3, 4 is valid when gravitational and electro-magnetic phenomena occur. The gravitational phenomena are taken into account in the structure of the metric tensor; electro-magnetic phenomena are accounted for by replacing $-pg^{ij}$ by a general stress tensor.

In concluding this section we summarize our results. Our basic problem is to determine a quasi-linear system of *necessary conditions* in terms of directional derivatives along bicharacteristic and related directions (the so-called characteristic variables) which are consequences of the flow equations in the non-steady case. These equations—the characteristic system—must consist of *three types* of relations: first, a set of equations defining the bicharacteristic and related directions (see (4.20)); second, the quasi-linear flow equations expressed in terms of the characteristic variables (see (4.36)–(4.39)); and, finally, the so-called “strip” conditions, which are integrability conditions (see (4.40) and (4.41)).

We note that the characteristic system in the case under discussion is over-determined. This is necessarily so, since the characteristic system is obtained in the desired quasi-linear form by reducing the non-linear algebraic equation ($\lambda^i \lambda_i = 1$) to a system of quasi-linear differential equations (4.42) and, further, adding the integrability conditions (4.31), (4.33). It should be noted that in the irrotational case, $\omega_{ij} = 0$, the conditions (4.31) or their equivalent (4.40) are the only integrability conditions.

From the point of view of *sufficient* conditions for a solution of the flow equations, the system of quasi-linear characteristic equations must be shown to (1) possess the same number of *independent* equations as unknown functions and (2) possess a unique solution. Whether or not these conditions are met in the present case remains to be determined from a study of the initial value problem. The characteristic system furnishes a starting point in this study. In the classical two-dimensional steady case, LEWY showed that although the characteristic system was over-determined from the point of view of *necessary* conditions, from the sufficiency point of view this system furnished the same number of independent equations as unknown functions and possessed a unique solution [17].

5. A Class of Coordinate Systems in Space-Time. In this section we shall study the second problem which we mentioned in the first paragraph of section 4. We shall show that a class of coordinate systems can be introduced in space-time for which the directions $t^i, 't^i, p^i, 'p^i$ determine coordinate curves. Two steps will be needed to verify this result. First, by use of a technique due to E. W. TRIT [4], we shall show that coordinate systems exist in which the coordinate curves lie along $t^i, 't^i$ and two directions in the plane of $p^i, 'p^i$. Then we shall show that the last two directions may be selected to be $p^i, 'p^i$.

By integrating the differential equations of the bicharacteristics (3.1) and the strip conditions (3.10), we obtain the equations of the two families of characteristic surfaces

$$(5.1) \quad x^i = x^i(\mu, x_0^i, n_0^i),$$

$$(5.2) \quad x^i = 'x^i(' \mu, x_0^i, 'n_0^i)$$

where $\mu, ' \mu$ are parameters along $t^i, 't^i$, respectively, and $x_0^i, n_0^i, 'n_0^i$ are the values of the coordinates and normal vectors of the two characteristic surfaces along a curve C_1 or $'C_1$ of S_3 .

If u, v, w are a set of parameters for the determination of the given hypersurface S_3 , then we may write

$$(5.3) \quad x_0^i = x_0^i(u, v, w).$$

Similarly, $n_0^i, 'n_0^i$ depend on u, v, w . We shall require that the curves C_1 of S_3 are given by $u = \text{constant}$, $w = \text{constant}$ and the curves $'C_1$ of S_3 are given by $v = \text{constant}$, $w = \text{constant}$. Thus the two-dimensional subspaces S_2 of S_3 which contain $C_1, 'C_1$ are given by $w = \text{constant}$. Further, each of the characteristic surfaces contains these S_2 . That is, if we substitute (5.3) into (5.1) and (5.2) and set $w = \text{constant}$ in the resulting equations, we obtain the equations of both characteristic surfaces. In order to distinguish between these surfaces, we shall replace one of these w 's by $'w$. The equations of the two families of characteristic surfaces become

$$(5.4) \quad x^i = x^i(\mu, u, v, w),$$

$$(5.5) \quad x^i = 'x^i(' \mu, u, v, 'w).$$

Each of these sets of equations determines a one-to-one mapping of space-time in the neighborhood of the given hypersurface S_3 onto the variables μ, u, v, w or $' \mu, u, v, 'w$. This follows from the fact that, by condition 2 of section 3, neither t^i nor $'t^i$ lies on S_3 , and hence the Jacobians

$$J \frac{(x^0, x^1, x^2, x^3)}{(\mu, u, v, w)}, \quad J \frac{(x^0, x^1, x^2, x^3)}{(' \mu, u, v, 'w)}$$

do not vanish.

Now we shall obtain a one-to-one mapping of the space-time neighborhood of S_3 onto the variables $u, v, w, 'w$. By equating two of the equations of (5.4), (5.5), say x^0, x^1 , and solving for $\mu, ' \mu$ in terms of $u, v, w, 'w$, we obtain

$$(5.6) \quad \mu = \mu(u, v, w, 'w),$$

$$(5.7) \quad ' \mu = ' \mu(u, v, w, 'w).$$

In the next paragraph we shall show that we can solve for $\mu, ' \mu$. If we substitute (5.6) into the equations x^2, x^3 of (5.4), we obtain

$$(5.8) \quad x^j = z^j(u, v, w, 'w), \quad j = 2, 3.$$

We complete the mapping by writing

$$(5.9) \quad x^0 = w, \quad x^1 = 'w.$$

Thus the two families of characteristic surfaces $w = \text{constant}$, $'w = \text{constant}$ are coordinate surfaces. The coordinate lines $w = \text{variable}$ (or $u, v, 'w$ all constants) are bicharacteristic curves (see (5.2), (5.5)). Similarly, the coordinate lines $'w = \text{variable}$ are bicharacteristic curves. Finally, the coordinate curves $u = \text{variable}$ lie in the two-dimensional subspaces L_2 , which are the intersections of the characteristic hypersurfaces $w = \text{constant}$, $'w = \text{constant}$. Similarly, the coordinate lines $v = \text{variable}$ lie in these L_2 .

Now we shall show that, by equating some pair of equations in (5.4) and (5.5), we can solve for $\mu, ' \mu$ (see (5.6), (5.7)). Let us assume the contrary. Then, if we let

$$(5.10) \quad F^i = x^i(\mu, u, v, w) - 'x^i(' \mu, u, v, 'w),$$

it follows that *all* the second order Jacobians

$$(5.11) \quad J \frac{(F^i, F^i)}{(\mu, ' \mu)} = \begin{vmatrix} \frac{\partial x^i}{\partial \mu} & -\frac{\partial x^i}{\partial ' \mu} \\ \frac{\partial x^j}{\partial \mu} & -\frac{\partial x^j}{\partial ' \mu} \end{vmatrix}$$

must vanish. Thus the vector t^i coincides with $'t^i$. By condition 2 of section 3, this is impossible. This completes our proof of the existence of the class of coordinate systems described in the last paragraph.

Now we prove that the coordinate system can be chosen so that the coordinate lines lie along $t^i, 't^i, p^i, 'p^i$. The vector p^i lies along the intersection of the local two-spaces orthogonal to $n^i, 'n^i$, respectively. Each of these two-spaces lies within the local three-space spanned by $t^i, 't^i, n^i, 'n^i$ (see (3.19)). The vector $'p^i$ is orthogonal to this local three-space. In the last paragraph we showed that the vectors $t^i, 't^i, q^i, 'q^i$ (where $q^i, 'q^i$ are two vectors orthogonal to both $n^i, 'n^i$) form a coordinate system. Hence $p^i, 'p^i, q^i, 'q^i$ are related by the equations

$$(5.12) \quad \begin{aligned} p^i &= Aq^i + B'q^i, \\ 'p^i &= Cq^i + D't^i, \end{aligned} \quad AD - BC \neq 0,$$

where A, B, C, D are scalar functions of x^i . On the initial hypersurface S_3 and in its neighborhood the vector fields $t^i, 't^i, n^i, 'n^i, p^i, 'p^i$ are given but $q^i, 'q^i$ are to be determined by specifying A, B, C, D properly. In the next few paragraphs we shall determine (1) necessary and sufficient conditions for $t^i, 't^i, p^i, 'p^i$ to form a coordinate system; (2) relations to be satisfied by A, B, C, D in order for the coordinate system of (1) to exist in the neighborhood of S_3 when $t^i, 't^i, q^i, 'q^i$ form a coordinate system.

Now we shall show that the vector fields $t^i, 't^i, p^i, 'p^i$ determine a coordinate system in space-time *if and only if every pair of vector fields of the set $t^i, 't^i, p^i, 'p^i$ forms a family of two-dimensional surfaces*. First, we prove that if the vectors $t^i, 't^i, p^i, 'p^i$ determine a coordinate system, and hence four families of hypersurfaces, then every two vector fields (in particular, say, $t^i, 't^i$) form a family

of two-dimensional surfaces. This follows from the fact that t^i , $'t^i$, p^i and t^i , $'t^i$, $'p^i$ form hypersurfaces and hence satisfy the integrability conditions [15]

$$(5.13) \quad t^i \nabla_i 't^k - 't^i \nabla_i t^k = A_{11}t^i + A_{12}'t^i + A_{13}p^i,$$

$$(5.14) \quad t^i \nabla_i 't^k - 't^i \nabla_i t^k = A_{21}t^i + A_{22}'t^i + A_{23}'p^i$$

(where A_{ii} are functions of x^i) and four similar relations. By equating the right hand sides of (5.13), (5.14) and noting that t^i , $'t^i$, p^i , $'p^i$ are independent, we find

$$(5.15) \quad A_{13} = A_{23} = 0, \quad A_{11} = A_{21}, \quad A_{12} = A_{22}$$

and hence (5.13), (5.14) reduce to the single relation

$$(5.16) \quad t^i \nabla_i 't^k - 't^i \nabla_i t^k = A_{11}t^k + A_{12}'t^k.$$

But (5.16) implies that t^k , $'t^k$ form a family of *two-dimensional surfaces*. Similarly, one can show that every pair of vectors of the set t^i , $'t^i$, p^i , $'p^i$ forms a family of two-dimensional surfaces. Conversely, if every pair of vectors of the set t^i , $'t^i$, p^i , $'p^i$ forms a family of two-dimensional surfaces, then every group of three vectors of the set forms a family of three-dimensional coordinate hypersurfaces. Hence the set of vectors determines a coordinate system.

Next we note that since q^i , t^i and $'q^i$, t^i form families of two-dimensional surfaces, then

$$(5.17) \quad \begin{aligned} q^i \nabla_i t^k - t^i \nabla_i q^k &= A_{31}q^k + A_{32}t^k, \\ 'q^i \nabla_i t^k - t^i \nabla_i 'q^k &= A_{41}'q^k + A_{42}t^k, \end{aligned}$$

where A_{ii} are known functions of A , B , C , D and x^i on S_3 and also in the neighborhood of the initial surface S_3 . Multiplying the first equation by A and the second equation by B and adding, we find, by use of (5.12),

$$(5.18) \quad \begin{aligned} p^i \nabla_i t^k - t^i \nabla_i p^k &= (A_{31}A + t^i \nabla_i A)q^k \\ &+ (A_{41}B + t^i \nabla_i B)'q^k + (A_{32}A + A_{42}B)t^k. \end{aligned}$$

Replacing q^k , $'q^k$ in the right hand side of (5.18) by their expressions in terms of p^k , $'p^k$ (see (5.12)) and setting the coefficient of $'p^k$ equal to zero, we obtain a first-order partial differential equation in the unknowns A , B , C , D . If we multiply the first equation of (5.17) by C and the second equation by D and add, we find that the vector pair p^i , t^i will determine a two-dimensional surface provided another first-order differential equation in the unknown A , B , C , D is satisfied. Similarly, if we consider the integrability relations corresponding to (5.17) for the pairs of vectors q^i , $'t^i$ and $'q^i$, t^i and proceed in the same manner, we find that p^i , $'t^i$ and $'p^i$, t^i form two families of two-dimensional surfaces when two additional differential equations are satisfied. Thus if A , B , C , D satisfy four first-order partial differential equations, then the vector pairs p^i , t^i ; $'p^i$, t^i ; p^i , $'t^i$; and $'p^i$, $'t^i$ form two-dimensional surfaces. Since the vector pair t^i , $'t^i$ determines a family of two-dimensional surfaces in the coordinate system de-

terminated by $t^i, 't^i, q^i, 'q^i$, the vectors $t^i, 't^i$ will determine a family of two-dimensional surfaces when considered as part of the system $t^i, 't^i, p^i, 'p^i$. Finally, we show that $p^i, 'p^i$ form a family of two-dimensional surfaces. By use of (5.12), we find that

$$(5.19) \quad p^i \nabla_i 'p^k - 'p^i \nabla_i p^k = (AD - BC)(q^i \nabla_i 'q^k - 'q^i \nabla_i q^k).$$

Thus the vectors $p^i, 'p^i$ form a family of two-dimensional surfaces. Hence the vectors $p^i, 'p^i, t^i, 't^i$ form a coordinate system in the space-time neighborhood of S_3 when (1) the S_3 curves $C_1, 'C_1$ of (5.3) with tangent vectors $q^i, 'q^i$, respectively, generate a coordinate system and (2) the $q^i, 'q^i$ and $p^i, 'p^i$ are related by (5.12) in space-time and A, B, C, D satisfy four first order partial differential equations. It should be noted that if these conditions are not satisfied, then we can only show that coordinate systems exist with coordinate lines along $t^i, 't^i, q^i, 'q^i$. The theory of section 4 must be altered so that $p^i, 'p^i$ are replaced by linear combinations of $q^i, 'q^i$ (see (5.12)). The resulting computation is rather complicated, in particular, (4.28) becomes very unwieldy. Again, the present system is a direct generalization of the non-steady two-dimensional case. In this case $'p^i$ is orthogonal to a three-dimensional hypersurface which contains $n^i, 'n^i, t^i, 't^i$, and p^i . Further, *in this case a coordinate system always exists with coordinate lines along $t^i, 't^i$ and p^i .*

In addition, we can always determine a local coordinate system along $p^i = q^i, 'p^i = 'q^i$ on the given initial hypersurface S_3 . Evidently, *along this S_3* , a coordinate system exists for which $p^i, 'p^i, t^i, 't^i$ are the directions of coordinate curves. Such a system is analogous to geodesic coordinates along a curve or at a point in Riemannian space. *Since the method of characteristics is primarily used in step by step calculations, such a coordinate system is all that is required in computational work.* Thus the main advantage in determining whether a coordinate system along $t^i, 't^i, p^i, 'p^i$ exists would lie in studying existence and uniqueness of solutions of the characteristic system of partial differential equations.

6. The Characteristic System for the Relativistic Case. Now we introduce a coordinate system with the following coordinate lines: u = variable, along t^i ; v = variable, along $'t^i$; w = variable, along p^i ; z = variable, along $'p^i$. Thus the surfaces u = constant and the surfaces v = constant are two families of characteristic surfaces. The first fundamental form for the u, v, w, z coordinate system will be

$$(6.1) \quad \sigma ds^2 = U^2 du^2 + V^2 dv^2 + W^2 dw^2 \\ - Z^2 dz^2 + 2M du dv + 2N du dw + 2Q dv dw$$

where U, V, W, Z, M, N, Q are the metric coefficients. Note $'p^i$ is orthogonal to $t^i, 't^i, p^i$ and hence (6.1) contains no terms of types $du dz, dv dz, dw dz$. The components of $t^i, 't^i, p^i, 'p^i$ are, in the x^i coordinate system of *special relativity* (see (3.2)),

$$(6.2) \quad \begin{aligned} t^i &= \frac{1}{U} \frac{\partial x^i}{\partial u}, & 't^i &= \frac{1}{V} \frac{\partial x^i}{\partial v}, \\ p^i &= \frac{1}{W} \frac{\partial x^i}{\partial w}, & 'p^i &= \frac{1}{Z} \frac{\partial x^i}{\partial z}. \end{aligned}$$

Further, from (3.16), (6.1), (6.2) we find

$$(6.3) \quad k = \frac{M}{UV}, \quad d = \frac{N}{UW} = \frac{Q}{VW}.$$

Again, we note that

$$(6.4) \quad \begin{aligned} t^i \nabla_i \lambda_k &= \frac{1}{U} \frac{\partial \lambda_k}{\partial u}, & 't^i \nabla_i \lambda_k &= \frac{1}{V} \frac{\partial \lambda_k}{\partial v}, \\ p^j \nabla_j \lambda_k &= \frac{1}{W} \frac{\partial \lambda_k}{\partial w}, & 'p^j \nabla_j \lambda_k &= \frac{1}{Z} \frac{\partial \lambda_k}{\partial z}. \end{aligned}$$

Finally, we can determine the metric coefficients by

$$(6.5) \quad Z^2 = -g_{ik} \frac{\partial x^i}{\partial z} \frac{\partial x^k}{\partial z}, \quad U^2 = g_{ik} \frac{\partial x^i}{\partial u} \frac{\partial x^k}{\partial u}, \quad M = g_{ik} \frac{\partial x^i}{\partial u} \frac{\partial x^k}{\partial v}, \quad \text{etc.}$$

Note

$$g_{ii} = 0 \text{ (for } i \neq j), \quad g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1.$$

Now we express the first characteristic relation (4.20) and the second characteristic relations (4.36)–(4.42) in terms of partial derivatives by use of (6.2), (6.4). Equation (4.20a) becomes

$$(6.6a) \quad \frac{(1 + k - 2d^2)}{B} \lambda^i = (1 - d^2) \left(\frac{1}{U} \frac{\partial x^i}{\partial u} + \frac{1}{V} \frac{\partial x^i}{\partial v} \right) + \frac{d(k - 1)}{W} \frac{\partial x^i}{\partial w}.$$

Also, (4.20b), (4.20c) become

$$(6.6b) \quad g_{ik} \frac{\partial x^j}{\partial u} \frac{\partial x^k}{\partial z} = g_{ik} \frac{\partial x^j}{\partial v} \frac{\partial x^k}{\partial z} = g_{ik} \frac{\partial x^j}{\partial w} \frac{\partial x^k}{\partial z} = 0,$$

$$(6.6c) \quad V g_{ik} \frac{\partial x^j}{\partial w} \frac{\partial x^k}{\partial u} = U g_{ik} \frac{\partial x^j}{\partial w} \frac{\partial x^k}{\partial v}.$$

Further, (4.36)–(4.41) become

$$(6.7) \quad \begin{aligned} & \frac{-B}{Z^2} \frac{\partial x^k}{\partial z} \frac{\partial \lambda_k}{\partial z} + \frac{B}{U} \left[\frac{(F - D^2)}{U} \frac{\partial x^k}{\partial u} + \frac{G}{W} \frac{\partial x^k}{\partial w} \right] \frac{\partial \lambda_k}{\partial u} \\ & + \frac{B}{W} \left(\frac{G}{U} \frac{\partial x^k}{\partial u} + \frac{J}{W} \frac{\partial x^k}{\partial w} \right) \frac{\partial \lambda_k}{\partial v} - \frac{a^2}{\rho c^2 U} \frac{\partial \rho}{\partial u} \\ & + \frac{2B(D^2 + 1)\omega_{jk}}{U(1 + k - 2d^2)} \frac{\partial x^k}{\partial u} \left[\frac{1 - d^2}{V} \frac{\partial x^j}{\partial v} + \frac{d(k - 1)}{W} \frac{\partial x^j}{\partial w} \right] \\ & + \frac{B^2}{\rho(1 + k - 2d^2)} \left[(1 - d^2) \left(\frac{1}{U} \frac{\partial \rho}{\partial u} + \frac{1}{V} \frac{\partial \rho}{\partial v} \right) + \frac{d(k - 1)}{W} \frac{\partial \rho}{\partial w} \right] = 0, \end{aligned}$$

$$\begin{aligned}
(6.8) \quad & \frac{-B}{Z^2} \frac{\partial x^k}{\partial z} \frac{\partial \lambda_k}{\partial z} + \frac{B}{V} \left[\frac{(F - D^2)}{V} \frac{\partial x^k}{\partial v} + \frac{G}{W} \frac{\partial x^k}{\partial w} \right] \frac{\partial \lambda_k}{\partial v} \\
& + \frac{B}{W} \left(\frac{G}{V} \frac{\partial x^k}{\partial v} + \frac{J}{W} \frac{\partial x^k}{\partial w} \right) \frac{\partial \lambda_k}{\partial w} - \frac{a^2}{\rho c^2 V} \frac{\partial \rho}{\partial v} \\
& + \frac{2B(D^2 + 1)\omega_{ik}}{V(1 + k - 2d^2)} \frac{\partial x^k}{\partial v} \left[\frac{(1 - d^2)}{U} \frac{\partial x^i}{\partial u} + \frac{d(k - 1)}{W} \frac{\partial x^i}{\partial w} \right] \\
& + \frac{B^2}{\rho(1 + k - 2d^2)} \left[(1 - d^2) \left(\frac{1}{U} \frac{\partial \rho}{\partial u} + \frac{1}{V} \frac{\partial \rho}{\partial v} \right) + \frac{d(k - 1)}{W} \frac{\partial \rho}{\partial w} \right] = 0,
\end{aligned}$$

$$(6.9) \quad \frac{2(1 - d^2)B\omega_{ik}}{(1 + k - 2d^2)W} \frac{\partial x^k}{\partial w} \left(\frac{1}{U} \frac{\partial x^i}{\partial u} + \frac{1}{V} \frac{\partial x^i}{\partial v} \right) - \frac{a^2}{\rho c^2 W} \frac{\partial \rho}{\partial w} = 0,$$

$$\begin{aligned}
(6.10) \quad & \frac{2B\omega_{ik}}{(1 + k - 2d^2)Z} \frac{\partial x^k}{\partial z} \left[(1 - d^2) \left(\frac{1}{U} \frac{\partial x^i}{\partial u} + \frac{1}{V} \frac{\partial x^i}{\partial v} \right) + \frac{d(k - 1)}{W} \frac{\partial x^i}{\partial w} \right] \\
& - \frac{a^2}{\rho c^2 Z} \frac{\partial \rho}{\partial z} = 0,
\end{aligned}$$

$$\begin{aligned}
(6.11) \quad & 2\omega_{ik} \frac{\partial x^i}{\partial u} \frac{\partial x^k}{\partial v} = \frac{\partial x^k}{[\partial v \partial u]} \frac{\partial \lambda_k}{\partial u}, \quad 2\omega_{ik} \frac{\partial x^i}{\partial u} \frac{\partial x^k}{\partial w} = \frac{\partial x^k}{[\partial w \partial u]} \frac{\partial \lambda_k}{\partial u}, \\
& 2\omega_{ik} \frac{\partial x^i}{\partial u} \frac{\partial x^k}{\partial z} = \frac{\partial x^k}{[\partial z \partial u]} \frac{\partial \lambda_k}{\partial u}, \quad 2\omega_{ik} \frac{\partial x^i}{\partial v} \frac{\partial x^k}{\partial w} = \frac{\partial x^k}{[\partial w \partial v]} \frac{\partial \lambda_k}{\partial v}, \\
& 2\omega_{ik} \frac{\partial x^i}{\partial v} \frac{\partial x^k}{\partial z} = \frac{\partial x^k}{[\partial z \partial v]} \frac{\partial \lambda_k}{\partial v}, \quad 2\omega_{ik} \frac{\partial x^i}{\partial w} \frac{\partial x^k}{\partial z} = \frac{\partial x^k}{[\partial z \partial w]} \frac{\partial \lambda_k}{\partial w},
\end{aligned}$$

$$\begin{aligned}
(6.12) \quad & \frac{\partial x^k}{\partial u} \frac{\partial x^q}{\partial z} \frac{\partial \omega_{kq}}{\partial w} = 0, \quad \frac{\partial x^k}{\partial v} \frac{\partial x^q}{\partial z} \frac{\partial \omega_{kq}}{\partial w} = 0, \\
& \frac{\partial x^k}{\partial w} \frac{\partial x^q}{\partial u} \frac{\partial \omega_{kq}}{\partial v} = 0, \quad \frac{\partial x^k}{\partial z} \frac{\partial x^q}{\partial u} \frac{\partial \omega_{kq}}{\partial v} = 0
\end{aligned}$$

where the brackets indicate alternation (see (3.15)), and (4.42) becomes

$$\begin{aligned}
(6.13) \quad & (1 - d^2) \left(\frac{1}{U} \frac{\partial \lambda_k}{\partial u} + \frac{1}{V} \frac{\partial \lambda_k}{\partial v} \right) + \frac{d(k - 1)}{W} \frac{\partial \lambda_k}{\partial w} \\
& - 2 \left[(1 - d^2) \left(\frac{1}{U} \frac{\partial x^i}{\partial u} + \frac{1}{V} \frac{\partial x^i}{\partial v} \right) + \frac{d(k - 1)}{W} \frac{\partial x^i}{\partial w} \right] \omega_{ik} = 0.
\end{aligned}$$

Equations (6.3) furnish d , k in terms of the metric coefficients which are determined by (6.5). The above system contains twenty-six *first order partial differential* equations (eight equations (6.6), four equations (6.7), (6.8), (6.9), (6.10), ten equations (6.11), (6.12), and the four equations (6.13)) in the fifteen unknowns λ_k , ρ , ω_{ik} , x^i . Since these equations are of the first order and u , v are characteristic variables, these equations form the *desired characteristic*

system. We shall not discuss existence and uniqueness of the solution. The characteristic system for general relativity is obtained by replacing $\partial\lambda_i/\partial x^k$ by $(\partial\lambda_i/\partial x^k) - \Gamma_{ik}^i\lambda_i$ and similarly for ω_{ik} , where Γ_{ik}^i are the Christoffel symbols.

In the non-steady, two-dimensional case, ρ , λ^i , ω_{ik} , x^i can be taken to be independent of the variable z .

7. Reduction of the Previous Relativistic Theory to Newtonian Mechanics.

In this section we shall discuss the following topics. First, we shall consider the well known technique for reducing the relativistic equations to their Newtonian analogs. Then we shall study the reduction of the various relativistic relations of sections 1–6 to their Newtonian analogs. In particular, we consider (1) the discontinuities in the derivatives of ρ , λ_i ; (2) the characteristic partial differential equation; (3) the time-like bicharacteristic vectors and the space-like normal vectors; (4) the relation between these vectors; and (5) the first and second characteristic relations in Newtonian form.

First, we shall briefly discuss the technique for reducing relativistic equations to their Newtonian analogs. Greek indices will run from 1 through 3. The transition to Newtonian mechanics will be made in two steps. *In the first step we go from general to special relativity* by use of the following replacements of general relativity quantities by *equivalent quantities in Newtonian terms*:

$$\begin{aligned} x^0 &\rightarrow ct, \\ (7.1) \quad g_{00} &\rightarrow 1, \quad g_{0\alpha} = g_{\alpha 0} \rightarrow 0, \\ g_{\alpha\beta} &\rightarrow -'g_{\alpha\beta}, \quad g^{\alpha\beta} \rightarrow -'g^{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, \end{aligned}$$

where $'g_{\alpha\beta}$ is the metric tensor of Euclidean three-dimensional space and t is the time. It should be noted that the arrows in (7.1), and in our future work, differ from equal signs. They connect corresponding quantities where the left hand side is a quantity in general or special relativity and the right hand side is a Newtonian quantity. We note that the element of arc in Euclidean space is dl where

$$(7.2) \quad dl^2 = 'g_{\alpha\beta} dx^\alpha dx^\beta$$

and $'g_{\alpha\beta}$ is independent of c . Now, in the second step, we go to Newtonian mechanics by the additional order assumption that if R , Q , etc. are any quantities independent of c then

$$(7.3) \quad \frac{R}{c^2} \rightarrow 0, \quad \frac{Q}{c^3} \rightarrow 0, \quad \text{etc.}$$

We shall show that from (7.1), (7.2), (7.3) it follows that

$$(7.4) \quad \lambda^0 \rightarrow 1, \quad \lambda^\alpha \rightarrow \frac{v^\alpha}{c}$$

where v^α are the components of the velocity vector in Newtonian mechanics. From (7.1), (7.2), (2.1) we see that for a time-like displacement

$$(7.5) \quad ds^2 = c^2 dt^2 - dl^2.$$

By use of the definitions of λ^i , v^α and q (the magnitude of v^α), we obtain

$$(7.6) \quad \lambda^i = \frac{dx^i}{ds}, \quad \frac{v^\alpha}{c} = \frac{1}{c} \frac{dx^\alpha}{dt}, \quad \frac{q}{c} = \frac{1}{c} \frac{dl}{dt}.$$

From (7.6), (7.5) we find

$$(7.7) \quad \lambda^i = \frac{dx^i}{dt} \left[c^2 - \left(\frac{dl}{dt} \right)^2 \right]^{-\frac{1}{2}}$$

or by use of (7.6), (7.1)

$$(7.8) \quad \lambda^0 = \left(1 - \frac{q^2}{c^2} \right)^{-\frac{1}{2}}, \quad \lambda^\alpha = \left(1 - \frac{q^2}{c^2} \right)^{-\frac{1}{2}} \frac{v^\alpha}{c}.$$

If we assume that v^α is independent of c and use the condition (7.3), then (7.4) follows from (7.8).

We shall consider the reduction of the magnitude of λ^i (2.4) and the equations of motion and continuity (see (2.5), (2.6)) to Newtonian mechanics. From (7.1), (7.4) we find that (2.4) becomes

$$(7.9) \quad 1 - \left(g_{\alpha\beta} \frac{v^\alpha v^\beta}{c^2} \right) = 1.$$

By use of our order condition (7.3) and the assumption that $'g_{\alpha\beta}$, v^α are independent of c , we find that (7.9) is an identity. Again, by use of (7.1), (7.4), the tensor T^{ik} of (2.5) becomes $'T^{ik}$ (or $T^{ik} \rightarrow 'T^{ik}$), where

$$(7.10) \quad 'T^{00} = c^2 \rho - p, \quad 'T^{0\alpha} = 'T^{\alpha 0} = c \rho v^\alpha, \quad 'T^{\alpha\beta} = 'T^{\beta\alpha} = \rho v^\alpha v^\beta + p 'g^{\alpha\beta}.$$

The equations of motion and continuity of (2.6) reduce to

$$(7.11) \quad \frac{\partial(c^2 \rho - p)}{c \partial t} + ' \nabla_\alpha (c \rho v^\alpha) = 0,$$

$$(7.12) \quad \frac{\partial(c \rho v^\alpha)}{c \partial t} + ' \nabla_\beta (\rho v^\alpha v^\beta + p 'g^{\alpha\beta}) = 0$$

where $' \nabla_\alpha$ is the covariant derivative with respect to $'g_{\alpha\beta}$. Since the Christoffel symbols contain only terms of the form $'g^{..} \partial'g^{..}/\partial x^..$,

$$(7.13) \quad \nabla_j \rightarrow ' \nabla_\alpha, \quad j = \alpha.$$

Dividing (7.11) by c and noting that $c^{-2} p \rightarrow 0$, by (7.3), we see that the equation (7.11) becomes the continuity relation in Newtonian mechanics. Equations

(7.12) are the equations of motion. The reductions of this and the preceding paragraph are well known. We have discussed the theory to illustrate the use of assumptions (7.1), (7.3). In the next paragraph we shall show how our relations for the discontinuities in the derivatives of ρ , λ_i and the characteristic relation reduce to well known results in Newtonian mechanics.

Let us determine the Newtonian expressions for L of (2.19), the jumps $[\nabla_i \rho]$, $[\nabla_i \lambda_k]$ of (2.30), (2.31), and the characteristic equation (2.18). By use of (7.1), (7.4), we find that $L \rightarrow {}'L$, where

$$(7.14) \quad P = \frac{\partial \phi}{\partial t} + v^\alpha \frac{\partial \phi}{\partial x^\alpha}, \quad {}'L = \frac{P}{c}$$

and P is independent of c . Again, we see that the jump $[\nabla_i \rho] \rightarrow (1/c) [\partial \rho / \partial t]$ when $j = 0$ and $[\nabla_i \rho] \rightarrow [\partial \rho / \partial x^\alpha]$ when $j = 1, 2, 3$. Thus (2.30) furnishes

$$(7.15) \quad \left[\frac{\partial \rho}{\partial t} \right] = A \frac{\partial \phi}{\partial t}, \quad \left[\frac{\partial \rho}{\partial x^\alpha} \right] = A \frac{\partial \phi}{\partial x^\alpha}.$$

Further, from (7.1), (7.4) we find

$$(7.16) \quad [\nabla_i \lambda^0] \rightarrow 0, \quad [\nabla_0 \lambda^\beta] \rightarrow \frac{1}{c^2} \frac{\partial v^\beta}{\partial t}, \quad [\nabla_\alpha \lambda^\beta] \rightarrow \frac{1}{c} [{}'\nabla_\alpha v^\beta].$$

Thus (7.16), (7.13), (2.31) furnish, by use of the order condition (7.3),

$$(7.17) \quad \left[\frac{\partial v^\beta}{\partial t} \right] = -\frac{a^2 A}{\rho P} {}'g^{\beta\alpha} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x^\alpha},$$

$$[{}'\nabla_\alpha v^\beta] = -\frac{a^2 A}{\rho P} {}'g^{\beta\mu} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\mu}.$$

The relations (7.15), (7.17) are *well known in the non-steady Newtonian case* [7]. By use of (7.1), (7.2) and the fact that (see (7.13))

$$L \rightarrow \frac{1}{c} \left(\frac{\partial \phi}{\partial t} + v^\alpha \frac{\partial \phi}{\partial x^\alpha} \right),$$

the equation (2.18) becomes, through use of the order condition (7.3),

$$(7.18) \quad \left(\frac{\partial \phi}{\partial t} + v^\alpha \frac{\partial \phi}{\partial x^\alpha} \right)^2 - a^2 {}'g^{\lambda\mu} \frac{\partial \phi}{\partial x^\lambda} \frac{\partial \phi}{\partial x^\mu} = 0.$$

This is the well known *characteristic relation in the Newtonian case* [1].

We consider the Newtonian form of the vectors n^i , t^i of section 2. First, we consider the normal vector n^i . Because of (7.1) and the order condition (7.3), the relation (2.20) determines the Newtonian unit normal vector N_α where

$$(7.19) \quad n_i \rightarrow N_\alpha, \quad j = \alpha,$$

and if $\phi = \text{constant}$ is a characteristic surface, then

$$(7.20) \quad \begin{aligned} n_0 &= \frac{\partial \phi}{\partial t} \Big/ c \left(g^{\alpha\beta} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} \right)^{\frac{1}{2}}, \\ N_\mu &= \frac{\partial \phi}{\partial x^\mu} \Big/ \left(g^{\alpha\beta} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} \right)^{\frac{1}{2}}. \end{aligned}$$

Since $n^k = g^{ki} n_i$, it follows from (7.1) that

$$(7.21) \quad n^k \rightarrow -N^\alpha, \quad k = \alpha.$$

The Newtonian bicharacteristic vector is T^α , which is determined by the relativity vector t^i where

$$(7.22) \quad t^i \rightarrow T^\alpha, \quad j = \alpha.$$

From the argument of (7.7) it follows, by use of t^i instead of λ^i , that

$$(7.23) \quad T^\alpha = \frac{1}{c} \frac{dx^\alpha}{dt}, \quad T^0 = 1.$$

In (7.23), dx^α/dt is not the velocity vector. By use of (4.3), (7.3), (7.4), (7.21), (7.23), the equations (3.6), (3.7) reduce to

$$(7.24) \quad v^\alpha = \frac{dx^\alpha}{dt} - aN^\alpha.$$

This is the *known relation connecting* T^α , v^α , N^α [1] in Newtonian mechanics.

Now we consider the Newtonian form of the relations of section 6. In particular, we are interested in the *first and second sets of characteristic relations* (6.6)–(6.13). We assume that the functions ρ , v^α , t , and x^α are differentiable functions of the independent variables u , v , w , cz and *these functions are independent of c* . The variables x^α are the coordinate variables of a Cartesian orthogonal coordinate system. The Newtonian equivalents of the metric coefficients U , M , etc. of (6.5) can be determined by use of the order condition (7.3) and the additional order assumption that the sound speed a is such that a/c is of order zero. We shall use this condition only once, namely in our analysis of (3.6), (3.7). The direction numbers of the time-like vectors λ^i , t^i , $'t^i$ in the four-space of special relativity are

$$\lambda^i : (\lambda^0, \lambda^\alpha), \quad t^i : \left(c \frac{\partial t}{\partial u}, \frac{\partial x^\alpha}{\partial u} \right), \quad 't^i : \left(c \frac{\partial t}{\partial v}, \frac{\partial x^\alpha}{\partial v} \right).$$

By use of the above order condition on a/c and (3.6), (3.7), we see that the vectors λ^i , t^i , $'t^i$ coincide, or *the above direction numbers are proportional*. To within terms of order $1/c^2$, the vector λ^i is given by $(1, v^\alpha/c)$, see (7.4). Hence, *the terms $\partial x^\alpha/\partial u$, $\partial x^\alpha/\partial v$ are of order $1/c$ as compared to $c\partial t/\partial u$, $c\partial t/\partial v$, respectively, for all α* . Now we consider the direction numbers of the space-like vector $'p^i$: $(c\partial t/\partial z, \partial x^\alpha/\partial z)$. As $'p^i$ is orthogonal to t^i , we may write

$$c^2 \frac{\partial t}{\partial z} \frac{\partial t}{\partial u} - \sum_{\alpha=1}^3 \frac{\partial x^\alpha}{\partial z} \frac{\partial x^\alpha}{\partial u} = 0.$$

Since, for all α , $\partial x^\alpha / \partial u$ is of order $1/c$ as compared to $c \partial t / \partial u$, it follows from the above orthogonality relation that $c \partial t / \partial z$ is of order $1/c$ as compared to $\partial x^\alpha / \partial z$, for at least one index α . Finally, from (4.20), it follows that the time-like vector p^i is a linear combination of $\lambda^i, t^i, 't^i$. Hence the direction numbers of p^i behave as those of $\lambda^i, t^i, 't^i$ or $\partial x^\alpha / \partial w$ are of order $1/c$ as compared to $c \partial t / \partial w$, for all α . These results and (7.3) lead to the following formulæ for U, V, W, Z , etc. of (6.5) when we introduce the independent variable $r = cz$:

$$(7.25) \quad \begin{aligned} \frac{U^2}{c^2} &\rightarrow \left(\frac{\partial t}{\partial u} \right)^2, & \frac{V^2}{c^2} &\rightarrow \left(\frac{\partial t}{\partial v} \right)^2, & \frac{W^2}{c^2} &\rightarrow \left(\frac{\partial t}{\partial w} \right)^2, \\ \frac{Z^2}{c^2} &\rightarrow 'g_{\alpha\beta} \frac{\partial x^\alpha}{\partial r} \frac{\partial x^\beta}{\partial r}, & \frac{M}{c^2} &\rightarrow \frac{\partial t}{\partial u} \frac{\partial t}{\partial v}, \\ \frac{N}{c^2} &\rightarrow \frac{\partial t}{\partial u} \frac{\partial t}{\partial w}, & \frac{Q}{c^2} &\rightarrow \frac{\partial t}{\partial v} \frac{\partial t}{\partial w}. \end{aligned}$$

By use of the above and (7.1) and (7.4), we can evaluate the Newtonian equivalent of a type of term which occurs in the characteristic relations in (6.7), (6.8). We find

$$(7.26) \quad \frac{c^3}{U^2} g_{i\alpha} \frac{\partial \lambda^\alpha}{\partial u} \rightarrow -'g_{\beta\mu} \frac{\partial v^\mu}{\partial u} \bigg/ \left(\frac{\partial t}{\partial u} \right)^2, \quad j = \beta.$$

From (3.7), (7.3) we obtain

$$(7.27) \quad B \rightarrow 1, \quad D \rightarrow \frac{c}{a}, \quad \frac{1}{D^2} \rightarrow \frac{a^2}{c^2},$$

or $1/D^2 \rightarrow 0$, by our order condition. From (7.27), (7.25) we can evaluate the Newtonian equivalents of three more terms which occur in (6.7), (6.8):

$$(7.28) \quad \frac{c^3}{UD^2} \frac{\partial \rho}{\partial u} \rightarrow a^2 \frac{\partial \rho}{\partial u} \bigg/ \frac{\partial t}{\partial u},$$

$$(7.29) \quad \frac{ca^2}{UD^2} \frac{\partial \rho}{\partial u} \rightarrow 0,$$

$$(7.30) \quad \frac{c^3}{Z^2 D^2} g_{i\alpha} \frac{\partial x^i}{\partial z} \frac{\partial \lambda^\alpha}{\partial z} \rightarrow -a^2 'g_{\alpha\beta} \frac{\partial x^\alpha}{\partial r} \frac{\partial v^\beta}{\partial r} \bigg/ 'g_{\alpha\beta} \frac{\partial x^\alpha}{\partial r} \frac{\partial x^\beta}{\partial r}.$$

Again, by use of (7.25), relation (6.3) leads to

$$(7.31) \quad k^2 \rightarrow 1, \quad d^2 \rightarrow 1.$$

The cases $k \rightarrow -1, d^2 \rightarrow 1$ are impossible (see (7.40) and the discussion preceding and following this equation). It should be noted that (7.31) is a degenerate

case of the theory discussed in section 4. Thus, before we can find Newtonian equations corresponding to (6.7), (6.8), we shall need to evaluate the terms

$$\lim_{d^2, k \rightarrow 1} \frac{F}{D^2}, \quad \lim_{d^2, k \rightarrow 1} \frac{J}{D^2}, \quad \lim_{d^2, k \rightarrow 1} \frac{G}{D^2}.$$

We note that F, J, G, D^2 are given by (4.16), (4.27). If we assume that

$$(7.32) \quad \lim_{d^2, k \rightarrow 1} \frac{k-1}{d^2-1} = -R,$$

where R is a function of u, v, w, r , then

$$(7.33) \quad \lim_{d^2, k \rightarrow 1} \frac{F}{D^2} = \lim_{d^2, k \rightarrow 1} \frac{J}{D^2} = \frac{R}{R+2}, \quad \lim_{d^2, k \rightarrow 1} \frac{G}{D^2} = \frac{\mp R}{R+2},$$

$$(7.34) \quad \lim_{d^2, k \rightarrow 1} \frac{1-d^2}{1+k-2d^2} = \frac{1}{R+2}, \quad \lim_{d^2, k \rightarrow 1} \frac{d(k-1)}{1+k-2d^2} = \frac{\pm R}{R+2};$$

the signs depend upon d . By use of (7.33), we can evaluate the Newtonian expressions for the terms of the following type needed in connection with (6.7), (6.8):

$$(7.35) \quad \lim_{d^2, k \rightarrow 1} \frac{c^3 F}{U^2 D^2} g_{i\alpha} \frac{\partial \lambda^\alpha}{\partial u} \rightarrow \frac{-R}{2+R} g_{\beta\alpha} \frac{\partial v^\alpha}{\partial u} \bigg/ \left(\frac{\partial t}{\partial u} \right)^2, \quad j = \beta.$$

Finally, by use of (4.31), (7.13), we find for the Newtonian vorticity tensor $'\omega_{\alpha\beta}$, where $2 '\omega_{\alpha\beta} = '\nabla_\alpha v_\beta - '\nabla_\beta v_\alpha$, that this tensor and ω_{jk} are related by

$$(7.36) \quad g^{nj} g^{lk} \omega_{jk} \rightarrow \frac{1}{c} g^{\alpha\gamma} g^{\beta\mu} '\omega_{\gamma\mu}, \quad n = \alpha, \quad l = \beta.$$

Further, from (7.4) and the fact that $g_{ik} \lambda^k = \lambda_i$ we see that

$$\omega_{00} \rightarrow 0, \quad g^{k\alpha} \omega_{0\alpha} \rightarrow \frac{-1}{2c^2} \frac{\partial v^\beta}{\partial t}, \quad k = \beta,$$

and hence

$$\frac{\partial x^0}{\partial v} \frac{\partial x^i}{\partial u} \omega_{0i} = g^{kl} g_{lp} \frac{\partial x^0}{\partial v} \frac{\partial x^p}{\partial u} \omega_{0k} \rightarrow \frac{-g_{\alpha\beta}}{2c} \frac{\partial t}{\partial v} \frac{\partial x^\alpha}{\partial u} \frac{\partial v^\beta}{\partial t}.$$

It should be noted that $'\omega_{\alpha\beta}$ involves the derivatives of v^α with respect to x^α . Similarly, we define the vector h^β by

$$(7.37) \quad \frac{\partial v^\beta}{\partial t} = h^\beta.$$

Then the two preceding relations become

$$(7.38) \quad \omega_{00} \rightarrow 0, \quad g^{k\alpha} \omega_{0\alpha} \rightarrow \frac{-1}{2c^2} h^\beta, \quad k = \beta,$$

$$\frac{\partial x^0}{\partial v} \frac{\partial x^i}{\partial u} \omega_{0i} \rightarrow \frac{-g_{\alpha\beta}}{2c} \frac{\partial t}{\partial v} \frac{\partial x^\alpha}{\partial u} h^\beta.$$

From (7.38) the Newtonian quantity corresponding to the following term pertaining to (6.7), (6.8) is seen to be

$$(7.39) \quad \frac{c^3}{UV} \omega_{ik} \frac{\partial x^j}{\partial v} \frac{\partial x^k}{\partial u} \rightarrow \left(' \omega_{\alpha\beta} \frac{\partial x^\alpha}{\partial v} \frac{\partial x^\beta}{\partial u} - \frac{1}{2} ' g_{\alpha\beta} \frac{\partial t}{\partial v} \frac{\partial x^\alpha}{\partial u} h^\beta + \frac{1}{2} ' g_{\alpha\beta} \frac{\partial t}{\partial u} \frac{\partial x^\alpha}{\partial v} h^\beta \right) / \frac{\partial t}{\partial u} \frac{\partial t}{\partial v}.$$

Now we shall express *the first and second characteristic relations of (6.6)–(6.13) in Newtonian terms*. If we let $j = 0$ in (6.6a), divide by $1 + k - 2d^2$, take limits as $d^2 \rightarrow 1$, $k^2 \rightarrow 1$, and use (7.25), (7.34), (7.4), we find that $d \rightarrow -1$, $k \rightarrow -1$ is impossible but both of the cases $d^2 \rightarrow 1$, $k \rightarrow 1$ are possible if

$$(7.40) \quad R = 0,$$

one of these cases, namely $d \rightarrow 1$, $k \rightarrow 1$, being possible for all R . Further, the equation (6.6), for $j = 1, 2, 3$ (or $j = \alpha$), furnishes the *first* characteristic relation in Newtonian form (in the case $d \rightarrow 1$, $k \rightarrow -1$ equation (6.6a) implies $v^\alpha = p^\alpha$, that is, that the bicharacteristic directions can be chosen so that the velocity vector has no component along either of them; this can be valid for only a very restricted family of flows). For $R = 0$, we find

$$(7.41a) \quad 2v^\alpha = \left(\frac{\partial x^\alpha}{\partial u} / \frac{\partial t}{\partial u} \right) + \left(\frac{\partial x^\alpha}{\partial v} / \frac{\partial t}{\partial v} \right).$$

The orthogonality relations (6.6b) vanish identically for two-dimensional non-steady flows. In the three-dimensional non-steady case, these relations furnish the following three relations, which are conditions on the relative magnitude of the components of the vector $(c\partial t/\partial r, \partial x^\alpha/\partial r)$ as compared to the components of $(c\partial t/\partial u, \partial x^\alpha/\partial u)$, $(c\partial t/\partial v, \partial x^\alpha/\partial v)$, $(c\partial t/\partial w, \partial x^\alpha/\partial w)$:

$$(7.41b) \quad \begin{aligned} c^2 \frac{\partial t}{\partial u} \frac{\partial t}{\partial r} - ' g_{\alpha\beta} \frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial r} &= 0, \\ c^2 \frac{\partial t}{\partial v} \frac{\partial t}{\partial r} - ' g_{\alpha\beta} \frac{\partial x^\alpha}{\partial v} \frac{\partial x^\beta}{\partial r} &= 0, \\ c^2 \frac{\partial t}{\partial w} \frac{\partial t}{\partial r} - ' g_{\alpha\beta} \frac{\partial x^\alpha}{\partial w} \frac{\partial x^\beta}{\partial r} &= 0; \end{aligned}$$

the relation (6.6c) vanishes identically.

By multiplying (6.7), (6.8) by c^3/D^2 , taking limits as $k, d^2 \rightarrow 1$, and using (7.26)–(7.39), we find (note $\lambda^0 = 1$, $R = 0$)

$$(7.42) \quad \begin{aligned} & \left[' g_{\beta\alpha} \frac{\partial v^\alpha}{\partial u} \frac{\partial x^\beta}{\partial u} / \left(\frac{\partial t}{\partial u} \right)^2 \right] + \left[a^2 ' g_{\alpha\beta} \frac{\partial x^\alpha}{\partial r} \frac{\partial v^\beta}{\partial r} / ' g_{\alpha\beta} \frac{\partial x^\alpha}{\partial r} \frac{\partial x^\beta}{\partial r} \right] \\ & + \left[\left(' \omega_{\alpha\beta} \frac{\partial x^\alpha}{\partial v} \frac{\partial x^\beta}{\partial u} - \frac{1}{2} ' g_{\alpha\beta} \frac{\partial t}{\partial v} \frac{\partial x^\alpha}{\partial u} h^\beta + \frac{1}{2} ' g_{\alpha\beta} \frac{\partial t}{\partial u} \frac{\partial x^\alpha}{\partial v} h^\beta \right) / \frac{\partial t}{\partial v} \frac{\partial t}{\partial u} \right] \\ & + \frac{a^2}{2\rho} \left[\left(\frac{\partial \rho}{\partial u} / \frac{\partial t}{\partial u} \right) + \left(\frac{\partial \rho}{\partial v} / \frac{\partial t}{\partial v} \right) \right] = 0, \end{aligned}$$

$$\begin{aligned}
(7.43) \quad & \left['g_{\alpha\beta} \frac{\partial v^\alpha}{\partial v} \frac{\partial x^\beta}{\partial v} \middle/ \left(\frac{\partial t}{\partial v} \right)^2 \right] + \left[a^2 'g_{\alpha\beta} \frac{\partial x^\alpha}{\partial r} \frac{\partial v^\beta}{\partial r} \middle/ 'g_{\alpha\beta} \frac{\partial x^\alpha}{\partial r} \frac{\partial x^\beta}{\partial r} \right] \\
& + \left[\left(' \omega_{\alpha\beta} \frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial v} - \frac{'g_{\alpha\beta}}{2} \frac{\partial t}{\partial u} \frac{\partial x^\alpha}{\partial v} h^\beta + \frac{'g_{\alpha\beta}}{2} \frac{\partial t}{\partial v} \frac{\partial x^\alpha}{\partial u} h^\beta \right) \middle/ \frac{\partial t}{\partial u} \frac{\partial t}{\partial v} \right] \\
& + \frac{a^2}{2\rho} \left[\left(\frac{\partial \rho}{\partial v} \middle/ \frac{\partial t}{\partial v} \right) + \left(\frac{\partial \rho}{\partial u} \middle/ \frac{\partial t}{\partial u} \right) \right] = 0.
\end{aligned}$$

Multiplying (6.9), (6.10) by c^3 and taking limits as d^2 , $k \rightarrow 1$, we obtain, by use of (7.39) and a similar relation in u, r ,

$$\begin{aligned}
(7.44) \quad & ' \omega_{\alpha\beta} \left[\left(\frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial w} \middle/ \frac{\partial t}{\partial u} \frac{\partial t}{\partial w} \right) + \left(\frac{\partial x^\alpha}{\partial v} \frac{\partial x^\beta}{\partial w} \middle/ \frac{\partial t}{\partial v} \frac{\partial t}{\partial w} \right) \right] \\
& - \frac{1}{2} 'g_{\alpha\beta} \left[\left(\frac{\partial x^\alpha}{\partial w} h^\beta \middle/ \frac{\partial t}{\partial u} \right) + \left(\frac{\partial x^\alpha}{\partial w} h^\beta \middle/ \frac{\partial t}{\partial v} \right) \right] \\
& + \frac{1}{2} 'g_{\alpha\beta} \left[\left(\frac{\partial x^\alpha}{\partial u} h^\beta \middle/ \frac{\partial t}{\partial w} \right) + \left(\frac{\partial x^\alpha}{\partial v} h^\beta \middle/ \frac{\partial t}{\partial w} \right) \right] - \frac{a^2}{\rho} \frac{\partial \rho}{\partial w} \middle/ \frac{\partial t}{\partial w} = 0,
\end{aligned}$$

$$\begin{aligned}
(7.45) \quad & ' \omega_{\alpha\beta} \left[\left(\frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial r} \middle/ \frac{\partial t}{\partial u} \right) + \left(\frac{\partial x^\alpha}{\partial v} \frac{\partial x^\beta}{\partial r} \middle/ \frac{\partial t}{\partial v} \right) \right] \\
& + \frac{1}{2} 'g_{\alpha\beta} \left[\left(\frac{\partial x^\alpha}{\partial u} \frac{\partial t}{\partial r} h^\beta \middle/ \frac{\partial t}{\partial u} \right) + \left(\frac{\partial x^\alpha}{\partial v} \frac{\partial t}{\partial r} h^\beta \middle/ \frac{\partial t}{\partial v} \right) \right] \\
& - 'g_{\alpha\beta} \frac{\partial x^\alpha}{\partial r} h^\beta - \frac{a^2}{\rho} \frac{\partial \rho}{\partial r} = 0.
\end{aligned}$$

By use of relations of the types (7.36), (7.38) with $\partial ' \omega_{\alpha\beta} / \partial w$, $\partial h^\beta / \partial t$ replacing $' \omega_{\alpha\beta}$, h^β , respectively, we obtain from (6.11), (6.12)

$$(7.46) \quad 2 ' \omega_{\alpha\beta} \frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial v} - 'g_{\alpha\beta} \left(h^\beta \frac{\partial t}{\partial u} \frac{\partial x^\alpha}{\partial v} - \frac{\partial x^\alpha}{\partial v} \frac{\partial v^\beta}{\partial u} \right) = 0,$$

$$(7.47) \quad 2 ' \omega_{\alpha\beta} \frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial w} - 'g_{\alpha\beta} \left(h^\beta \frac{\partial t}{\partial u} \frac{\partial x^\alpha}{\partial w} - \frac{\partial x^\alpha}{\partial w} \frac{\partial v^\beta}{\partial u} \right) = 0,$$

$$(7.48) \quad 2 ' \omega_{\alpha\beta} \frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial r} - 'g_{\alpha\beta} \left(h^\beta \frac{\partial t}{\partial u} \frac{\partial x^\alpha}{\partial r} - \frac{\partial x^\alpha}{\partial r} \frac{\partial v^\beta}{\partial u} \right) = 0,$$

$$(7.49) \quad 2 ' \omega_{\alpha\beta} \frac{\partial x^\alpha}{\partial v} \frac{\partial x^\beta}{\partial w} - 'g_{\alpha\beta} \left(h^\beta \frac{\partial t}{\partial v} \frac{\partial x^\alpha}{\partial w} - \frac{\partial x^\alpha}{\partial w} \frac{\partial v^\beta}{\partial v} \right) = 0,$$

$$(7.50) \quad 2 ' \omega_{\alpha\beta} \frac{\partial x^\alpha}{\partial v} \frac{\partial x^\beta}{\partial r} - 'g_{\alpha\beta} \left(h^\beta \frac{\partial t}{\partial v} \frac{\partial x^\alpha}{\partial r} - \frac{\partial x^\alpha}{\partial r} \frac{\partial v^\beta}{\partial v} \right) = 0,$$

$$(7.51) \quad 2 ' \omega_{\alpha\beta} \frac{\partial x^\alpha}{\partial w} \frac{\partial x^\beta}{\partial r} - 'g_{\alpha\beta} \left(h^\beta \frac{\partial t}{\partial w} \frac{\partial x^\alpha}{\partial r} - \frac{\partial x^\alpha}{\partial r} \frac{\partial v^\beta}{\partial w} \right) = 0,$$

$$(7.52) \quad \frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial r} \frac{\partial}{\partial w} \frac{\omega_{\alpha\beta}}{\partial w} - {}'g_{\alpha\beta} \frac{\partial t}{\partial u} \frac{\partial x^\alpha}{\partial r} \frac{\partial h^\beta}{\partial w} = 0,$$

$$(7.53) \quad \frac{\partial x^\alpha}{\partial v} \frac{\partial x^\beta}{\partial r} \frac{\partial}{\partial w} \frac{\omega_{\alpha\beta}}{\partial w} - {}'g_{\alpha\beta} \frac{\partial t}{\partial v} \frac{\partial x^\alpha}{\partial r} \frac{\partial h^\beta}{\partial w} = 0,$$

$$(7.54) \quad \frac{\partial x^\alpha}{\partial w} \frac{\partial x^\beta}{\partial u} \frac{\partial}{\partial v} \frac{\omega_{\alpha\beta}}{\partial v} - {}'g_{\alpha\beta} \frac{\partial t}{\partial w} \frac{\partial x^\alpha}{\partial u} \frac{\partial h^\beta}{\partial v} = 0,$$

$$(7.55) \quad \frac{\partial x^\alpha}{\partial r} \frac{\partial x^\beta}{\partial u} \frac{\partial}{\partial v} \frac{\omega_{\alpha\beta}}{\partial v} - {}'g_{\alpha\beta} \frac{\partial t}{\partial r} \frac{\partial x^\alpha}{\partial u} \frac{\partial h^\beta}{\partial v} = 0.$$

Finally, by use of (7.4), (7.34), (7.38), (7.39), we find that the Newtonian equivalent of (6.13) is (for superindex $k = 0, k = \alpha$)

$$(7.56) \quad {}'g_{\alpha\beta} h^\beta \left[\left(\frac{\partial x^\alpha}{\partial u} \middle/ \frac{\partial t}{\partial u} \right) + \left(\frac{\partial x^\alpha}{\partial v} \middle/ \frac{\partial t}{\partial v} \right) \right] = 0,$$

$$(7.57) \quad \left[\left(\frac{\partial v^\alpha}{\partial u} \middle/ \frac{\partial t}{\partial u} \right) + \left(\frac{\partial v^\alpha}{\partial v} \middle/ \frac{\partial t}{\partial v} \right) \right] + 2 {}'g^{\alpha\beta} \left[\left(\omega_{\beta\gamma} \frac{\partial x^\gamma}{\partial u} - \frac{1}{2} {}'g_{\beta\gamma} h^\gamma \frac{\partial t}{\partial u} \right) \middle/ \frac{\partial t}{\partial u} \right. \\ \left. + \left(\omega_{\beta\gamma} \frac{\partial x^\gamma}{\partial v} - \frac{1}{2} {}'g_{\beta\gamma} h^\gamma \frac{\partial t}{\partial v} \right) \middle/ \frac{\partial t}{\partial v} \right] = 0.$$

The equations (7.41)–(7.57) are a system of twenty-four equations in fourteen unknowns v^α , ρ , $\omega_{\alpha\beta}$, h^β , t , x^α ; they constitute the characteristic system in the Newtonian case.

8. Existence and Uniqueness of Solution. E. W. TRIT showed that the characteristic system associated with a general hyperbolic equation (and hence the equation itself) in three independent variables, for properly given boundary data, possesses a unique solution [4]. This theorem on existence and uniqueness of solution can be applied (as we shall show) to the equations of motion and continuity for the non-steady flow in two spacial dimensions of Newtonian mechanics when the fluid is an irrotational, isentropic, polytropic gas.

Evidently, our problem is to show that the *equations of motion and continuity* for the non-steady flow in two spacial dimensions of an irrotational, isentropic polytropic gas are *equivalent to a single hyperbolic equation* in three independent variables. Then, by TRIT's work, *this system possesses a unique solution*. In Newtonian mechanics these equations of motion and continuity are

$$(8.1) \quad \frac{\partial v_\lambda}{\partial t} + v^\mu \nabla_\mu v_\lambda = -a^2 \nabla_\lambda \ln \rho, \quad a^2 = \frac{dp}{d\rho},$$

$$(8.2) \quad \frac{\partial \ln \rho}{\partial t} + v^\alpha \nabla_\alpha \ln \rho + \nabla_\alpha v^\alpha = 0, \quad \alpha, \lambda, \mu = 1, 2,$$

where the local sound speed a is related to the density ρ by (see (2.7))

$$(8.3) \quad a^2 = K \rho^{\gamma-1}.$$

In (8.3), K is some constant and γ is the ratio of the specific heats of the gas. For irrotational isentropic flow, a velocity potential ϕ exists such that

$$(8.4) \quad v_\lambda = \frac{\partial \phi}{\partial x^\lambda}.$$

Further, the Bernoulli relation is [16]

$$(8.5) \quad \frac{\partial \phi}{\partial t} + \frac{1}{2} 'g^{\alpha\mu} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\mu} + \frac{a^2}{\gamma - 1} = 0.$$

This relation is a first integral of (8.1). That is, by differentiation of (8.5) and use of

$$(8.6) \quad \nabla_\lambda \frac{\partial \phi}{\partial t} = \frac{\partial v_\lambda}{\partial t}, \quad \frac{1}{2} \nabla_\lambda \left('g^{\alpha\mu} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\mu} \right) = v^\mu \nabla_\mu v_\lambda,$$

$$(8.7) \quad \nabla_\lambda \frac{a^2}{\gamma - 1} = \nabla_\lambda h = \frac{1}{\rho} \nabla_\lambda p = a^2 \nabla_\lambda \ln \rho,$$

where h is the enthalpy, we obtain (8.1). Thus if a^2 , or ρ (see (8.3)), is determined by (8.5), then (8.1) is satisfied and only (8.2) must be discussed. From (8.1), (8.4) we find

$$(8.8) \quad a^2 \nabla_\lambda \ln \rho = - \frac{\partial^2 \phi}{\partial x^\lambda \partial t} - 'g^{\alpha\mu} \frac{\partial \phi}{\partial x^\alpha} \left(\nabla_\mu \frac{\partial \phi}{\partial x^\lambda} \right).$$

By substituting (8.3) into (8.5) and using logarithmic differentiation, we obtain

$$(8.9) \quad (\gamma - 1) \frac{\partial}{\partial t} \ln \rho = \left(\frac{\partial^2 \phi}{\partial t^2} + 'g^{\alpha\mu} \frac{\partial^2 \phi}{\partial t \partial x^\alpha} \frac{\partial \phi}{\partial x^\mu} \right) / \left(\frac{\partial \phi}{\partial t} + \frac{'g^{\alpha\mu}}{2} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\mu} \right).$$

Substituting (8.8), (8.9), (8.4) into (8.2), we determine the desired partial differential equation

$$(8.10) \quad \begin{aligned} & a^2 \left(\frac{\partial^2 \phi}{\partial t^2} + 'g^{\alpha\mu} \frac{\partial \phi}{\partial x^\mu} \frac{\partial^2 \phi}{\partial t \partial x^\alpha} \right) \\ & + (\gamma - 1) \left[\frac{\partial \phi}{\partial t} + \frac{'g^{\alpha\mu}}{2} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\mu} \right] \left[a^2 'g^{\alpha\beta} \left(\nabla_\alpha \frac{\partial \phi}{\partial x^\beta} \right) \right. \\ & \left. - 'g^{\alpha\lambda} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial^2 \phi}{\partial x^\lambda \partial t} - 'g^{\alpha\mu} 'g^{\lambda\beta} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} \left(\nabla_\mu \frac{\partial \phi}{\partial x^\lambda} \right) \right] = 0. \end{aligned}$$

It should be noted that the theory of irrotational waves in shallow water is covered by the case $\gamma = 2$ of the non-steady motion in two spacial dimensions of an isentropic polytropic gas.

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