On Rigidity Properties Of Developable Surfaces

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INTRODUCTION

Consider a surface S containing a curve C . Let us imagine that S is made of some flexible but inextensible material. Then S is capable only of isometric deformations (transformations under which arc-length on S is invariant). If one further assumes that the curve C on S is kept point-wise fixed, then the possible deformations of S are further restricted, and for certain choices of S and C it may happen that all these restrictions imply that S is rigid (is capable of no deformations at all). Questions of this type are related to issues in the theory of rigid surfaces (for a general account of this theory, see, for example, [1]). However, in the theory of rigid surfaces most of the results are concerned with infinitesimal rigidity (see, for example, [2], [3]) and there arises the question whether analogous results are valid for rigidity concepts which are more closely related to physically relevant approaches. The purpose of this note is to study this question for the case of developable surfaces S .

In this type of study, the regularity assumptions concerning the curves, surfaces, and deformations under consideration are essential, as simple examples show. The regularity assumptions relate partly to differentiability assumptions and partly to the absence of geometrical singularities. We shall have to be quite careful in the following discussion as regards the exclusion of geometrical singularities, and, in any case, more careful than is customary in differential geometry. On the other hand, to keep the presentation reasonably brief, we shall assume that all the functions and transformations occurring in the sequel are continuously differentiable to the extent sufficient for our purposes. The determination of the precise necessary and sufficient differentiability conditions in this sort of study may lead to quite interesting issues (see, for example, [2], [3]), and we hope to elaborate this point on another occasion.

1. PRELIMINARIES

1.1. Notations. Vector notation will be used throughout. If a, b are vectors, then ab and $a \times b$ are the scalar product and the vector product of a and b

respectively. Thus $aa = a^2 = |a|^2$, where $|a|$ is the length of the vector a . If a, b, c are three vectors, then (a, b, c) denotes the determinant whose rows consist of the components of the vectors a, b, c respectively. Constant use will be made of the identities

$$(a \times b)(c \times d) = (ac)(bd) - (ad)(bc),$$

$$(a \times b) \times c = (ac)b - (bc)a.$$

1.2. The curve C . We assume that we are given, in Euclidean three-space, a curve

$$C: \quad \mathfrak{r} = \mathfrak{y}(s), \quad 0 \leq s \leq l,$$

where $\mathfrak{r}$ is the vector from the origin to a generic point of C , s is the arc-length of C , and $\mathfrak{y}(s)$ is a vector-function, the position-vector of C . Since s is the arc length of C , we have then $|\mathfrak{y}'| = 1$, and hence $\mathfrak{y}'\mathfrak{y}'' = 0$. We assume that $|\mathfrak{y}''| > 0$. It follows that

$$|\mathfrak{y}' \times \mathfrak{y}''|^2 = |\mathfrak{y}'|^2 |\mathfrak{y}''|^2 - (\mathfrak{y}'\mathfrak{y}'')^2 = |\mathfrak{y}''|^2 > 0.$$

The curvature κ and the torsion τ of C are given by the formulas

$$\kappa = |\mathfrak{y}''|, \quad \tau = \frac{(\mathfrak{y}', \mathfrak{y}'', \mathfrak{y}''')}{(\mathfrak{y}' \times \mathfrak{y}'')^2}.$$

Thus $\kappa > 0$, since by assumption $|\mathfrak{y}''| > 0$.

The unit tangent, principal normal, and binormal vectors ξ_1, ξ_2, ξ_3 of C are given by the formulas

$$\xi_1 = \mathfrak{y}', \quad \xi_2 = \frac{1}{\kappa} \mathfrak{y}'', \quad \xi_3 = \frac{1}{\kappa} (\mathfrak{y}' \times \mathfrak{y}'').$$

We have then the identities

$$\begin{aligned} \xi_1 \times \xi_2 &= \xi_3, & \xi_2 \times \xi_3 &= \xi_1, & \xi_3 \times \xi_1 &= \xi_2, & (\xi_1, \xi_2, \xi_3) &= 1, \\ |\xi_i| &= 1, & i &= 1, 2, 3; & \xi_i \xi_j &= 0 & \text{if } i \neq j. \end{aligned}$$

Furthermore, we have the Frenet formulas

$$\xi_1' = \kappa \xi_2, \quad \xi_2' = -\kappa \xi_1 + \tau \xi_3, \quad \xi_3' = -\tau \xi_2.$$

1.3. Admissible developable surface S through C . Such a surface is assumed to be given in terms of parameters s, t in the form

$$(1) \quad S: \quad \mathfrak{r} = \mathfrak{y}(s) + t\mathfrak{z}(s),$$

where $\mathfrak{y}(s)$ is the position-vector of the curve C and $\mathfrak{z}(s)$ is a vector-function which lies in the generator of S passing through the point $\mathfrak{y}(s)$ of C . We take $\mathfrak{z}(s)$ as a unit vector. Thus

$$(2) \quad |\mathfrak{z}| = 1 \quad \text{and hence} \quad \mathfrak{z}\mathfrak{z}' = 0.$$

To exclude singularities along C , we assume that

$$(3) \quad \xi_3 \mathfrak{z} \neq 0.$$

Geometrically, this means that the tangent plane to S at points of C should not coincide with the osculating plane of C . The assumption that S is developable is equivalent to the condition

$$(4) \quad (\mathfrak{z}, \mathfrak{z}', \xi_1) = 0.$$

On representing $\mathfrak{z}$ in the form

$$(5) \quad \mathfrak{z} = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3, \quad \alpha_i = \xi_i \mathfrak{z}, \quad i = 1, 2, 3,$$

the conditions (2), (3), (4) are found to be equivalent to the following ones in terms of $\alpha_1, \alpha_2, \alpha_3$:

$$(6) \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$

$$(7) \quad \alpha_3 \neq 0,$$

$$(8) \quad (\alpha_2^2 + \alpha_3^2)\tau - \alpha_1\alpha_3\kappa + \alpha_2\alpha'_3 - \alpha_3\alpha'_2 = 0.$$

From (6) and (7) it follows that

$$(9) \quad |\xi_1 \mathfrak{z}| = |\alpha_1| < 1.$$

Now since

$$(\xi_1 \times \mathfrak{z})^2 = \xi_1^2 \mathfrak{z}^2 - (\xi_1 \mathfrak{z})^2 = 1 - \alpha_1^2,$$

we conclude from (9) that

$$(10) \quad \xi_1 \times \mathfrak{z} \neq 0.$$

From (1) we obtain

$$\mathfrak{r}_s = \xi_1 + t\mathfrak{z}', \quad \mathfrak{r}_t = \mathfrak{z}.$$

For the first fundamental quantities E, F, G there follow the formulas

$$E = 1 + 2t\xi_1\mathfrak{z}' + t^2(\mathfrak{z}')^2, \quad F = \xi_1\mathfrak{z}, \quad G = 1,$$

$$W^2 = EG - F^2 = 1 - (\xi_1\mathfrak{z})^2 + 2t\xi_1\mathfrak{z}' + t^2(\mathfrak{z}')^2.$$

Since $|\xi_1\mathfrak{z}| < 1$ by (9), we have $W > 0$ for $t = 0$. Hence, if the absolute value of t is sufficiently small, then the representation (1) is regular in the sense that $W > 0$ throughout.

If (2), (3), (4) hold and $|t|$ is so restricted that $W > 0$, then the surface S given by (1) will be termed an admissible developable surface through the curve C .

1.4. The unit normal vector ξ . Given an admissible developable surface S through C , the condition 1.3(4) implies that the unit normal vector

$$(1) \quad \xi = \frac{\mathfrak{r}_s \times \mathfrak{r}_t}{W}$$

is independent of t . Accordingly, if we calculate $\mathfrak{x}_s$, $\mathfrak{x}_t$ from 1.3(1) for $t = 0$, we obtain

$$(2) \quad \xi = \frac{\xi_1 \times \mathfrak{z}}{|\xi_1 \times \mathfrak{z}|}.$$

In terms of the representation 1.3(5) of $\mathfrak{z}$ we obtain further

$$(3) \quad \xi = \frac{\alpha_2 \xi_3 - \alpha_3 \xi_2}{(\alpha_2^2 + \alpha_3^2)^{\frac{1}{2}}}.$$

We wish to express $\mathfrak{z}$ in terms of ξ . Differentiation of the relation $\xi \xi_1 = 0$ with respect to s yields

$$(4) \quad \xi' \xi_1 + \kappa \xi \xi_2 = 0.$$

Now $\kappa > 0$ by assumption, and by (3) we have

$$\xi \xi_2 = -\frac{\alpha_3}{(\alpha_2^2 + \alpha_3^2)^{\frac{1}{2}}} \neq 0$$

in view of 1.3(7). Hence, by (4), $\xi' \xi_1 \neq 0$, and thus

$$(5) \quad \xi' \neq 0.$$

Next, the relation $(\mathfrak{z}, \mathfrak{z}', \xi_1) = 0$ yields $\mathfrak{z}'(\xi_1 \times \mathfrak{z}) = 0$. As $\xi_1 \times \mathfrak{z} \parallel \xi$ by (2), it follows that

$$(6) \quad \xi \mathfrak{z}' = 0.$$

Differentiating the relation $\xi \mathfrak{z} = 0$ (see (2)) with respect to s , we obtain $\xi' \mathfrak{z} + \xi \mathfrak{z}' = 0$. In view of (6) it follows that

$$(7) \quad \xi' \mathfrak{z} = 0.$$

By (5) it follows that

$$(8) \quad \xi \times \xi' \neq 0.$$

Let us put

$$\sigma = (\alpha_2^2 + \alpha_3^2)^{\frac{1}{2}}.$$

By (3) we have then

$$\sigma \xi = \alpha_2 \xi_3 - \alpha_3 \xi_2.$$

Differentiation with respect to s yields

$$\sigma' \xi + \sigma \xi' = \alpha_3 \kappa \xi_1 - (\alpha'_3 + \alpha_2 \tau) \xi_2 + (\alpha'_2 - \alpha_3 \tau) \xi_3.$$

In this equation we multiply on the left by ξ (in the vector-product sense) and use (3) in computing the products $\xi \times \xi_1$, $\xi \times \xi_2$, $\xi \times \xi_3$, obtaining

$$\sigma^2(\xi \times \xi') = \alpha_3 \kappa(\alpha_2 \xi_2 + \alpha_3 \xi_3) + [(\alpha_2^2 + \alpha_3^2) \tau + \alpha_2 \alpha'_3 - \alpha_3 \alpha'_2] \xi_1.$$

By 1.3(8), the coefficient of ξ_1 is equal to $\alpha_1\alpha_3\kappa$. Hence, in view of 1.3(5), we obtain

$$(9) \quad \sigma^2(\xi \times \xi') = \alpha_3\kappa\mathfrak{z}.$$

Taking the lengths, we get

$$(10) \quad \sigma^2 |\xi \times \xi'| = |\alpha_3| \kappa.$$

Now $\sigma, \kappa, \alpha_3, |\xi \times \xi'|$ are all different from zero (see (8) and 1.3(7)). Thus (9) and (10) yield

$$\mathfrak{z} = \frac{\xi \times \xi'}{|\xi \times \xi'|} \operatorname{sgn} \alpha_3.$$

By (2) we have $\operatorname{sgn} \alpha_3 = -\operatorname{sgn} (\xi\xi_2)$. Thus we obtain finally

$$(11) \quad \mathfrak{z} = -\frac{\xi \times \xi'}{|\xi \times \xi'|} \operatorname{sgn} (\xi\xi_2).$$

1.5. Construction of S in terms of ξ . We wish to determine an admissible developable surface S through the curve C such that the unit normal vector of S along C coincides with a prescribed vector-function $\xi = \xi(s)$. In view of 1.3(7) and 1.4(3) the following conditions are necessary for the existence of S :

$$(1) \quad |\xi| = 1.$$

$$(2) \quad \xi\xi_1 = 0.$$

$$(3) \quad \xi\xi_2 \neq 0.$$

We proceed to verify the sufficiency of these conditions and the uniqueness of the corresponding admissible developable surface S . Differentiating (2) with respect to s we obtain

$$\xi'\xi_1 + \kappa\xi\xi_2 = 0.$$

Since $\kappa > 0$ by assumption, and $\xi\xi_2 \neq 0$ by (3), it follows that

$$(4) \quad \xi' \neq 0.$$

Hence

$$(5) \quad \xi \times \xi' \neq 0.$$

Now, if a corresponding surface S exists at all, then, by 1.4(11), at each point of C the generator of S passing through that point is parallel to the vector $\xi \times \xi' \neq 0$. Thus, if S exists at all, then S is uniquely determined by ξ . To show the existence of S , we introduce along C the vector-function (see (5), (3))

$$(6) \quad \mathfrak{z} = -\frac{\xi \times \xi'}{|\xi \times \xi'|} \operatorname{sgn} (\xi\xi_2),$$

and proceed to verify that the surface

$$(7) \quad S: \quad \mathbf{r} = \mathfrak{r}(s) + t\mathfrak{z}(s)$$

is an admissible developable surface through C whose unit normal vector along C coincides with the given ξ . From (6) we have

$$(8) \quad \xi_3 = 0.$$

Let us put

$$\lambda = -\frac{\operatorname{sgn}(\xi\xi_2)}{|\xi \times \xi'|}.$$

Then $\mathfrak{z} = \lambda(\xi \times \xi')$. Differentiation with respect to s yields

$$\mathfrak{z}' = \lambda'(\xi \times \xi') + \lambda(\xi \times \xi'').$$

Hence

$$(9) \quad \xi_3' = 0.$$

Thus the vectors ξ_1 , $\mathfrak{z}$, $\mathfrak{z}'$ are all orthogonal to ξ (see (2), (8), (9)), and hence

$$(10) \quad (\mathfrak{z}, \mathfrak{z}', \xi_1) = 0.$$

In view of (1), (2), (3), we have for ξ the representation

$$(11) \quad \xi = \beta_2 \xi_2 + \beta_3 \xi_3, \quad \beta_2 = \xi \xi_2, \quad \beta_3 = \xi \xi_3,$$

$$(12) \quad \beta_2^2 + \beta_3^2 = 1, \quad \beta_2 \neq 0.$$

Differentiation with respect to s yields

$$(13) \quad \xi' = -\beta_2 \kappa \xi_1 + (\beta_2' - \tau \beta_3) \xi_2 + (\beta_3' + \tau \beta_2) \xi_3.$$

There follows, in view of (11), the formula

$$(14) \quad \xi \times \xi' = (\beta_2 \beta_3' - \beta_3 \beta_2' + \tau) \xi_1 - \beta_2 \beta_3 \kappa \xi_2 + \beta_2^2 \kappa \xi_3.$$

From (6) and (14) we obtain

$$\xi_3 \mathfrak{z} = -\frac{\beta_2^2 \kappa}{|\xi \times \xi'|} \operatorname{sgn}(\xi \xi_2) = -\frac{\beta_2^2 \kappa}{|\xi \times \xi'|} \operatorname{sgn} \beta_2.$$

Since κ and β_2 are both different from zero, it follows that

$$(15) \quad \xi_3 \mathfrak{z} \neq 0.$$

From (6) obviously

$$(16) \quad |\mathfrak{z}| = 1.$$

In view of (16), (15) and (10), the vector-function $\mathfrak{z}$ satisfies the conditions (2), (3), (4) in 1.3. There remains to show (see 1.4(2)) that

$$(17) \quad \xi = \frac{\xi_1 \times \mathfrak{z}}{|\xi_1 \times \mathfrak{z}|}.$$

From (6) we obtain

$$\xi_1 \times \mathfrak{z} = -(\mathfrak{z} \times \xi_1) = \frac{(\xi \times \xi') \times \xi_1}{|\xi \times \xi'|} \operatorname{sgn} \beta_2 = \frac{(\xi \xi_1) \xi' - (\xi' \xi_1) \xi}{|\xi \times \xi'|} \operatorname{sgn} \beta_2 .$$

Now $\xi \xi_1 = 0$ by (2), and $\xi' \xi_1 = -\beta_2 \kappa$ by (13). Hence

$$(18) \quad \xi_1 \times \mathfrak{z} = \frac{|\beta_2| \kappa}{|\xi \times \xi'|} \xi,$$

and consequently

$$(19) \quad |\xi_1 \times \mathfrak{z}| = \frac{|\beta_2| \kappa}{|\xi \times \xi'|} .$$

Since β_2 and κ are both different from zero, division of (18) by (19) yields (17).

1.6. Standard isometric map of S . Given an admissible developable surface S through C (see 1.3), the geodesic curvature κ_g of C relative to S is given by the formula

$$(1) \quad \kappa_g = \kappa \xi \xi_3 .$$

In view of 1.4(3), 1.5(11), 1.3(5), we have for κ_g the alternative formulas

$$(2) \quad \kappa_g = \kappa \beta_3 ,$$

$$(3) \quad \kappa_g = \frac{\kappa \alpha_2}{(\alpha_2^2 + \alpha_3^2)^{\frac{1}{2}}} .$$

We now take an auxiliary (u, v) -plane, and in this plane we determine a curve C^* as follows. We first determine in the (u, v) -plane two vector-functions

$$\xi_1^*(s) = (\xi_{11}^*(s), \xi_{12}^*(s)), \quad \xi_2^*(s) = (\xi_{21}^*(s), \xi_{22}^*(s)), \quad 0 \leq s \leq l,$$

from the system of differential equations

$$(4) \quad \xi_1^{*'} = \kappa_g \xi_2^* ,$$

$$(5) \quad \xi_2^{*'} = -\kappa_g \xi_1^* ,$$

with the initial conditions

$$(6) \quad \xi_1^*(0) = (1, 0), \quad \xi_2^*(0) = (0, 1).$$

Next we determine the vector-function $\eta^*(s)$ in the (u, v) -plane by the conditions

$$(7) \quad \eta^{*'} = \xi_1^* , \quad \eta^*(0) = (0, 0).$$

The curve C^* is then defined by the representation

$$(8) \quad C^*: \quad \mathfrak{r}^* = \eta^*(s), \quad 0 \leq s \leq l.$$

In view of (6) and the special form of the system (4), (5), it follows readily that the matrix

$$\begin{vmatrix} \xi_{11}^* & \xi_{12}^* \\ \xi_{21}^* & \xi_{22}^* \end{vmatrix}$$

is orthogonal (with determinant equal to $+1$). Hence we have identically in s

$$(9) \quad |\xi_1^*| = 1, \quad |\xi_2^*| = 1, \quad \xi_1^* \xi_2^* = 0.$$

From (7) and (9) it follows that $|\eta^{*'}| = 1$, and thus s is the arc-length of C^* . From (7), (4), and (9) we obtain for the curvature κ^* of C^* the formula

$$(10) \quad \kappa^* = |\kappa_\sigma|.$$

We now define a mapping from S into the (u, v) -plane as follows. We had (see 1.3)

$$(11) \quad \mathfrak{z} = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3.$$

In the (u, v) -plane we introduce the auxiliary vector-function $\mathfrak{z}^* = \mathfrak{z}^*(s)$ by the formula

$$(12) \quad \mathfrak{z}^* = \alpha_1 \xi_1^* + (1 - \alpha_1^2)^{\frac{1}{2}} \xi_2^*.$$

Let us recall that (see 1.3(9))

$$(13) \quad |\alpha_1| < 1.$$

Using the representation 1.3(1) of S , we associate with the point $\mathfrak{z} = \eta(s) + t\mathfrak{z}(s)$ the point

$$(14) \quad \mathfrak{z}^* = \eta^*(s) + t\mathfrak{z}^*(s)$$

of the (u, v) -plane. We shall verify presently that the resulting mapping from S into the (u, v) -plane is isometric (and obviously carries the point $\eta(s)$ of C into the point $\eta^*(s)$ of C^*). Indeed, we can interpret (14) as a parametric representation for a portion of the (u, v) -plane, the corresponding first fundamental quantities E^* , F^* , G^* being given by the formulas

$$E^* = 1 + 2t\xi_{1\mathfrak{z}}^{**'} + t^2(\mathfrak{z}^{*'})^2, \quad F^* = \xi_{1\mathfrak{z}}^{**}, \quad G^* = 1.$$

Accordingly, in view of the formulas for E , F , G in 1.3, our assertion will be proved if we show that

$$(15) \quad \xi_{1\mathfrak{z}}^{**} = \xi_{1\mathfrak{z}},$$

$$(16) \quad \xi_{1\mathfrak{z}}^{**'} = \xi_{1\mathfrak{z}}',$$

$$(17) \quad (\mathfrak{z}^{*'})^2 = (\mathfrak{z}')^2.$$

Now by (12) and (11) we have

$$(18) \quad \xi_{1\mathfrak{z}}^{**} = \alpha_1, \quad \xi_{1\mathfrak{z}} = \alpha_1,$$

and (15) follows. Differentiating the preceding relations (18) with respect to s we obtain (using the Frenet formulas as well as (4), (5), (11), (12))

$$\kappa_g \xi_2^* \xi_3^* + \xi_1^* \xi_3^{*'} = \alpha_1', \quad \kappa \xi_2 \xi_3 + \xi_1 \xi_3' = \alpha_1',$$

and hence

$$\kappa_g(1 - \alpha_1^2)^{\frac{1}{2}} + \xi_1^* \xi_3^{*'} = \alpha_1' = \kappa \alpha_2 + \xi_1 \xi_3'.$$

Since, by (3),

$$\kappa_g(1 - \alpha_1^2)^{\frac{1}{2}} = \kappa_g(\alpha_2^2 + \alpha_3^2)^{\frac{1}{2}} = \kappa \alpha_2,$$

the formula (16) follows. Now let us recall (see 1.3(4)) that $(\xi_1, \xi_2, \xi_3) = 0$. Since $\xi_1 \times \xi_3 \neq 0$ (see 1.3(10)), it follows that ξ_3' is a linear combination of ξ_1 and ξ_3 :

$$(19) \quad \xi_3' = \lambda \xi_1 + \mu \xi_3.$$

Scalar multiplication by ξ_1 and ξ_3 yields

$$(20) \quad \lambda + \mu \alpha_1 = \xi_1 \xi_3', \quad \lambda \alpha_1 + \mu = 0.$$

Also, from (12), $\xi_1^* \times \xi_3^* = (1 - \alpha_1^2)^{\frac{1}{2}} (\xi_1^* \times \xi_2^*)$, and

$$(\xi_1^* \times \xi_2^*)^2 = 1$$

by (9). Since $|\alpha_1| < 1$, it follows that $\xi_1^* \times \xi_3^* \neq 0$. Hence $\xi_3^{*'}$ can be expressed as a linear combination of ξ_1^* and ξ_3^* :

$$(21) \quad \xi_3^{*'} = \lambda^* \xi_1^* + \mu^* \xi_3^*.$$

Scalar multiplication by ξ_1^* and ξ_3^* yields

$$(22) \quad \lambda^* + \mu^* \alpha_1 = \xi_1^* \xi_3^{*'}, \quad \lambda^* \alpha_1 + \mu^* = 0.$$

Now we already proved that $\xi_1^* \xi_3^{*'} = \xi_1 \xi_3'$. Since $|\alpha_1| < 1$, it follows from (20) and (22) that $\lambda^* = \lambda$, $\mu^* = \mu$. Hence, by (19) and (21),

$$\xi_3' = \lambda \xi_1 + \mu \xi_3, \quad \xi_3^{*'} = \lambda \xi_1^* + \mu \xi_3^*.$$

Since $\xi_1 \xi_3 = \alpha_1 = \xi_1^* \xi_3^*$, it follows that

$$(\xi_3')^2 = \lambda^2 + 2\alpha_1 \lambda \mu + \mu^2 = (\xi_3^{*'})^2,$$

and (17) is proved.

The isometric mapping so defined will be referred to as the *standard isometric mapping* for S (corresponding to the representation 1.3(1)).

2. ISOMETRIC MAPPINGS

2.1. Definitions. Given an admissible developable surface S through C as in 1.3, we shall say that S is of *special character* (relative to C) if C is a geodesic on S . This is the case if and only if we have $\kappa_g = \kappa \xi \xi_3 = 0$ along C . Since $\kappa > 0$

by assumption, this condition is equivalent to any one of the following (see 1.6(2), (3)):

$$(1) \quad \xi\xi_3 = 0,$$

$$(2) \quad \beta_3 = 0,$$

$$(3) \quad \alpha_2 = 0.$$

Disregarding intermediate cases, we shall consider only the further case when $\kappa_r \neq 0$ all along C , when we shall say that S is of *general character* (relative to C). Referring again to 1.6(2), (3) and also to 1.3(7), 1.5(12), we have in this case

$$(4) \quad \xi\xi_3 \neq 0,$$

$$(5) \quad \beta_3 \neq 0, \quad \beta_2 \neq 0,$$

$$(6) \quad \alpha_2 \neq 0, \quad \alpha_3 \neq 0.$$

Consider the case when S is of general character. In terms of the corresponding unit normal vector

$$(7) \quad \xi = \beta_2\xi_2 + \beta_3\xi_3, \quad \beta_2^2 + \beta_3^2 = 1, \quad \beta_2 \neq 0, \quad \beta_3 \neq 0,$$

we define along C a vector-function $\bar{\xi} = \bar{\xi}(s)$ by the formula

$$(8) \quad \bar{\xi} = \bar{\beta}_2\xi_2 + \bar{\beta}_3\xi_3 \quad \text{with} \quad \bar{\beta}_2 = -\beta_2, \quad \bar{\beta}_3 = \beta_3.$$

Then, in view of (7), we have $|\bar{\xi}| = 1$, $\bar{\xi}\xi_1 = 0$, $\bar{\xi}\xi_2 \neq 0$. Hence, by 1.5, there exists a unique admissible developable surface $\tilde{S}$ through C whose unit normal vector coincides with $\bar{\xi}$. Note that in starting with S we could use $-\xi$ instead of ξ . Clearly, $\bar{\xi}$ would then be replaced by $-\bar{\xi}$ (see (7), (8)), and thus we would get the same surface $\tilde{S}$ as before. The surface $\tilde{S}$ so obtained will be termed the surface *associated* with S (relative to C). In view of (7), (8), the associated surface $\tilde{S}$ is again of general character, and the geodesic curvature of C relative to S and $\tilde{S}$ is the same. Let us also note that $\tilde{S} \neq S$. Indeed, $\tilde{S} = S$ would imply that $\bar{\xi} = \pm\xi$. But $\bar{\xi} = \xi$ would yield $\beta_2 = 0$, and $\bar{\xi} = -\xi$ would yield $\beta_3 = 0$, in contradiction with (7).

2.2. Isometric mappings. Given the admissible developable surface S through C , we want to determine all those isometric mappings m of S which leave C point-wise fixed. The image $\tilde{S}$ of S under such a mapping m is again a developable surface through C , and on denoting by $\bar{\kappa}_r$, κ_r the geodesic curvatures of C relative to $\tilde{S}$, S respectively, we have

$$(1) \quad |\bar{\kappa}_r| = |\kappa_r|.$$

To exclude singularities, we restrict m by the requirement that the image surface $\tilde{S} = m(S)$ should be admissible relative to C , in the sense of 1.3. On using representations

$$(2) \quad S: \quad \mathfrak{x} = \mathfrak{v}(s) + t_3(s),$$

$$(3) \quad \tilde{S}: \quad \mathfrak{x} = \mathfrak{v}(s) + t_3^{\sim}(s),$$

$$(4) \quad \mathfrak{z} = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3,$$

$$(5) \quad \tilde{\mathfrak{z}} = \tilde{\alpha}_1 \xi_1 + \tilde{\alpha}_2 \xi_2 + \tilde{\alpha}_3 \xi_3,$$

we have then (see 1.3)

$$(6) \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \quad \alpha_3 \neq 0,$$

$$(7) \quad \tilde{\alpha}_1^2 + \tilde{\alpha}_2^2 + \tilde{\alpha}_3^2 = 1, \quad \tilde{\alpha}_3 \neq 0.$$

For the unit normal vectors $\xi, \tilde{\xi}$ we have (see 1.5)

$$(8) \quad \xi = \beta_2 \xi_2 + \beta_3 \xi_3, \quad \beta_2^2 + \beta_3^2 = 1, \quad \beta_2 \neq 0,$$

$$(9) \quad \tilde{\xi} = \tilde{\beta}_2 \xi_2 + \tilde{\beta}_3 \xi_3, \quad \tilde{\beta}_2^2 + \tilde{\beta}_3^2 = 1, \quad \tilde{\beta}_2 \neq 0.$$

The isometric mapping m of S (leaving C point-wise fixed) is termed *trivial* or *non-trivial* according as it does or does not reduce to the identity mapping on S .

Case 1. S is of general character relative to C . Then $\kappa_c \neq 0$, and hence by (1) also $v_c \neq 0$, and thus $\tilde{S}$ is also of general character relative to C . Accordingly, by 2.1, we have now the further relations

$$(10) \quad \alpha_2 \neq 0, \quad \tilde{\alpha}_2 \neq 0, \quad \beta_3 \neq 0, \quad \tilde{\beta}_3 \neq 0.$$

By replacing, if necessary, $\mathfrak{z}$ by $-\mathfrak{z}$ and $\tilde{\mathfrak{z}}$ by $-\tilde{\mathfrak{z}}$, we can therefore assume that (see 1.4(3))

$$(11) \quad \alpha_2 > 0, \quad \tilde{\alpha}_2 > 0, \quad \beta_3 > 0, \quad \tilde{\beta}_3 > 0.$$

Then $\kappa_c = \kappa\beta_3 > 0$, and $\tilde{\kappa}_c = \kappa\tilde{\beta}_3 > 0$, and hence, by (1), $\kappa\beta_3 = \kappa\tilde{\beta}_3$. Since $\kappa > 0$, it follows that $\beta_3 = \tilde{\beta}_3$, and hence (see (8), (9)) either $\tilde{\beta}_2 = \beta_2$ or else $\tilde{\beta}_2 = -\beta_2$ all along C . In the first case, $\xi = \tilde{\xi}$ and hence (see 1.5) $\tilde{S} = S$. In the second case $\tilde{S} = \bar{S}$, the surface associated with S , in the sense of 2.1.

Case 1.1. $m(S) = S$. Let then f be the standard isometric mapping from S into the (u, v) -plane (see 1.6). For the image C^* of C under f we have then (see 1.6(10)) $\kappa^* = |\kappa_c| > 0$, and thus C^* is not a straight segment. The mapping fmf^{-1} is then an isometric mapping of a portion of the plane upon itself, leaving C^* point-wise fixed. Since such a mapping is distance-preserving, and since C^* is not a straight segment, it follows that $fmf^{-1} = i$, the identity mapping in the plane. Clearly, it follows that m is the identity mapping on S , and hence in Case 1.1 the only mapping m available is the trivial one.

Case 1.2. $m(S) = \bar{S}$, the surface associated with S . Let $f, \bar{f}$ be the standard isometric mappings from $S, \bar{S}$ respectively into the (u, v) -plane, in the sense of 1.6. Since now $\kappa_c = \tilde{\kappa}_c > 0$, by the formulas in 1.6, it follows that $f, \bar{f}$ agree along C , yielding the same image curve C^* with $\kappa^* > 0$. Thus C^* is not a straight segment.

The mapping $\bar{f}mf^{-1}$ is then an isometric mapping of a portion of the (u, v) -plane upon itself, leaving C^* point-wise fixed. As under Case 1.1, it follows that $\bar{f}mf^{-1} = i$, the identity mapping in the (u, v) -plane, and hence $m = \bar{f}^{-1}f$. Since $f, \bar{S}, \bar{f}$ are uniquely determined by S (and since $\bar{f}^{-1}f$ is clearly an isometric mapping from S , leaving C point-wise fixed), it follows that in Case 1.2 there exists exactly one mapping m , and this unique m is $\bar{f}^{-1}f$. Since (see 2.1) $\bar{S} \neq S$, this mapping m is clearly nontrivial (does not reduce to the identity mapping on S).

Case 2. S is of special character relative to C . Then $\kappa_\sigma = 0$ and hence by (1) also $\bar{\kappa}_\sigma = 0$. Since $\kappa_\sigma = \kappa\beta_3$, $\bar{\kappa}_\sigma = \kappa\bar{\beta}_3$, $\kappa > 0$, it follows that $\bar{\beta}_3 = \beta_3 = 0$ and hence $|\frac{1}{2}\beta_2| = |\beta_2| = 1$ all along C . Hence, by (8) and (9), $\xi = \pm\xi$, and consequently $S = \bar{S}$ (see 1.5). Thus in Case 2 we have necessarily $m(S) = S$, and also (by the preceding comments) $\beta_3 = 0, \beta_2 = \pm 1$ all along C . By replacing, if necessary, ξ by $-\xi$, we can assume that $\beta_2 = 1$, and hence (see (8)) $\xi = \xi_2$. Let again f be the standard isometric mapping from S into the (u, v) -plane. Since now $\kappa_\sigma = 0$, from the formulas in 1.6 it follows that the image $C^* = f(C)$ is a straight segment on the u -axis. The mapping fmf^{-1} is an isometric mapping of a portion of the (u, v) -plane upon itself, leaving the straight segment C^* on the u -axis point-wise fixed. Hence either $fmf^{-1} = i$, the identity mapping in the (u, v) -plane, or else $fmf^{-1} = \rho$, the reflection upon the u -axis in the (u, v) -plane.

Case 2.1. $fmf^{-1} = i$. Then clearly m is the identity mapping on S , and thus m is trivial.

Case 2.2. $fmf^{-1} = \rho$. Then $m = f^{-1}\rho f$. Since f is uniquely determined, and since $f^{-1}\rho f$ is clearly an isometric mapping of S onto itself leaving C point-wise fixed, it follows that there exists now a unique mapping m , and this unique m is given by $f^{-1}\rho f$. Furthermore, this m is non-trivial. Indeed, if $m = f^{-1}\rho f$ were the identity mapping on S , then it would follow that ρ is the identity mapping in the (u, v) -plane, a contradiction.

2.3. Rigid mappings. A mapping of S is termed *rigid* if the distance of each pair of points of S is preserved. There arises the question whether the unique non-trivial isometric mapping m of S , leaving C point-wise fixed, is rigid. Suppose m is rigid. If C would contain four non-coplanar points, then m would have to reduce to the identity mapping, in contradiction with the fact that m is non-trivial. Thus C must be a plane curve. Conversely, if C is a plane curve, then the reflection upon the plane of C is clearly a non-trivial isometric mapping of S , leaving C point-wise fixed (note that S is admissible, in the sense of 1.3).

2.4. Summary. Given the admissible developable surface S through C as in 1.3, by the analysis in 2.2, 2.3 there exists exactly one mapping m of S with the following properties:

- (a) m does not reduce to the identity mapping on S .
- (b) m is isometric and leaves C point-wise fixed.
- (c) The image surface $m(S)$ is admissible relative to C , in the sense of 1.3.

For this unique m the following statements hold, by 2.2, 2.3:

(i) If S is of general character relative to C , then the image surface $m(S)$ is distinct from S and coincides with the surface $\bar{S}$ associated with S (in the sense of 2.1). Furthermore, if $f, \bar{f}$ are the standard isometric mappings from $S, \bar{S}$ respectively into the (u, v) -plane (in the sense of 1.6), then $m = \bar{f}^{-1}f$.

(ii) If S is of special character relative to C (that is, if C is a geodesic on S) then $m(S) = S$. Furthermore, if f is the standard isometric mapping from S into the (u, v) -plane, then $m = f^{-1}\rho f$, where ρ is the reflection upon the u -axis in the (u, v) -plane.

(iii) m is a rigid mapping if and only if C is a plane curve, and in this case it coincides with the reflection upon the plane of C .

Let us recall that the curve C is assumed to have positive curvature. Clearly, if C were permitted to reduce to a straight segment, then the preceding statements would be false (consider rotations around C of arbitrary angle).

3. RIGIDITY PROPERTIES

Given a surface S containing a curve C , let $\mathfrak{R}(S, C)$ be the class of those isometric mappings m of S which leave C point-wise fixed. A mapping $m \in \mathfrak{R}(S, C)$ is termed *trivial* or *non-trivial* according as it does or does not reduce to the identity mapping on S .

An *isometric deformation* of S , leaving C point-wise fixed, is a one-parameter family $m_t \in \mathfrak{R}(S, C)$ such that m_0 is the identity mapping on S . If $m_t \equiv m_0$, then m_t is termed *trivial*; otherwise m_t is termed *non-trivial*.

As regards the concept of S being *rigid* for C point-wise fixed, several choices are apparently available. Let us introduce the following terminology:

S is said to possess the property $R(C)$ if S admits only of the trivial isometric deformation leaving C point-wise fixed.

S is said to possess the property $R(C, n)$, where n is a positive integer, if the class $\mathfrak{R}(S, C)$ defined above contains not more than n elements (note that $\mathfrak{R}(S, C)$ contains at least one element, namely, the identity mapping on S).

Clearly, if for some positive integer n the surface S possesses the property $R(C, n)$, then S also possesses the property $R(C)$. Also, if $n' < n''$, then clearly the property $R(C, n')$ implies the property $R(C, n'')$.

In the way of illustration, consider the case when S is a sphere and C is a circle on S . Then the reflection on the plane of C yields a non-trivial isometric mapping of S which leaves C point-wise fixed. Thus S does not possess the property $R(C, 1)$. In view of this simple example, it would seem that the property $R(C, 1)$ represents an unreasonable requirement, and thus one may consider the property $R(C, 2)$ as a significant rigidity property of a surface (relative to a curve C).

The results in 2.4 may now be re-stated as follows: Given a developable surface S through a curve C as in 1.3 (thus, in particular, the curvature of C is different from zero), S possesses the property $R(C, 2)$. We hope to establish similar results for other significant classes of surfaces on another occasion.

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