Interior Estimates for Parabolic Systems of Partial Differential Equations

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Communicated by E. Hopf

Introduction. Interior estimates for second order elliptic equations have been derived by J. SCHAUDER [8][9]. More recently A. DOUGLIS & L. NIRENBERG [2] have extended these estimates to elliptic systems of great generality. Their method, specialized to second order equations, is a simplification and modification of SCHAUDER's method, some of the ideas occurring already in an earlier paper of E. HOPF [6]. They have also used their interior estimates to prove differentiability of solutions of elliptic systems, both linear and nonlinear.

Interior estimates for solutions of second order parabolic equations were derived by RICHARD B. BARRAR in his unpublished thesis in 1952. His method is patterned on SCHAUDER's, the results also being completely analogous to SCHAUDER's results. The special case of one space-dimension was derived more recently also by C. CILIBERTO [1].

In this paper we shall obtain interior estimates analogous to SCHAUDER's for solutions of general parabolic systems. Our method is patterned on that of DOUGLIS & NIRENBERG rather than on SCHAUDER's and, therefore, specialized to second order parabolic equations, it presents a simplification of the results of BARRAR and CILIBERTO.

In §1 we introduce an appropriate distance-function in the (x, t) -space and define accordingly the notion of Hölder continuity. Then we introduce norms in (1.5), (1.6), the latter being somewhat different from the analogous norm in [2]. We mention several elementary but useful lemmas.

In §2 we define the notion of linear parabolic systems and obtain estimates for the solutions of homogeneous systems with constant coefficients (Theorem 1). Theorem 1 is the analogue of a potential-theoretic theorem [2; Theorem 2]. Most ideas in the proof occur in [2] and [6], but some technical differences arise from the different nature of the singularity of the fundamental solutions.

In §3 we prove the main result of this paper (Theorem 3). The proof is analogous to that of [2], via Theorem 1 of §2.

Prepared under ONR Contract Nonr—908(09), NR 041 037 with Indiana University.

In §§4 and 5 we give some applications to the question of differentiability of solutions. In §4 we consider linear systems and our results generalize all the known results in this direction (see EIDELMAN [3]). In §5 we prove differentiability of solutions of nonlinear systems. The proofs of Theorems 4 and 7 are patterned on the corresponding proofs for the elliptic case [2] (see also [6]).

1. Notation and Preliminary Lemmas. We consider the following linear system of equations

$$(1.1) \quad \sum_{\mu=1}^N \sum_{0 \leq \sum_{i,j} i_j \leq m} a_{i_1 \dots i_n}^{\lambda_\mu}(x, t) \frac{\partial^i u_\mu}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} - \frac{\partial u_\lambda}{\partial t} = f_\lambda(x, t)$$

$$(\lambda = 1, \dots, N)$$

in an $(n + 1)$ -dimensional domain D of the variables $(x, t) = (x_1, \dots, x_n, t)$. The system (1.1) is called *parabolic* if the real parts of the characteristic roots of the matrix

$$((-i)^m \sum_{\sum_{i,j} i_j = m} a_{i_1 \dots i_n}^{\lambda_\mu}(x, t) \xi_1^{i_1} \dots \xi_n^{i_n}) \quad (\xi \text{ real}, \quad |\xi| = 1)$$

are negative. For parabolic systems m is necessarily even. We shall be interested in deriving Schauder type estimates in the interior for the parabolic system (1.1). With this purpose in mind, we shall define in this section an appropriate concept of Hölder continuity, introduce appropriate norms, and mention a few simple properties which will be used in the following sections.

We shall assume throughout this paper that D is a bounded domain. The *distance* between two points $P = (x, t)$ and $Q = (x', t')$ is defined by

$$(1.2) \quad d(P, Q) = (|x - x'|^2 + (t - t')^{2/m})^{\frac{1}{2}},$$

where $|x - x'|$ denotes the Euclidean distance between x and x' . It is easy to show that the function $d(P, Q)$ defines a proper metric. Let $R = (\xi, \tau)$ belong to D , and denote by Δ_τ the boundary of D which lies in the half space $t < \tau$. Denote by d_R the distance from R to the set Δ_τ . If P and Q belong to D , denote $d_{PQ} = \min(d_P, d_Q)$. We say that S is a *semi-cube* with *top* $P = (x^0, t^0)$ and *edge* δ , if S is defined by

$$x_i^0 - \delta < x_i < x_i^0 + \delta \quad (i = 1, \dots, n),$$

$$t^0 - \delta^m < t < t^0 \quad (x^0 = (x_1^0, \dots, x_n^0)).$$

Note that if Q belongs to S , then $d(P, Q) < (n + 1)^{\frac{1}{2}}\delta$.

A function $g(P)$ is *Hölder continuous* (exponent α) in the domain D if for every closed subset of D there exists a constant H such that, for all pairs of points P, Q in the closed subset,

$$\frac{|g(P) - g(Q)|}{d(P, Q)^\alpha} \leq H.$$

Denote by $D_x^i g$ any i^{th} partial x -derivative of g , and let $D_t g = \partial g / \partial t$. We define the following norms (see analogous norms in [2]):

$$(1.3) \quad M_{p,i}[g] = \text{l.u.b.}_{P \in D} d_P^{p+i} |D_x^i g(P)|,$$

$$(1.4) \quad M_{p,i+\alpha}[g] = \text{l.u.b.}_{P, Q \in D} d_{PQ}^{p+i+\alpha} \frac{|D_x^i g(P) - D_x^i g(Q)|}{d(P, Q)^\alpha} \quad (0 < \alpha < 1),$$

where p, i are integers, $p + i \geq 0, i \geq 0$ and all the i^{th} x -derivatives of g are supposed to exist and to be continuous in D . The l.u.b. is first taken with respect to all the derivatives D_x^i of order i . We also define the norms

$$(1.5) \quad |g|_{p,i} = \sum_{j=\max(0, -p)}^i M_{p,i}[g],$$

$$(1.6) \quad |g|_{p,i+\alpha} = |g|_{p,i} + \sum_{j=\max(0, -p)}^i M_{p,i+\alpha}[g] \quad (0 < \alpha < 1).$$

Here we assume that p, i are integers, $i \geq 0, p + i \geq 0$; if $p + i < 0$, we define $|g|_{p,i} = 0$.

We mention a few properties of the norms $|g|_{p,i}$ which are analogous to those mentioned in [2]. Proofs will be omitted.

Lemma 1. Assume $p \geq 0$ and $i \geq 1$ are integers. Then, for every positive $\epsilon < 1$,

$$(1.7) \quad |u|_{p,i-1} \leq \epsilon M_{p,i}[u] + CM_{p,0}[u],$$

provided $M_{p,i}[u]$ and $M_{p,0}[u]$ are finite. C is a constant depending only on ϵ, p, i .

Lemma 2. For $a \geq 0$ and integers $p, q, i, j \geq 0$,

$$(1.8) \quad |D_x^i u|_{a,a} \leq C_1 |u|_{a-j, a+i},$$

$$(1.9) \quad |uw|_{p+a,i} \leq C_2 |u|_{p,i} |v|_{q,a},$$

$$(1.10) \quad |uw|_{p+a,a} \leq C_3 |u|_{p,a} |v|_{q,a},$$

$$(1.11) \quad |u|_{p,a+1} \leq C_4 |u|_{p,a-[a]} + C_5 \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|_{p+1,a},$$

where $[a]$ is the largest integer $\leq a$ and C_k are constants depending on a, p, q, i, j .

For any subdomain A of D we shall denote any norm, such as $| \cdot |_{p,a}, M_{p,a}$, defined with respect to A by $| \cdot |_A^{p,a}, M_{p,a}^A$. Clearly

$$(1.12) \quad |u|_{p,a}^A \leq |u|_{p,a}.$$

Lemma 3. Let S be a semi-cube in D with top P and edge $r = d_P/2(n+1)^{\frac{1}{2}}$. Then

$$r |u|_{p,a}^S \leq |u|_{p+1,a} \quad (p \text{ integer, } a \text{ real}).$$

Lemma 4. Let u be defined in D and suppose for some nonnegative p, a there exists a constant H such that, for every point P of D , $r |u|_{p,a}^S \leq H$ holds, where S

is the semi-cube with top P and edge $r = d_P/2(n+1)^{\frac{1}{2}}$. Then $|u|_{p+1,a}$ exists and is bounded by CH , where C is a constant depending on p and a .

2. An Auxiliary Theorem. We shall use the following notation:

$$i = (i_1, \dots, i_n), \quad D_x^i = \frac{\partial^i}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}, \quad a_i^{\lambda\mu} = a_{i_1 \dots i_n}^{\lambda\mu}, \quad \xi^i = \xi_1^{i_1} \dots \xi_n^{i_n}.$$

Then (1.1) takes the form

$$(2.1) \quad L_\lambda u \equiv \sum_{\mu=1}^N \sum_{0 \leq i \leq m} a_i^{\lambda\mu}(x, t) D_x^i u_\mu - D_t u_\lambda = f_\lambda(x, t) \quad (\lambda = 1, \dots, N).$$

In this section we derive some estimates for the solution u_μ , under the following assumptions on L_λ and f_λ :

- (a) All the coefficients $a_i^{\lambda\mu}(x, t)$ are constants, and $a_i^{\lambda\mu} = 0$ if $i < m$.
- (b) The real parts of the characteristic roots of the matrix

$$((-1)^{\frac{1}{2}m} \sum_m a_m^{\lambda\mu} \xi^m) \quad (\xi \text{ real}, \quad |\xi| = 1)$$

are smaller than $-\alpha < 0$.

- (c) $|a_m^{\lambda\mu}|$ ($\lambda, \mu = 1, \dots, N$) is bounded by β ; $f_\lambda(x, t)$ ($\lambda = 1, \dots, N$) is Hölder continuous (exponent α) in a semi-cube S with top P and edge d .

In what follows we shall denote by C constants depending on n, m , and by K constants depending on n, m, α and β . We also use the notation

$$H_{R,S}[g] = \text{l.u.b.}_{B \in S} \frac{|g(R) - g(B)|}{d(R, B)^\alpha}.$$

Theorem 1. Assume that the assumptions (a), (b), (c) hold. If $u_\mu(x, t)$ ($\mu = 1, \dots, N$) is a solution of (2.1) in S with m continuous x -derivatives, then $D_x^i u_\mu$ ($0 \leq i \leq m$) is Hölder continuous (exponent α) in S and there exists a constant K such that, for $\mu = 1, \dots, N$ and $0 \leq i \leq m$,

$$(2.2) \quad |D_x^i u_\mu(P)| \leq d^{-i} K \text{l.u.b.}_S |u_\lambda| + d^{m-i} K \text{l.u.b.}_S |f_\lambda| + d^{m-i+\alpha} K \sum_{\lambda=1}^N H_{P,S}[f_\lambda] \equiv KI,$$

$$(2.3) \quad d^\alpha \frac{|D_x^i u_\mu(P) - D_x^i u_\mu(Q)|}{d(P, Q)^\alpha} \leq KI + K d^{m-i+\alpha} \sum_{\lambda=1}^N H_{Q,S}[f_\lambda] \quad \text{if } d(P, Q) \leq \frac{d}{4}.$$

Proof. The proof will be given for the case $N = 1$; the proof of the general case is quite analogous. (2.1) takes the form

$$(2.4) \quad Lu \equiv \sum_m a_m D_x^m u - D_t u = f(x, t).$$

Denote by S_η the semi-cube with top P and edge ηd , and let $\varphi(Q)$ be an infinitely differentiable function defined in S , which satisfies the conditions:

$$(2.5) \quad \varphi(Q) = \begin{cases} 1 & \text{if } Q \in S_{\frac{1}{2}} \\ 0 & \text{if } Q \in S - S_{\frac{1}{2}}, \end{cases}$$

$$(2.6) \quad |D_i^l D_x^k \varphi(x, t)| \leq A_{lk} d^{-m l - k} \quad (k, l = 0, 1, \dots; A_{lk} \text{ constants}).$$

Let $P = (x^0, t^0 + d^m)$ and denote by R the lower basis of S . Let $G(x, t)$ be the fundamental solution of $L^* u = 0$ (L^* the adjoint of L) with the singularity at the origin (for $N = 1$ see [7], for general N see [4]), and apply Green's identity to the functions $u(x, t)$ and $\varphi(x, t)G(x - \xi, t - \tau)$. Integrating this identity and using (2.4) and (2.5), we get

$$(2.7) \quad \begin{aligned} u(\xi, \tau) &= \int_{t^0}^{\tau} \int_R \varphi(x, t) f(x, t) G(x - \xi, t - \tau) dx dt \\ &\quad - \int_{t^0}^{\tau} \int_R u(x, t) L^* [\varphi(x, t) G(x - \xi, t - \tau)] dx dt = H_0 - J_0, \end{aligned}$$

where H_0 and J_0 denote the first and second integrals respectively. Later we shall make use of the inequalities ([4][7])

$$(2.8) \quad |D_i^j D_x^j G(x - \xi, t - \tau)| \leq \frac{K}{(\tau - t)^{(n+m+i+j)/m}} \exp \left\{ \frac{-K |x - \xi|^{m/(m-1)}}{(\tau - t)^{1/(m-1)}} \right\} \\ (i, j = 0, 1, \dots)$$

where K depends on i and j . We shall first prove (2.2), (2.3) for $i = m$. Denoting

$$(2.9) \quad H = D_x^m H_0, \quad J = D_x^m J_0$$

we proceed to estimate J and H at all points $A = (\xi, \tau)$ whose distance from P is less than $\frac{1}{4}d$. If we denote by Ω the domain $\{S_{\frac{1}{2}} - S_{\frac{1}{2}}\} \cap \{t^0 < t < \tau\}$, then

$$(2.10) \quad J = \int_{\Omega} u(x, t) \left\{ \left(\sum_m a_m D_x^m + D_t \right) [\varphi(x, t) D_t^m G(x - \xi, t - \tau)] \right\} dx dt.$$

Using (2.6), (2.8), we obtain

$$(2.11) \quad |J| \leq K \text{ l.u.b.}_S |u| \sum_{i=0}^m d^{-m+i} \cdot \int_{\Omega} \frac{1}{(\tau - t)^{(n+m+i)/m}} \exp \left\{ -K \frac{|x - \xi|^{m/(m-1)}}{(\tau - t)^{1/(m-1)}} \right\} dx dt.$$

Write the last integral as $\int_{\Omega_1} + \int_{\Omega_2}$, where Ω_1 is the subset of Ω in which $\frac{1}{2}d \leq |x - x^0| \leq \frac{3}{4}d$, and $\Omega_2 = \Omega - \Omega_1$. Substituting $z = d^m/(\tau - t)$ in $\int_{\Omega_1}$, we obtain

$$(2.12) \quad \int_{\Omega_1} \leq K d^n \int_0^{\infty} \frac{\exp \{ -K z^{1/(m-1)} \}}{d^{n+m+i}} z^{(n+m+i)/m} \frac{d^m}{z^2} dz \leq \frac{K}{d^i}.$$

For $\int_{\Omega_2}$ we have

$$(2.13) \quad \int_{\Omega_2} \leq K \frac{1}{d^{n+m+i}} \text{vol. } \Omega_2 \leq \frac{K}{d^i}.$$

Combining (2.12), (2.13), we conclude from (2.11) that

$$(2.14) \quad |J| \leq K d^{-m} \text{l.u.b.}_s |u| \quad (J = J(A)).$$

To estimate H , we use the following equality:

$$(2.15) \quad \begin{aligned} H &= \int_{t^0}^{\tau} \int_R D_{\xi}^m G(x - \xi, t - \tau) [\varphi(x, t) f(x, t) - \varphi(\xi, \tau) f(\xi, \tau)] dx dt \\ &\quad + f(\xi, \tau) D_{\xi}^{m-1} \int_{t^0}^{\tau} \int_R D_{\xi} G(x - \xi, t - \tau) dx dt = H_1 + H_2 \end{aligned}$$

where H_1 and H_2 denote the first and second terms respectively. The proof of (2.15) is given in the Appendix.

Using the simple inequality

$$(2.16) \quad \begin{aligned} |\varphi(B)f(B) - \varphi(A)f(A)| &\leq C d(A, B)^{\alpha} H_{A, s}[f] \\ &\quad + C \left(\frac{|x - \xi|}{d} + \frac{\tau - t}{d^m} \right) \text{l.u.b.}_s |f|, \end{aligned}$$

where $B = (x, t)$, we have

$$(2.17) \quad \begin{aligned} H_1 &\leq K H_{A, s}[f] \int_{t^0}^{\tau} \int_R \frac{1}{(\tau - t)^{(n+m)/m}} \exp \left\{ -K \frac{|x - \xi|^{m/(m-1)}}{(\tau - t)^{1/(m-1)}} \right\} \\ &\quad \cdot [|x - \xi|^{\alpha} + (\tau - t)^{\alpha/m}] dx dt \\ &\quad + K \text{l.u.b.}_s |f| \int_{t^0}^{\tau} \int_R \frac{1}{(\tau - t)^{(n+m)/m}} \exp \left\{ -K \frac{|x - \xi|^{m/(m-1)}}{(\tau - t)^{1/(m-1)}} \right\} \\ &\quad \cdot \left[\frac{|x - \xi|}{d} + \frac{\tau - t}{d^m} \right] dx dt. \end{aligned}$$

Substituting $|x - \xi| = \rho(\tau - t)^{1/m}$ in each of the above integrals and noting that $\tau - t^0 < d^m$, we easily find that

$$(2.18) \quad |H_1| \leq K d^{\alpha} H_{A, s}[f] + K \text{l.u.b.}_s |f|.$$

To estimate H_2 we use the inequality

$$(2.19) \quad \int_{t^0}^{\tau} \int_{\dot{R}} |D_{\xi}^{m-1} G(x - \xi, t - \tau)| dS_x dt \leq K,$$

where $\dot{R}$ is the boundary of R and dS_x is the volume element on $\dot{R}$. To prove (2.19) we use (2.8), then substitute $z = d^m/(\tau - t)$ and note that (in the above

integral) $|x - \xi| \sim d$, $\tau - t^0 \sim d^m$. Using (2.19) and the definition of H_2 , we conclude that

$$(2.20) \quad |H_2| \leq K \underset{s}{\text{l.u.b.}} |f|.$$

Combining (2.17) with (2.15) and (2.20), we obtain

$$(2.21) \quad |H| \leq K d^\alpha H_{A,s}[f] + K \underset{s}{\text{l.u.b.}} |f| \quad (H = H(A)).$$

Taking the point $A = (\xi, \tau)$ to be equal to P , using (2.7), (2.9) and (2.14), (2.21), the proof of (2.2), for $i = m$, is completed.

Derivation of (2.3) for $i = m$. To prove (2.3) ($i = m$), write

$$(2.22) \quad d^\alpha \frac{|D_\xi^m u(P) - D_\xi^m u(Q)|}{d(P, Q)^\alpha} \leq d^\alpha \frac{|H(P) - H(Q)|}{d(P, Q)^\alpha} + d^\alpha \frac{|J(P) - J(Q)|}{d(P, Q)^\alpha} = H' + J'$$

where H' and J' denote the first and second terms respectively. We proceed to estimate J' .

Let $P = (\xi, \tau)$ (note that $\xi = x^0$, $\tau = t^0 + d^m$), $Q = (\xi', \tau')$, so that $\tau \geq \tau'$. Using the definition of J in (2.10), we get

$$(2.23) \quad \begin{aligned} |J'| &\leq \frac{d^\alpha}{d(P, Q)^\alpha} \left(\sum_{i,m} |a_m| |D_x^{m-i} \varphi| \right) C \underset{s}{\text{l.u.b.}} |u| \\ &\quad \cdot \int_{\Omega'} |D_x^{m+i} G(x - \xi, t - \tau) - D_x^{m+i} G(x - \xi', t - \tau')| dx dt \\ &\quad + \frac{d^\alpha}{d(P, Q)^\alpha} \left(\sum_{i,m} |a_m| |D_x^{m-i} \varphi| \right) C \underset{s}{\text{l.u.b.}} |u| \\ &\quad \cdot \int_{\Omega'',} |D_x^{m+i} G(x - \xi, t - \tau)| dx dt \\ &\quad + \frac{d^\alpha}{d(P, Q)^\alpha} |D_t \varphi| \underset{s}{\text{l.u.b.}} |u| \\ &\quad \cdot \int_{\Omega'} |D_x^m G(x - \xi, t - \tau) - D_x^m G(x - \xi', t - \tau')| dx dt \\ &\quad + \frac{d^\alpha}{d(P, Q)^\alpha} |D_t \varphi| \underset{s}{\text{l.u.b.}} |u| \\ &\quad \cdot \int_{\Omega'',} |D_x^m G(x - \xi, t - \tau)| dx dt = J'_1 + J'_2 + J'_3 + J'_4, \end{aligned}$$

where J'_i denotes the i^{th} term on the right side of the inequality, and

$$\Omega' = \{S_1 - S_1\} \cap \{t_0 < t < \tau'\}$$

$$\Omega'' = \{S_1 - S_1\} \cap \{\tau' < t < \tau\}.$$

Clearly, it is enough to estimate J'_1 , J'_2 . For the first integral in (2.23) we use the Mean Value Theorem and obtain

$$\int_{\Omega'} \leq \int_{\Omega'} [|D_x^{m+i+1} G| |\xi - \xi'| + |D_x^{m+i} D_t G| (\tau - \tau')] dx dt.$$

We now split Ω' into two parts, as in the case of evaluating the integrals of (2.11), and make use of (2.8). We then obtain (in analogy with (2.12), (2.13))

$$(2.24) \quad \int_{\Omega'} \leq K |\xi - \xi'| d^{-i-1} + K(\tau - \tau') d^{-i-m}.$$

Using (2.6) and the inequalities

$$0 \leq (\tau - \tau')^{1/m} \leq d(P, Q) \leq C d, \quad |\xi - \xi'| \leq d(P, Q) \leq C d,$$

we conclude from (2.24) that

$$(2.25) \quad J'_1 \leq K d^{-m} \text{l.u.b.}_s |u|.$$

We proceed to estimate J'_2 . Noting that $|x - \xi| \sim d$ in Ω'' , we get

$$(2.26) \quad J'_2 \leq K \frac{d^\alpha}{d(P, Q)^\alpha} \sum_i d^{i-m} (\text{l.u.b.}_s |u|) d^n \cdot \int_{\tau'}^\tau \frac{1}{(\tau - t)^{(n+m+i)/m}} \exp \left\{ -\frac{K d^{m/(m-1)}}{(\tau - t)^{1/(m-1)}} \right\} dt.$$

Substituting $z = [d^m/(\tau - t)]^{1/(m-1)}$ we find that the last integral is bounded by

$$\frac{C}{d^{n+i}} \int_A^\infty e^{-Kz} z^{[(n-m+i)/m](m-1)} z^{m-2} dz \leq \frac{C}{d^{n+i}} e^{-\frac{1}{2}KA} \leq \frac{C'}{d^{n+i}} \left(\frac{2}{KA} \right)^\mu$$

where $A = [d^m/(\tau - \tau')]^{1/(m-1)}$, μ is any positive number and $C' = C'(\mu)$. Taking $\mu = (m-1)\alpha/m$ we obtain

$$(2.27) \quad J'_2 \leq K d^{-m} \text{l.u.b.}_s |u|.$$

Combining (2.27) with (2.25) and using (2.23) we get

$$(2.28) \quad |J'| \leq K d^{-m} \text{l.u.b.}_s |u|.$$

Estimation of H' defined by (2.22), (2.15). It remains to estimate H' . Using the definition of H' ((2.22), (2.15)), one can easily verify that

$$\begin{aligned}
 H' &\leq \frac{d^\alpha}{d(P, Q)^\alpha} \left| \int_{t^0}^{\tau'} \int_R \{ D_\xi^m G(x - \xi, t - \tau) \right. \\
 &\quad \cdot [\varphi(x, t)f(x, t) - \varphi(\xi, \tau)f(\xi, \tau)] - D_\xi^m G(x - \xi', t - \tau') \\
 &\quad \cdot [\varphi(x, t)f(x, t) - \varphi(\xi', \tau')f(\xi', \tau')] \} dx dt \Big| \\
 &+ \frac{d^\alpha}{d(P, Q)^\alpha} \int_{\tau'}^\tau \int_R |D_\xi^m G(x - \xi, t - \tau) \\
 (2.29) \quad &\quad \cdot [\varphi(x, t)f(x, t) - \varphi(\xi, \tau)f(\xi, \tau)]| dx dt \\
 &+ \frac{d^\alpha}{d(P, Q)^\alpha} \left| f(\xi, \tau) D_\xi^{m-1} \int_{t^0}^{\tau'} \int_R D_\xi G(x - \xi, t - \tau) dx dt \right. \\
 &\quad \left. - f(\xi', \tau') D_\xi^{m-1} \int_{t^0}^{\tau'} \int_R D_\xi G(x - \xi', t - \tau') dx dt \right| \\
 &+ \frac{d^\alpha}{d(P, Q)^\alpha} \left| f(\xi, \tau) D_\xi^{m-1} \int_{\tau'}^\tau \int_R D_\xi G(x - \xi, t - \tau) dx dt \right| \\
 &= W_1 + W_2 + W_3 + W_4,
 \end{aligned}$$

where W_i denotes the i^{th} term on the right side of the inequality. We shall estimate W_2 , W_4 , W_3 and finally W_1 . To estimate W_2 we use (2.16), (2.8); then substitute $|x - \xi| = (\tau - t)^{1/m} \rho$ and note that $(\tau - \tau')^{1/m} \leq d(P, Q)$. We then obtain (in analogy with (2.18))

$$(2.30) \quad W_2 \leq K d^\alpha H_{P,s}[f] + K \underset{s}{\text{l.u.b.}} |f|.$$

To estimate W_4 we transform the volume integral which appears in W_4 into a surface integral. We then notice that $|x - \xi| \sim d$ and substitute $z = [d^m/(\tau - t)]^{1/(m-1)}$. In analogy with the estimates which follow (2.26), we find that

$$(2.31) \quad W_4 \leq K \underset{s}{\text{l.u.b.}} |f|.$$

We proceed to estimate W_3 .

$$\begin{aligned}
 W_3 &\leq \frac{d^\alpha}{d(P, Q)^\alpha} (\underset{s}{\text{l.u.b.}} |f|) \left| D_\xi^{m-1} \int_{t^0}^{\tau'} \int_R D_\xi G(x - \xi, t - \tau) dx dt \right. \\
 &\quad \left. - D_\xi^{m-1} \int_{t^0}^{\tau'} \int_R D_\xi G(x - \xi', t - \tau') dx dt \right| \\
 (2.32) \quad &+ \frac{d^\alpha}{d(P, Q)^\alpha} H_{P,s}[f] d(P, Q)^\alpha \\
 &\quad \cdot \left| D_\xi^{m-1} \int_{t^0}^{\tau'} \int_R D_\xi G(x - \xi, t - \tau) dx dt \right| = W'_3 + W''_3.
 \end{aligned}$$

From (2.19) we get

$$(2.33) \quad W_3'' \leq K d^\alpha H_{P,s}[f].$$

If we prove that

$$(2.34) \quad W_3' \leq K \text{l.u.b.}_s |f|$$

then, together with (2.32), we conclude that

$$(2.35) \quad W_3 \leq K d^\alpha H_{P,s}[f] + K \text{l.u.b.}_s |f|.$$

Proof of (2.34). For simplicity we first consider two special cases.

Case (i): $\tau = \tau'$. Using the Mean Value Theorem we find that the integral of W_3' is bounded by

$$\int_{t^*}^{\tau} \int_{\dot{K}} |D_{\xi}^m G(x - \xi, t - \tau)| |\xi' - \xi| dS_x dt,$$

where ξ is an appropriate point and $|x - \xi| \sim d$. Estimating the last integral as in (2.19) and noting that $d(P, Q) = |\xi' - \xi|$, we obtain, on using the definition of W_3' , that

$$W_3' \leq K \frac{d^\alpha}{d(P, Q)^\alpha} (\text{l.u.b.}_s |f|) \frac{d(P, Q)}{d} \leq K \text{l.u.b.}_s |f|.$$

Case (ii): $\xi = \xi'$. Writing W_3' in the form

$$W_3' = \frac{d^\alpha}{d(P, Q)^\alpha} (\text{l.u.b.}_s |f|) \left| \int_{\tau'}^{\tau} d\sigma \int_{\dot{K}} D_{\xi}^{m-1} D_{\sigma} G(x - \xi, t - \sigma) dS_x \right|$$

and using (2.8), we obtain

$$W_3' \leq K \frac{d^\alpha}{d(P, Q)^\alpha} (\text{l.u.b.}_s |f|) d^{n-1} \cdot \int_{\tau'}^{\tau} d\sigma \int_{t^*}^{\tau'} \frac{1}{(\sigma - \tau)^{(n+2m-1)/m}} \exp \left\{ -\frac{K d^{m/(m-1)}}{(\sigma - t)^{1/(m-1)}} \right\} dt.$$

Substituting $z = d^m/(\sigma - t)$, we find that the inner integral is bounded by K/d^{n+m-1} . Since $\tau - \tau' = d(P, Q)^m$, we obtain

$$W_3' \leq K \frac{d^\alpha}{d(P, Q)^\alpha} (\text{l.u.b.}_s |f|) \frac{\tau - \tau'}{d^m} \leq K \text{l.u.b.}_s |f|.$$

General case. Writing the integral of W_3' in the form

$$\begin{aligned}
& \left[D_{\xi}^{m-1} \int_{t^{\circ}}^{\tau'} \int_R D_{\xi} G(x - \xi, t - \tau) dx dt \right. \\
& \quad \left. - D_{\xi}^{m-1} \int_{t^{\circ}}^{\tau'} \int_R D_{\xi} G(x - \xi, t - \tau') dx dt \right] \\
& + \left[D_{\xi}^{m-1} \int_{t^{\circ}}^{\tau'} \int_R D_{\xi} G(x - \xi, t - \tau') dx dt \right. \\
& \quad \left. - D_{\xi'}^{m-1} \int_{t^{\circ}}^{\tau'} \int_R D_{\xi'} G(x - \xi', t - \tau') dx dt \right]
\end{aligned}$$

and applying the above two special cases, the proof of (2.34) is completed.

Estimation of W_1 . To complete the estimation of H' (see (2.29)) it remains to estimate W_1 . We split W_1 into the sum of four terms and thus obtain (on using (2.16))

$$\begin{aligned}
(2.36) \quad W_1 & \leq \frac{d^{\alpha}}{d(P, Q)^{\alpha}} \int_{t^{\circ}}^{\tau'-\eta} \int_R |D_{\xi}^m G(x - \xi, t - \tau) - D_{\xi'}^m G(x - \xi', t - \tau')| \\
& \quad \cdot \left[C(|x - \xi|^{\alpha} + (\tau - t)^{\alpha/m}) H_{P,s}[f] \right. \\
& \quad \left. + C(\text{l.u.b.}_s |f|) \left(\frac{|x - \xi|}{d} + \frac{\tau - t}{d^m} \right) \right] dx dt \\
& + \frac{d^{\alpha}}{d(P, Q)^{\alpha}} \left| \int_{t^{\circ}}^{\tau'-\eta} \int_R D_{\xi'}^m G(x - \xi', t - \tau') dx dt \right| \\
& \quad \cdot \left[C(|\xi' - \xi|^{\alpha} + (\tau - \tau')^{\alpha/m}) H_{P,s}[f] \right. \\
& \quad \left. + C(\text{l.u.b.}_s |f|) \left(\frac{|\xi' - \xi|}{d} + \frac{\tau - \tau'}{d^m} \right) \right] dx dt \\
& + \frac{d^{\alpha}}{d(P, Q)^{\alpha}} \int_{\tau'-\eta}^{\tau'} \int_R |D_{\xi}^m G(x - \xi, t - \tau)| \\
& \quad \cdot \left[C(|x - \xi|^{\alpha} + (\tau - t)^{\alpha/m}) H_{P,s}[f] \right. \\
& \quad \left. + C(\text{l.u.b.}_s |f|) \left(\frac{|x - \xi|}{d} + \frac{\tau - t}{d^m} \right) \right] dx dt \\
& + \frac{d^{\alpha}}{d(P, Q)^{\alpha}} \int_{\tau'-\eta}^{\tau'} \int_R |D_{\xi}^m G(x - \xi', t - \tau')| \\
& \quad \cdot \left[C(|x - \xi'|^{\alpha} + (\tau' - t)^{\alpha/m}) H_{Q,s}[f] \right. \\
& \quad \left. + C(\text{l.u.b.}_s |f|) \left(\frac{|x - \xi'|}{d} + \frac{\tau' - t}{d^m} \right) \right] dx dt \\
& = V_1 + V_2 + V_3 + V_4,
\end{aligned}$$

where V_i denotes the i^{th} term on the right side of the inequality, and $\eta = \frac{1}{4}d(P, Q)^m$.

Using (2.8) and substituting $|x - \xi| = \rho(\tau - t)^{1/m}$, we get

$$(2.37) \quad V_3 \leq K d^\alpha H_{P,s}[f] + K \underset{s}{\text{l.u.b.}} |f|.$$

Similarly,

$$(2.38) \quad V_4 \leq K d^\alpha H_{Q,s}[f] + K \underset{s}{\text{l.u.b.}} |f|.$$

Further, using (2.19) we immediately get

$$(2.39) \quad V_2 \leq K d^\alpha H_{P,s}[f] + K \underset{s}{\text{l.u.b.}} |f|.$$

It remains to estimate V_1 . For simplicity we shall first consider two special cases.

Case (i): $\tau = \tau'$. For simplicity we shall consider only the case in which the brackets in the integrand of V_1 are replaced by $|x - \xi|^\alpha + (\tau - t)^{\alpha/m}$. Denote the expression obtained from V_1 (as a result of this replacement) by $\bar{V}_1$. Let ζ be a point which varies on the segment (ξ, ξ') . Then

$$\begin{aligned} \bar{V}_1 \leq C \frac{d^\alpha}{d(P, Q)^\alpha} \int_{\xi}^{\xi'} d\zeta \int_{t^0}^{\tau'-\eta} dt \int_R |D_{\xi}^{m+1} G(x - \zeta, t - \tau)| \\ \cdot [(|x - \zeta|^\alpha + (\tau - t)^{\alpha/m}) + |\zeta - \xi|^\alpha] dx. \end{aligned}$$

Using (2.8) and substituting $|x - \zeta| = \rho(\tau - t)^{1/m}$, we obtain

$$(2.40) \quad \bar{V}_1 \leq K \frac{d^\alpha}{d(P, Q)^\alpha} \int_{\xi}^{\xi'} \left(\frac{1}{\eta^{(1-\alpha)/m}} + \frac{|\zeta - \xi|^\alpha}{\eta^{1/m}} \right) d\zeta \leq K d^\alpha.$$

By the same method we can prove the desired inequality:

$$(2.41) \quad V_1 \leq K d^\alpha H_{P,s}[f] + K \underset{s}{\text{l.u.b.}} |f|.$$

Case (ii): $\xi = \xi'$. We again consider the expression $\bar{V}_1$. Writing

$$\begin{aligned} \bar{V}_1 \leq C \frac{d^\alpha}{d(P, Q)^\alpha} \int_{\tau'}^{\tau} d\sigma \int_{t^0}^{\tau'-\eta} dt \int_R |D_{\xi}^m D_{\sigma} G(x - \xi, t - \sigma)| \\ \cdot [(|x - \xi|^\alpha + (\sigma - t)^{\alpha/m}) + (\tau - \sigma)^{\alpha/m}] dx, \end{aligned}$$

we can proceed as in the previous case and thus prove (2.40), (2.41).

General Case. We decompose $\bar{V}_1$ into a sum of three terms and obtain the inequality

$$\begin{aligned}
\bar{V}_1 \leq & C \frac{d^\alpha}{d(P, Q)^\alpha} \int_{t_0}^{\tau'-\eta} \int_R |D_\xi^m G(x - \xi, t - \tau) \\
& - D_\xi^m G(x - \xi, t - \tau')| [|x - \xi|^\alpha + (\tau - t)^{\alpha/m}] dx dt \\
& + C \frac{d^\alpha}{d(P, Q)^\alpha} \int_{t_0}^{\tau'-\eta} \int_R |D_\xi^m G(x - \xi, t - \tau') \\
& - D_\xi^m G(x - \xi', t - \tau')| [|x - \xi'|^\alpha + (\tau' - t)^{\alpha/m}] dx dt \\
& + C \frac{d^\alpha}{d(P, Q)^\alpha} \left\{ \int_{t_0}^{\tau'-\eta} \int_R |D_\xi^m G(x - \xi, t - \tau') \right. \\
& \left. - D_\xi^m G(x - \xi', t - \tau')| dx dt \right\} [|\xi - \xi'|^\alpha + (\tau - \tau')^{\alpha/m}].
\end{aligned}$$

The first two terms are estimated as in the above two special cases. The last term is bounded by

$$C d^\alpha \int_\xi^{\xi'} d\zeta \int_{t_0}^{\tau'-\eta} dt \int_R |D_\zeta^{m+1} G(x - \zeta, t - \tau')| dx.$$

Similarly to Case (i), we find that the last function is bounded by

$$K d^\alpha \int_\xi^{\xi'} \frac{d\zeta}{\eta^{1/m}} \leq 4K d^\alpha \frac{|\xi - \xi'|}{d(P, Q)} \leq 4K d^\alpha,$$

and the proof of (2.40), and similarly of (2.41), is completed for the general case.

Combining (2.36) with (2.37), (2.38), (2.39), (2.41), we get

$$(2.42) \quad W_1 \leq K d^\alpha H_{P,s}[f] + K d^\alpha H_{Q,s}[f] + K \text{l.u.b.}_s |f|.$$

Combining (2.29) with (2.30), (2.31), (2.35), (2.42), we get an estimate for H' which, when combined with (2.28), completes the proof of (2.3) for $i = m$.

To prove (2.2), (2.3) for $0 \leq i < m$, we apply D_ξ^i to both sides of (2.7) and proceed step by step along the previous proof of the case $i = m$. Making use of (2.8) the desired estimates readily follow.

Let u be a solution of (2.4) with $f = 0$. If we differentiate (2.7) p times with respect to t and q times with respect to x , we get, in analogy with (2.13),

$$(2.43) \quad |D_t^p D_x^q u| \leq K d^{-mp+q} \text{l.u.b.}_s |u|.$$

With the aid of (2.43) we shall prove the following Liouville type theorem.

Theorem 2. *Let u_μ be a solution in the half space $t < 0$ of equations (2.1) with $f_\lambda = 0$ and with coefficients satisfying the assumptions (a), (b). If*

$$(2.44) \quad \sum_{\mu=1}^N |u_\mu(x, t)| \leq \text{const.} (1 + |t|)^\delta (1 + |x|)^\gamma$$

holds for all $t < 0$, where β, γ are nonnegative constants, then each u_μ is a polynomial of degree $[\gamma]$ in $x_1, \dots, x_n$ and of degree $\min([\beta], [\gamma/m])$ in t .

Proof. For simplicity we assume $N = 1$. Using (2.43) and (2.44) with $d \rightarrow \infty$ we easily conclude that u is a polynomial in x, t . Since this polynomial satisfies (2.44), it must be of degree $[\gamma]$ in x and of degree $[\beta]$ in t . Writing, finally, u in the form $\sum P_i(x)t^i$ and using the fact that u is a solution of (2.4) with $f = 0$, we conclude that $P_i(x)$ is a polynomial of degree $[\gamma] - mi$; hence u is a polynomial in t of degree $\min([\beta], [\gamma/m])$, and the proof is completed.

Theorem 2, which is the analogue of [2; Theorem 3] was proved by a somewhat different method by EIDELMAN [4].

3. Schauder Type Interior Estimates. We shall make the following assumptions:

(A) The functions $a_i^{\lambda\mu}$ are Hölder continuous (exponent α) in D , and

$$(3.1) \quad \text{l.u.b.}_{\lambda, \mu, i} |a_i^{\lambda\mu}|_{m-i, \alpha} \leq K_1.$$

(B) The real parts of the characteristic roots of the matrix

$$((-1)^{\frac{1}{2}m} \sum_m a_m^{\lambda\mu}(x, t)\xi^m),$$

when $(x, t) \in D$ and ξ real, $|\xi| = 1$, are smaller than some negative constant $-K_2$.

(C) The functions f_λ are Hölder continuous (exponent α) and

$$(3.2) \quad |f|^* = \text{l.u.b.}_\lambda |f_\lambda|_{m, \alpha} \quad \text{is finite.}$$

We shall denote by C constants depending on n, m , and by K constants depending on n, m, K_1 and K_2 .

The basic interior estimates, which are the analogues of the Schauder type estimates for elliptic equations, are contained in the following theorem.

Theorem 3. Let u_λ be a solution of the system (2.1) and assume that the assumptions (A), (B), (C) hold. Assume that u_λ has Hölder continuous (exponent α) x -derivatives up to order m and one continuous t -derivative, and assume that $\text{l.u.b.}_D |u|$ is finite. Then, for $\mu = 1, \dots, N$,

$$(3.3) \quad |u_\mu|_{0, m+\alpha} \leq K \left(\text{l.u.b.}_D \sum_{\lambda=1}^N |u_\lambda| + |f|^* \right).$$

Proof. For simplicity we give the proof in the case $N = 1$, the proof of the general case is quite analogous. From Lemma 1 it follows that it is sufficient to prove that

$$(3.4) \quad M \equiv M_{0, m}[u] \leq K(\text{l.u.b.}_D |u| + |f|_{m, \alpha}),$$

$$(3.5) \quad M'_i \equiv M_{0, i+\alpha}[u] \leq K(\text{l.u.b.}_D |u| + |f|_{m, \alpha}) \quad (i = 0, 1, \dots, m).$$

Without loss of generality we may assume that M, M'_i are finite since, in the contrary case, we take subdomains D_ϵ of D with $D_\epsilon \rightarrow D$ as $\epsilon \rightarrow 0$ and prove (3.4), (3.5) in D_ϵ ; then, on letting $\epsilon \rightarrow 0$, we obtain the finiteness of M, M'_i and the desired estimates.

Since M is finite, there exists a point P of D and a certain derivative D_x^m such that

$$(3.6) \quad \frac{1}{2}M \leq d_P^m |D_x^m u(P)|.$$

Let S be a semi-cube with top P and edge λd_P , where λ is to be determined later. Writing equation (2.4) in the form

$$(3.7) \quad \sum_m a_m(P) D_x^m u - D_t u = \sum_m [a_m(P) - a_m(x, t)] D_x^m u - \sum_{i \leq m-1} a_i(x, t) D_x^i u + f(x, t) \equiv F(x, t)$$

and applying Theorem 1, we get

$$(3.8) \quad |D_x^m u(P)| \leq K d^{-m} \text{l.u.b.}_S |u| + K \text{l.u.b.}_S |F| + d^\alpha H_{P,S}[F] \equiv KI.$$

In the sequel we shall make use of the following simple inequality:

$$(3.9) \quad H_{P,S}[gh] \leq |g(P)| H_{P,S}[h] + (\text{l.u.b.}_S |h|) H_{P,S}[g].$$

To estimate I we begin by noting that

$$(3.10) \quad d^{-m} \text{l.u.b.}_S |u| \leq \lambda^{-m} d_P^{-m} \text{l.u.b.}_D |u|.$$

Next, using the assumption (A) and Lemma 1, we have

$$(3.11) \quad \begin{aligned} \text{l.u.b.}_S |F| &\leq \text{l.u.b.}_S \{ |f| + \sum_{i \leq m-1} |a_i(x, t)| |D_x^i u| \\ &\quad + \sum_m |a_m(P) - a_m(x, t)| |D_x^m u| \} \\ &\leq d_P^{-m} (|f|_{m,\alpha} + K \sum_{i \leq m-1} |u|_{0,i}) + \frac{(\lambda d_P)^\alpha K_1}{[C(1-\lambda) d_P]^\alpha} d_P^{-m} M \\ &\leq C d_P^{-m} (|f|_{m,\alpha} + K M_{0,m-1}[u] + K \text{l.u.b.}_D |u| + K \lambda^\alpha M). \end{aligned}$$

Using the assumption (A), Lemma 1 and (3.9), we have

$$(3.12) \quad \begin{aligned} d^\alpha H_{P,S}[F] &\leq C d^\alpha d_P^{-m-\alpha} |f|_{m,\alpha} + C d^\alpha \sum_{i \leq m-1} |a_i(P)| H_{P,S}[D_x^i u] \\ &\quad + C d^\alpha \sum_{i \leq m} (\text{l.u.b.}_S |D_x^i u|) H_{P,S}[a_i] \leq C \lambda^\alpha d_P^{-m} |f|_{m,\alpha} \\ &\quad + K \lambda^\alpha d_P^\alpha \sum_{i \leq m-1} d_P^{-m+i} H_{P,S}[D_x^i u] + K \lambda^\alpha d_P^{-m} (|u|_{0,m} + \text{l.u.b.}_D |u|). \end{aligned}$$

Since $H_{P,s}[D'_x u] \leq C d_P^{-\alpha} M_{0,i+\alpha}[u]$, we get

$$(3.13) \quad d^\alpha H_{P,s}[F] \leq C \lambda^\alpha d_P^{-m} (|f|_{m,\alpha} + K |u|_{0,m} + K \text{l.u.b.}_D |u| + K \sum_{i=0}^{m-1} M_{0,i+\alpha}[u]).$$

Combining (3.10), (3.11), (3.13) and using the definition of I (in (3.8)), of M (in (3.4)) and of M'_i (in (3.5)), we get

$$(3.14) \quad d_P^m I \leq K \lambda^{-m} \text{l.u.b.}_D |u| + K |u|_{0,m-1} + K \lambda^\alpha M + K \lambda^\alpha \sum_{i=0}^{m-1} M'_i + C |f|_{m,\alpha}.$$

By Lemma 1 we have, for any $\epsilon > 0$,

$$(3.15) \quad |u|_{0,m-1} \leq \epsilon M + C' \text{l.u.b.}_D |u|$$

for some appropriate constant C' . If we make use of (3.15) in (3.14), and then use the definition of I in (3.8) and the inequality (3.6), we get, on taking ϵ to be sufficiently small (depending on K_1, K_2),

$$(3.16) \quad M \leq K \lambda^{-m} \text{l.u.b.}_D |u| + K |f|_{m,\alpha} + K \lambda^\alpha \left(M + \sum_{i=0}^{m-1} M'_i \right).$$

We shall now estimate the M'_i . We start with M'_m . There exist points P, Q (P has the t -coordinate greater than the t -coordinate of Q) and a derivative D_x^m such that

$$(3.17) \quad \frac{1}{2} M'_m \leq d_P^{m+\alpha} \frac{|D_x^m u(P) - D_x^m u(Q)|}{d(P, Q)^\alpha},$$

where we assumed (without restricting the generality) that $d_{PQ} = d_P$. We shall consider separately the cases $d(P, Q) \geq \lambda d_P / 4(n+1)^{\frac{1}{2}}$.

Case (i): $d(P, Q) \leq \lambda d_P / 4(n+1)^{\frac{1}{2}}$.

$$(3.18) \quad M'_m \leq C d_P^m \lambda^{-\alpha} |D_x^m u(P)| + C d_Q^m \lambda^{-\alpha} |D_x^m u(Q)| \leq C \lambda^{-\alpha} M.$$

Case (ii): $d(P, Q) < \lambda d_P / 4(n+1)^{\frac{1}{2}}$. Let S be the semi-cube with top P and edge λd_P . Then, applying Theorem 1 with $d = \lambda d_P / (n+1)^{\frac{1}{2}}$ to the equation (3.7), we have

$$(3.19) \quad d^\alpha \frac{|D_x^m u(P) - D_x^m u(Q)|}{d(P, Q)^\alpha} \leq KI + K d^\alpha H_{Q,S}[F].$$

Since I was already estimated (see (3.14), (3.15)), it remains to estimate $d^\alpha H_{Q,S}[F]$. In analogy with (3.12), (3.13), we find

$$(3.20) \quad \begin{aligned} d^\alpha H_{Q,S}[F] &\leq C\lambda^\alpha d_P^{-m}(|f|_{m,\alpha} + K|u|_{0,m} + K \text{l.u.b.}_D |u| \\ &\quad + K \sum_{i=0}^{m-1} M_{0,i+\alpha}[u]) + C d^\alpha \sum_m |a_m(P) - a_m(Q)| H_{Q,S}[D_x^m u]. \end{aligned}$$

The last term on the right side of (3.20) is bounded by

$$(3.21) \quad d^\alpha d_P^{-\alpha} K d(P, Q)^\alpha H_{Q,S}[D_x^m u] \leq K\lambda^{2\alpha} d_P^{-m} M_{0,m+\alpha}[u].$$

Substituting (3.21) in (3.20), then substituting (3.20) in (3.19), we obtain, on making use of (3.17) and the estimates derived previously for I ($d_P^m I \leq M$, and M is estimated in (3.16)),

$$(3.22) \quad \begin{aligned} M'_m &\leq K\lambda^{-\alpha} d_P^m I + K\lambda^{-\alpha} d_P^m d^\alpha H_{Q,S}[F] \\ &\leq K\lambda^{-m-\alpha} \text{l.u.b.}_D |u| + K\lambda^{-\alpha} |f|_{m,\alpha} + K \left(M + \sum_{i=0}^{m-1} M'_i \right) + K\lambda^\alpha M'_m. \end{aligned}$$

By the same method, one can show that either

$$(3.23) \quad \begin{aligned} M'_i &\leq K\lambda^{-i-\alpha} \text{l.u.b.}_D |u| \\ &\quad + K\lambda^{m-i-\alpha} |f|_{m,\alpha} + K\lambda^{m-i} \left(M + \sum_{i=0}^{m-1} M'_i \right) + K\lambda^{m-i+\alpha} M'_m \end{aligned}$$

or

$$(3.24) \quad M'_i \leq C\lambda^{-\alpha} M_{0,i}[u] \quad (0 \leq i \leq m-1).$$

Hence M'_i is always bounded by the *sum* of the right sides of (3.23) and (3.24). With the estimates (3.16), (3.18), (3.22), (3.23), (3.24) we now proceed to complete the proof of Theorem 3.

In Case (i) we estimate the M'_m in (3.23) by (3.18) and, on using (3.16), (3.23), (3.24), we obtain

$$(3.25) \quad \begin{aligned} M + \sum_{i=0}^{m-1} M'_i &\leq K\lambda^{-m} \text{l.u.b.}_D |u| + K |f|_{m,\alpha} \\ &\quad + K\lambda^\alpha \left(M + \sum_{i=0}^{m-1} M'_i \right) + K\lambda^{-\alpha} \sum_{i=0}^{m-1} M_{0,i}[u]. \end{aligned}$$

Taking λ to be such that $K\lambda^\alpha < \frac{1}{4}$, and using Lemma 1 to conclude that

$$K\lambda^{-\alpha} \sum_{i=0}^{m-1} M_{0,i}[u] \leq \frac{1}{4}M + K\lambda^{-\alpha} \text{l.u.b.}_D |u|,$$

we obtain

$$(3.26) \quad M + \sum_{i=0}^{m-1} M'_i \leq K\lambda^{-m} \text{l.u.b.}_D |u| + K |f|_{m,\alpha}.$$

Using finally (3.18), the proof of (3.4) and (3.5) is completed.

It remains to consider Case (ii). Taking in (3.22) $K\lambda^\alpha < \frac{1}{2}$ we obtain an estimate on M'_m . Substituting it into (3.23), we get

$$(3.27) \quad M'_i \leq K\lambda^{-i-\alpha} \text{l.u.b.}_D |u| + K\lambda^{m-i-\alpha} |f|_{m,\alpha} + K\lambda^{m-i} \left(M + \sum_{i=0}^{m-1} M'_i \right).$$

Combining (3.27), (3.24) with (3.16) we find that

$$(3.28) \quad \begin{aligned} M + \sum_{i=0}^{m-1} M'_i &\leq K\lambda^{-m} \text{l.u.b.}_D |u| + K |f|_{m,\alpha} \\ &\quad + K\lambda^\alpha \left(M + \sum_{i=0}^{m-1} M'_i \right) + C\lambda^{-\alpha} \sum_{i=0}^{m-1} M_{0,i}[u]. \end{aligned}$$

Taking $K\lambda^\alpha < \frac{1}{4}$ and using Lemma 1 to estimate the last term on the right side of (3.28), (3.4) and (3.5) (for $0 \leq i \leq m-1$) follow. The proof of (3.5) for $i = m$ follows from (3.22) combined with the result just proved.

Using the equations (2.1), the estimates (3.4), (3.5) and Lemma 2 ((1.10), (1.8)) one can easily prove

Corollary. *Under the assumptions of Theorem 2,*

$$(3.29) \quad |D_i u_\mu|_{m,\alpha} \leq K \left(\text{l.u.b.}_D \sum_{\lambda=1}^N |u_\lambda| + |f|^* \right) \quad (\mu = 1, \dots, N).$$

4. Existence and Estimates of Higher Derivatives. We shall make the following assumptions:

(A_σ) The functions $a_i^{\lambda_\mu}$ have Hölder continuous (exponent α) x -derivatives up to order σ , and

$$(4.1) \quad \text{l.u.b.}_{\lambda,\mu,i} |a_i^{\lambda_\mu}|_{m-i,\sigma+\alpha} \leq K'_\sigma.$$

(C_σ) The functions f_λ have Hölder continuous (exponent α) x -derivatives up to order σ , and

$$(4.2) \quad |f|^*_\sigma = \text{l.u.b.}_\lambda |f_\lambda|_{m,\sigma+\alpha} \quad \text{is finite.}$$

Theorem 4. *Let u_λ be a solution of (2.1) in D and assume that (A_σ), (B), (C_σ) hold. Assume that u_λ has Hölder continuous (exponent α) x -derivatives up to order m and one continuous t -derivative. Assume further that $\text{l.u.b.}_D |u_\lambda|$ is finite. Then u_λ has Hölder continuous (exponent α) x -derivatives up to order $m + \sigma$ and, for $\mu = 1, \dots, N$,*

$$(4.3) \quad |u_\mu|_{0,m+\sigma+\alpha} \leq K_\sigma \left(\text{l.u.b.}_D \sum_{\lambda=1}^N |u_\lambda| + |f|^*_\sigma \right),$$

where K_σ depends on m, n, σ, K_2 and K'_σ .

Proof. It will be enough to prove the theorem in the case $\sigma = 1$; the proof for $\sigma = 2$ is obtained by first differentiating (2.1) once with respect to x and

then applying to the resulting system the proof of the case $\sigma = 1$ and Lemma 2, (1.11); the general case can similarly be derived by induction. The proof, for $\sigma = 1$, will be given in the case $N = 1$.

We first prove that u has Hölder continuous (exponent α) x -derivatives of order $m + 1$. Let A, B, C be subdomains of D which satisfy

$$\bar{A} \subset B \subset \bar{B} \subset C \subset \bar{C} \subset D.$$

For any point $P = (x_1, \dots, x_n, t)$ denote by P_h the point $(x_1 + h, x_2, \dots, x_n, t)$. If $|h|$ is small and P belongs to B , then P_h belongs to C . Denote

$$g^h(P) = \frac{g(P_h) - g(P)}{h}.$$

Following the method of DOUGLIS & NIRENBERG [2], we shall derive an estimate for $|u^h|_{0, m+\alpha}$ independent of h . From equation (2.4) we find

$$(4.4) \quad \sum_{i \leq m} a_i(P) D_x^i u^h(P) - D_t u^h(P) = f^h(P) - \sum_{i \leq m} a_i^h(P) D_x^i u(P_h) \equiv g.$$

Writing

$$f^h(P) = \frac{1}{h} \int_P^{P_h} D_t f(\zeta, x_2, \dots, x_n, t) d\zeta,$$

one can easily show that

$$(4.5) \quad |f^h|_{m, \alpha}^B \leq \text{const.} \cdot |f|_{m-1, 1+\alpha}^C.$$

Further, using Lemma 2 ((1.10), (1.8)) and the assumption (A_σ) , we obtain

$$(4.6) \quad \begin{aligned} \left| \sum_{0 \leq i \leq m} a_i^h(P) D_x^i u(P_h) \right|_{m, \alpha}^B &\leq C \sum_{1 \leq i \leq m} |a_i^h(P)|_{m-i+1, \alpha}^B |D_x^i u(P_h)|_{i-1, \alpha}^B \\ &+ C |a_0^h(P)|_{m, \alpha} |u(P_h)|_{0, \alpha} \leq K \sum_{1 \leq i \leq m} |D_x^i u(P)|_{i-1, \alpha}^C + K |u(P)|_{0, \alpha}^C. \end{aligned}$$

Here we have used the inequalities

$$|a_i^h|_{m-i+1, \alpha}^B \leq \text{const.} \cdot |a_i|_{m-i, 1+\alpha}^C.$$

Combining (4.5), (4.6), using the definition of g in (4.4), and applying Lemma 2 ((1.8)), we get

$$(4.7) \quad |g|_{m, \alpha}^B \leq \text{const.} \left(|f|_{m-1, 1+\alpha}^C + \sum_{i=1}^m |u|_{-1, i+\alpha}^C + |u|_{0, \alpha}^C \right).$$

Applying Theorem 3 to Equation (4.4), we have

$$(4.8) \quad |u^h|_{m, \alpha}^B \leq K(\text{l.u.b.}_B |u^h| + |g|_{m, \alpha}^B).$$

Since $\text{l.u.b.}_B |u^h| \leq |u|_{-1, 1}^C < \infty$, and since (by (4.7)) $|g|_{m, \alpha}^B$ is finite, we get that $|u^h|_{m, \alpha}^B$ is bounded by a constant independent of h .

We can now use a theorem of ARZELÀ to derive the existence and the Hölder

continuity (exponent α) of $D_x^m (\partial u / \partial x_1)$. A similar result can be proved with respect to x_i , $i = 2, \dots, n$.

To complete the proof of Theorem 4, it remains to prove that

$$(4.9) \quad |u|_{0, m+1+\alpha} \leq K(\text{l.u.b.}_D |u| + |f|^*).$$

Let S be a semi-cube with top P and edge $r = d_P/2(n+1)^{\frac{1}{2}}$. Applying D_x to (2.4), we obtain

$$(4.10) \quad \sum_{i \leq m} a_i D_x^i (D_x u) - D_i (D_x u) = - \sum_{i \leq m} D_x a_i \cdot D_x^i u + D_x f \equiv g.$$

Using Lemma 2 ((1.8), (1.10)), we get

$$(4.11) \quad |D_x f|_{m, \alpha}^S \leq C |f|_{m-1, 1+\alpha}^S,$$

$$(4.12) \quad |D_x a_i \cdot D_x^i u|_{m, \alpha}^S \leq C |D_x a_i|_{m-i+1, \alpha}^S |D_x^i u|_{i-1, \alpha}^S \\ \leq K |u|_{-1, i+\alpha}^S \quad (1 \leq i \leq m).$$

Using Lemma 2 ((1.8), (1.10)) and Lemma 3, we have

$$(4.13) \quad |D_x a_0 \cdot u|_{m, \alpha}^S \leq \frac{C}{r} (r |D_x a_0|_{m, \alpha}^S) |u|_{0, \alpha} \leq \frac{C}{r} |a_0|_{m, 1+\alpha} |u|_{0, \alpha} \leq \frac{K}{r} |u|_{0, \alpha}.$$

Using Lemma 3, the definition of g and (4.11), (4.12), (4.13), we get

$$(4.14) \quad |g|_{m, \alpha}^S \leq \frac{K}{r} (|u|_{0, m+\alpha} + |f|_{m, 1+\alpha}^S).$$

By Theorem 3 we have

$$(4.15) \quad |D_x u|_{0, m+\alpha}^S \leq K(\text{l.u.b.}_S |D_x u| + |g|_{m, \alpha}^S).$$

Noting that

$$\text{l.u.b.}_S |D_x u| \leq \text{const.} |u|_{-1, m}^S \leq \frac{C}{r} |u|_{0, m}$$

and using (4.14) and Theorem 3, we conclude from (4.15) that

$$(4.16) \quad r |D_x u|_{0, m+\alpha}^S \leq K(\text{l.u.b.}_D |u| + |f|^*).$$

Since P is arbitrary, we can apply Lemma 4 and get

$$|D_x u|_{1, m+\alpha} \leq K(\text{l.u.b.}_D |u| + |f|^*).$$

Finally, using Lemma 2 ((1.11)) and Theorem 3 ((3.5) with $i = 0$), the proof of (4.9) is completed.

We shall apply Theorem 4 to derive a general regularity theorem for linear parabolic systems.

Theorem 5. Assume that the functions

$$(4.17) \quad D_t^i D_x^k a_i^{\lambda\mu}, \quad D_t^i D_x^k f_\lambda$$

$$(0 \leq k \leq (q-j)m; \quad 0 \leq j \leq q_1 \text{ where } q_1 \leq q)$$

are Hölder continuous (exponent α) in D and that the system (2.1) is parabolic. If u_λ is a solution of (2.1) which has Hölder continuous (exponent α) x -derivatives up to order m and one continuous t -derivative, then the functions

$$(4.18) \quad D_x^i u_\lambda \quad (0 \leq i \leq (q+1)m),$$

$$D_t^{i+1} D_x^j u_\lambda \quad (0 \leq i \leq (q-j)m, 0 \leq j \leq q_1)$$

exist and are Hölder continuous (exponent α) in D .

Proof. For simplicity we give the proof in the case $N = 1$. The Hölder continuity of $D_x^i u$ ($0 \leq i \leq (q+1)m$) is included in Theorem 4. Applying D_x^i ($0 \leq i \leq qm$) to equation (2.4) we conclude that $D_t D_x^i u$ ($0 \leq i \leq qm$) is Hölder continuous (exponent α). Applying $D_t D_x^i$ ($0 \leq i \leq (q-1)m$) to (2.4) and using the previous result, we conclude that $D_t^2 D_x^i u$ ($0 \leq i \leq (q-1)m$) is Hölder continuous (exponent α). Continuing in this way, the proof of Theorem 5 is easily completed.

Using an inductive argument similar to that of the proof of Theorem 5 and using (4.3), we can establish estimates on the norms of the functions (4.18). We only state here the final result.

Theorem 6. Under the assumptions of Theorem 5, let

$$(4.19) \quad \text{l.u.b.}_{\lambda, \mu, i} |D_t^i a_i^{\lambda\mu}|_{m-i+jm, (q-j)m+\alpha} \leq K'_{qa_1},$$

$$\text{l.u.b.}_{\lambda} |D_t^i f_\lambda|_{m+jm, (q-j)m+\alpha} \leq |f|_{qa_1}^*.$$

$$(0 \leq j \leq q_1)$$

Then, for $\mu = 1, \dots, N$,

$$(4.20) \quad |D_t^{j+1} u_\mu|_{(j+1)m, (q-j)m+\alpha} \leq K \left(\text{l.u.b.}_D \sum_{\lambda=1}^N |u_\lambda| + |f|_{qa_1}^* \right) \quad (0 \leq j \leq q_1),$$

provided $\text{l.u.b.}_D |u_\lambda|$ is finite. Here K is a constant depending on m, n, q, q_1, K_2 and K'_{qa_1} .

5. Differentiability for Nonlinear Systems. If $b(x, t)$ is a positive function, and if $D_t^i D_x^j b(x, t)$ ($0 \leq i \leq i_0, 0 \leq j \leq j_0$) are Hölder continuous (exponent α), then $1/b(x, t)$ has the same properties. From this remark it follows that Theorem 5 remains true for systems of the form (2.1) in which $D_t u_\lambda$ is replaced by $b_\lambda(x, t) D_t u_\lambda$, provided the b_λ have the same properties as the coefficients $a_i^{\lambda\mu}$. An analogue of Theorem 6 can also be proved.

We shall now consider nonlinear systems of the form

$$(5.1) \quad F_\lambda(x, t, u_1, \dots, u_N, D_x u_1, \dots, D_x^m u_N, D_t u_\lambda) = 0 \quad (\lambda = 1, \dots, N)$$

where $D_t u_\lambda$ appears as an argument only in F_λ . The system (5.1) is called *parabolic* with respect to a given solution u_μ if the first variation of the system, with $u_1, \dots, u_N$ inserted in the coefficients, takes the form (2.1) with $D_t u_\lambda$ replaced by $b_\lambda D_t u_\lambda$, where the b_λ are positive functions, and if the $a_m^{\lambda\mu}$ satisfy the condition (B) (§3) of (linear) parabolicity. A function $\Phi(x, t, u_1, u_2, \dots)$ is said to be Hölder continuous (exponent α) if it is Hölder continuous (exponent α) with respect to the distance-function

$$[|x - x'|^2 + (t - t')^{2/m} + (u_1 - u'_1)^2 + (u_2 - u'_2)^2 + \dots]^{1/2}.$$

We shall now prove a differentiability theorem for the solutions of (5.1).

Theorem 7. *Assume that u_λ is a solution of the system (5.1) and that (5.1) is parabolic. Assume further that F_λ has Hölder continuous (exponent α) derivatives with respect to the arguments $x, u_1, \dots, D_x^m u_N, D_t u_\lambda$ up to order σ . If the u_λ have Hölder continuous (exponent α) x -derivatives up to order m and Hölder continuous (exponent α) t -derivative of the first order, then the u_λ have Hölder continuous (exponent α) x -derivatives up to order $m + \sigma$.*

Proof. We only give the proof for the case $\sigma = 1, N = 1$. The proof for general σ then follows by differentiating the system and applying Theorem 4. Replacing P by P_h in (5.1) (see §4 for definition of P_h) and subtracting, one obtains a linear parabolic equation for $u^h = [u(P_h) - u(P)]/h$. Dividing this equation by the (positive) coefficient of $D_t u^h$ we can obtain (see the remark at the beginning of the present section), as in the proof of Theorem 4,

$$(5.2) \quad |u^h|_{0, m+\alpha'}^B \leq C$$

where C is independent of h , B is a subdomain of D with $\bar{B} \subset D$, and $0 < \alpha' < \alpha$. Letting $h \rightarrow 0$ we get (on using a theorem of ARZELÀ)

$$|D_x u|_{0, m+\alpha'}^B \leq C.$$

Since $|u|_{0, \alpha'}^B$ is finite, Lemma 2 ((1.11)) can be applied to deduce that

$$(5.3) \quad |u|_{0, m+1+\alpha'}^B < \infty.$$

From the corollary to Theorem 3 we also have

$$(5.4) \quad |D_t u^h|_{m, \alpha'}^B \leq C.$$

Letting $h \rightarrow 0$ we get (on using a theorem of ARZELÀ)

$$(5.5) \quad |D_t D_x u|_{m, \alpha'}^B \leq C.$$

Using the assumption $|D_t u|_{m, \alpha}^B < \infty$ and (5.5), we conclude, with the aid of Lemma 2 ((1.11)), that

$$(5.6) \quad |D_t u|_{m, 1+\alpha'}^B < \infty.$$

Using the information (5.3), (5.6) we can again prove (5.2), (5.4), but this time with $\alpha' = \alpha$. On letting $h \rightarrow 0$ in (5.2), (5.4) (with $\alpha' = \alpha$) the proof is completed for $\sigma = 1$.

Combining the method of the proof of Theorem 5 with the result of Theorem 7, one can prove the following general differentiability theorem.

Theorem 8. Assume that $\partial^i F_\lambda / \partial t^i$ ($0 \leq j \leq q_1$; $1 \leq \lambda \leq N$) has derivatives up to the order $(q - j)m$ with respect to the arguments $x, u_1, \dots, D_x^m u_N, D_t u_\lambda$, and that all these derivatives are Hölder continuous (exponent α). Let u_λ be a solution of (5.1), which is supposed to be parabolic, and assume that u_λ has Hölder continuous (exponent α) x -derivatives up to order m and Hölder continuous (exponent α) t -derivative of the first order. Then the functions

$$D_x^i u_\lambda \quad (0 \leq i \leq (q + 1)m), \quad D_t^{i+1} D_x^i u_\lambda \quad (0 \leq i \leq (q - j)m, 0 \leq j \leq q_1)$$

exist and are Hölder continuous (exponent α) in D .

Remark 1. Theorems 7, 8 are true for more general nonlinear parabolic systems, but we shall not go into more details here.

Remark 2. In [5] we have proved that the infinitely differentiable (C^∞) solutions of certain nonlinear systems of parabolic type are analytic (or belong to classes $C\{M_a\}$) provided the functions of the system (the F_λ in (5.1)) are analytic (or belong to $C\{M_a\}$). Using Theorem 8 we conclude that the assumption on the C^∞ of the solutions can be replaced by the much weaker assumption that $D_x^i u_\lambda$ ($0 \leq i \leq m$) and $D_t u_\lambda$ are Hölder continuous. Furthermore, in considering the analyticity (or $C\{M_a\}$) only with respect to x , it is enough to assume that F_λ satisfies the assumptions of Theorem 7 and is analytic (or $C\{M_a\}$) with respect to $x, u_1, \dots, D_x^m u_N, D_t u_\lambda$. No additional regularity properties of F_λ with respect to t are needed.

6. Appendix. Proof of (2.15). Denote $g = \varphi f$; then g is Hölder continuous with exponent α . Let $(\xi, \tau) \in D$. To prove (2.15) we first note that

$$(6.1) \quad H = D_\xi \int_{t_0}^\tau \int_R D_\xi^{m-1} G(x - \xi, t - \tau) g(x, t) dx dt.$$

Denoting

$$(6.2) \quad E_\eta = \int_{\tau-\eta}^\tau \int_R D_\xi^{m-1} G(x - \xi, t - \tau) g(x, t) dx dt, \quad F_\eta = H - D_\xi E_\eta,$$

we have

$$(6.3) \quad F_\eta = \int_{t_0}^{\tau-\eta} \int_R D_\xi^m G(x - \xi, t - \tau) g(x, t) dx dt,$$

or

$$(6.4) \quad \begin{aligned} F_\eta = & \int_{t_0}^{\tau-\eta} \int_R D_\xi^m G(x - \xi, t - \tau) [g(x, t) - g(\xi, \tau)] dx dt \\ & + g(\xi, \tau) \int_{t_0}^{\tau-\eta} \int_R D_\xi^m G(x - \xi, t - \tau) dx dt. \end{aligned}$$

From (6.4) we see that $\lim F_\eta$ (as $\eta \rightarrow 0$) tends to the right side of (2.15). Here we make use of the absolute convergence of

$$\int_{t^0}^{\tau} \int_R D_\xi^m G(x - \xi, t - \tau) [g(x, t) - g(\xi, \tau)] dx dt$$

(which follows on using (2.8), (2.16)). Furthermore, on writing $g(x, t) = [g(x, t) - g(\xi, \tau)] + g(\xi, \tau)$ and using the Mean Value Theorem (in its integral form) one can show that

$$\int_{t^0}^{\tau-\eta} \int_R \left\{ \frac{1}{h} [D_{\xi'}^{m-1} G(x - \xi', t - \tau) - D_{\xi}^{m-1} G(x - \xi, t - \tau)] - D_{\xi}^m G(x - \xi, t - \tau) \right\} g(x, t) dx dt \quad (\xi' = \xi_h)$$

tends to zero provided $h = \eta^{1/m}$ and η tends to zero. Hence from (6.2) we conclude that all one has to show is that

$$(6.5) \quad B_\eta = \frac{|E_\eta(\xi_h) - E_\eta(\xi)|}{|h|}$$

becomes arbitrarily small as $h = \eta^{1/m}$ approaches zero. For simplicity we shall prove the last statement for $n = 1$. Suppose $R = (-l, l)$, then

$$B_\eta = \left| \frac{1}{h} \int_{\tau-\eta}^{\tau} \int_{-l}^l g(x, t) D_{\xi}^{m-1} G(x - \xi - h, t - \tau) dx dt - \frac{1}{h} \int_{\tau-\eta}^{\tau} \int_{-l}^l g(x, t) D_{\xi}^{m-1} G(x - \xi, t - \tau) dx dt \right|.$$

Substituting $x' = x - h$ in the first integral, we get

$$(6.6) \quad B_\eta \leq \int_{\tau-\eta}^{\tau} \int_{-l}^{l-h} \frac{1}{|h|^{1-\alpha}} |D_{\xi}^{m-1} G(x - \xi, t - \tau)| \left| \frac{g(x+h) - g(x)}{h^\alpha} \right| dx dt + \frac{C}{|h|} \int_{\tau-\eta}^{\tau} \left[\int_{-l-h}^{-l} + \int_{l-h}^l |D_{\xi}^{m-1} G(x - \xi, t - \tau)| dx \right] dt,$$

where C is an appropriate constant. Using (2.8) and the Hölder continuity (exponent α) of g we find that the first integral in (6.6) is bounded by $\text{const. } \eta^{\alpha/m}$. Since in the second integral the integrand tends to zero as $\eta \rightarrow 0$, provided $|x - \xi| > 0$, for all x which satisfy $l - |x| < h$ (and therefore $|x - \xi| > 0$ if $|h|$ is small enough), the proof concerning B_η is completed.

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