*On Elliptic Partial Differential Equations and Uniqueness Theorems for Closed Surfaces*¹

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1. Theorems in differential geometry. This paper deals with the conditions of differentiability for the validity of several theorems concerning closed, 2-dimensional surfaces in Euclidean 3-space. The emphasis in this paper is on the portion of the proofs involving elliptic partial differential equations. In particular, most of the paper deals with the behavior of a solution of a linear, elliptic partial differential equation in the vicinity of a zero.

Consider the following uniqueness statement:

(A) Let $F(H, K, \lambda, \mu, \nu)$ be defined on the set $H > 0, H^2 \geq K > 0, \lambda^2 + \mu^2 + \nu^2 = 1$ and satisfy

$$(1) \quad \frac{\partial F}{\partial k_1} \frac{\partial F}{\partial k_2} > 0, \quad \text{where} \quad 2H = k_1 + k_2 \quad \text{and} \quad K = k_1 k_2.$$

Then, up to translations, there is at most one closed surface S of positive Gaussian curvature such that the mean and Gaussian curvatures, H and K , at the point of S where the inward unit normal of S is (λ, μ, ν) satisfy the relation $F = 0$.

This assertion is due to A. D. ALEXANDROFF when F and S are analytic. It was proved by POGORELOFF under the conditions that F and S are sufficiently differentiable. His assumptions have been lightened successively by WINTNER and myself ([9], where earlier references are given) and by BERS [2]. In both [2] and [9], the assumptions imply that S is of class C^3 (in fact, S has local parametrizations in terms of functions possessing Hölder continuous third derivatives). The question was raised in [9] whether (A) holds under hypotheses which do not imply that S is of class C^3 . This question will be answered below in the affirmative.²

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² Added March 17, 1958. Since this paper has been submitted, additional results of ALEXANDROFF [*Vestnik*, Leningrad University, 1956, No. 19, pp. 1-17] and POGORELOFF

(I) *Assertion (A) holds if S is of class C^2 and F is of class C^1 (or, more generally, if F satisfies a uniform Hölder condition with respect to λ, μ, ν for fixed H, K and possesses continuous partial derivatives F_H, F_K).*

In the parenthetical part of (I), the Hölder condition on F (for fixed H, K) is assumed to be uniform with respect to H, K on compact subsets of the set $H > 0, H^2 \geq K > 0$. It is not assumed that F_H, F_K are Hölder continuous. Incidentally, the assumptions of (I) do not imply that S is of class C^3 , but do imply that S has local parametrizations in terms of functions having Hölder continuous second order derivatives; cf. [16] when F is of class C^1 and [7] when F satisfies the more general condition of (I).

The results to be obtained below also permit an improvement of results (see [10], [12], [6]) on special W -surfaces (as defined by CHERN). For example, one obtains the following version of a theorem due to H. HOPF [11] under assumptions of analyticity:

(II) *Let $F(H, K)$ be of class C^1 on the portion $H^2 \geq K$ of the (H, K) -plane and satisfy (1) when $H^2 = K$. Let S be a (small piece of a) surface of class C^2 and let its mean and Gaussian curvatures satisfy the relation $F = 0$. Then, unless S is part of a plane or of a sphere, the umbilical points of S (if any) are isolated and their indices are negative.*

In earlier versions, the assumptions imply that S is of class C^3 in a vicinity of its umbilical points; cf. [10], [12], [6].

There also will be obtained the following form of a theorem due to A. D. ALEXANDROFF under assumptions of analyticity (see [13], where H. HOPF gives an exposition of ALEXANDROFF's proof and suggests that the analyticity assumptions can be lightened, but not as far as in the following theorem):

(III) *Let $F(H, K)$ be of class C^1 on the portion $H^2 \geq K$ of the (H, K) -plane and satisfy (1) there. Let S be a closed surface of class C^2 without self-intersections and of arbitrary genus and let its mean and Gaussian curvatures satisfy the relation $F = 0$. Then S is a sphere.*

(III) is a consequence of (II) when it is supposed that S is a simple closed surface of genus 0 (in which case, it is sufficient to require (1) only when $H^2 = K$). In this case, the assertion (III) was first proved by CHERN [5] under the assumptions that $K > 0$ and that S is of class C^4 .

[*Uspehi Mat. Nauk (N.S.)*, 11 (1956), no. 5 (71), pp. 203-206] have come to my attention. (The reference to the first of these was supplied to me by Professor BUSEMANN). ALEXANDROFF's paper implies that (I) remains valid if it is merely assumed that F is continuous, has continuous partial derivatives with respect to H and to K and (1) holds. POGORELOFF's paper implies that if S has positive Gaussian curvature (in particular, S is of genus 0), then (III) remains valid if the relation $F = 0$ is replaced by $W(k_1, k_2) = 0$, where W is any increasing function of each of the variables k_1, k_2 . (Note that this result does not imply corresponding theorems of [10] nor their improvements via (II); for the latter involve conditions on S only near umbilical points.)

The theorems on elliptic partial differential equations which are needed for the proofs of (I), (II), (III) will be stated in the next two sections. By standard arguments (*cf.*, *e.g.*, [14], [9], [10]), (I) and (II) are consequences of (*) below. The proof of (III) will be indicated in Section 11 below and will depend on (*) and (iii).

2. A non-linear elliptic equation. One of the theorems to be proved below is the following:

(*) *Let $G(x, y, u, p, q, r, s, t)$ be defined on an 8-dimensional domain and have the properties that for some λ , $0 < \lambda < 1$,*

$$|G(x_2, y_2, u, p, q, r, s, t) - G(x_1, y_1, u, p, q, r, s, t)| \leq \text{const.} (|x_1 - x_2|^\lambda + |y_1 - y_2|^\lambda),$$

that the partial derivatives $G_u, G_p, G_q, G_r, G_s, G_t$ exist and are continuous and that $4G, G_t - G_s^2 > 0$. Let $u = u_1(x, y)$, $u = u_2(x, y)$ be C^2 -solutions of $G(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$ in a vicinity of some point, say $(x, y) = (0, 0)$, and let the difference $\varphi = u_2 - u_1$ satisfy

$$(2) \quad \varphi(0, 0) = \varphi_x(0, 0) = \varphi_y(0, 0) = 0.$$

Then, unless $\varphi \equiv 0$, there is a punctured vicinity of $(0, 0)$ in which $\varphi_x^2 + \varphi_y^2$ and $\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}$ are positive. Furthermore, there exists a positive integer n such that, as $\rho = (x^2 + y^2)^{\frac{1}{2}} \rightarrow 0$,

$$(3) \quad \text{l.u.b.} \{ \gamma | \varphi = O(\rho^\gamma) \} = n + 1,$$

$$(4) \quad \log (\varphi_x^2 + \varphi_y^2) / \log \rho^2 \rightarrow n,$$

$$(5) \quad \log \Phi / \log \rho^2 \rightarrow n - 1,$$

where, in (5),

$$(6_1) \quad \Phi = \varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}$$

or

$$(6_2) \quad \Phi = (\varphi_{xx} \cos \tau + \varphi_{xy} \sin \tau)^2 + (\varphi_{xy} \cos \tau + \varphi_{yy} \sin \tau)^2,$$

and, for the second choice of Φ , (5) holds uniformly in τ . In addition, $\Phi(0, 0) = 0$ implies that $n > 1$ and that the index of the net

$$(7) \quad \varphi_{xx}(dx)^2 + 2\varphi_{xy} dx dy + \varphi_{yy}(dy)^2$$

at the origin is $\frac{1}{2}(1 - n) < 0$.

The novelty of (*) lies, for the most part, in its assertions concerning the second order partial derivatives of φ under the very mild assumptions on G ; *cf.* the remarks in Section 4.

For the definition of the index of a net of directions, see [14], p. 751. The

inequality $\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy} > 0$ for small $x^2 + y^2 > 0$ implies that the index of (7) at the origin is non-positive ([14], p. 756); this fact is sufficient for the proof of (I).

3. Linear elliptic equations. The difference φ satisfies a homogeneous, linear elliptic partial differential equation

$$(8) \quad A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy} + D\varphi_x + E\varphi_y + F\varphi = 0$$

with continuous coefficients. Accordingly, (*) follows from assertions (i) and (ii) below.

Theorems (i)–(v) are concerned with the local behavior of a solution of (8) satisfying (2). The proofs of (i)–(iv) will depend on results of MORREY [15], of BERS & NIRENBERG [3], [4] and of [7]. In particular, the representation theorem [3], p. 116, will be a starting point for most of the considerations.

Below, φ will usually denote a solution of (8) satisfying (2). The derivatives of φ will be abbreviated by $p = \varphi_x$, $q = \varphi_y$, $r = \varphi_{xx}$, $s = \varphi_{xy} = \varphi_{yx}$, $t = \varphi_{yy}$. Functions of (x, y) will sometimes be written as functions of $z = x + iy = \rho e^{i\theta}$.

(i) *Let $A, B, \dots, F$ be continuous functions of (x, y) in a vicinity of the origin and let*

$$(9) \quad AC - B^2 > 0.$$

Let $\varphi = \varphi(x, y) \equiv 0$ be a C^2 -solution of (8) satisfying (2). Then the assertions of () concerning φ and $\varphi_x^2 + \varphi_y^2$ hold. In addition,*

$$(10) \quad \log \iint_{|z| < \rho} \rho^{-1} \Phi \, dx \, dy / \log \rho^2 \rightarrow n - \frac{1}{2} \quad \text{as} \quad \rho \rightarrow 0,$$

if $\Phi = \varphi_{xx}^2 + 2\varphi_{xy}^2 + \varphi_{yy}^2$ or $\Phi = \varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}$.

In (i), φ need not be of class C^2 but can be assumed to be of class C^1 and to possess strong L^2 second order derivatives satisfying (8) almost everywhere. Correspondingly, it need only be assumed, for example, that $A, B, \dots, F$ are bounded measurable functions and that A, B, C are continuous at the origin.

In order to sharpen the conclusions of (i) dealing with second order partials of φ to those of (*), an additional assumption will be made.

(ii) *In addition to the conditions of (i), assume that φ has second order partial derivatives satisfying a uniform Hölder condition. Then the conclusions of (*) concerning the functions Φ in (6) and the net in (7) hold.*

The assumptions of (i) do not imply that a C^2 -solution of (8) has Hölder continuous second order partials. In (*), however, $\varphi = u_2 - u_1$ does have this property since the solutions $u = u_1, u_2$ of $G = 0$ do; [7]. Thus (*), and so (I)–(II) follows from (i)–(ii).

The example with $m = 2$ in Section 12 shows that, without the additional assumption that r, s, t are Hölder continuous in (ii), the assertion that “ $r = s = t = 0$ at $(0, 0)$ implies $n > 1$ ” is false [correspondingly, the index of (7) need

not be negative (but only non-positive)]. Whether or not the conclusion that the zeros of $r^2 + s^2 + t^2$ are isolated becomes false will remain undecided.

In the proof of (III), there will be needed the following assertion:

(iii) *Under the conditions of (ii), there exists an $\epsilon > 0$ (depending only on $A, B, \dots, F$ and not on the solution φ) with the property that every wedge $|\theta_0 - \arg z| \leq \frac{1}{2}(\pi - \epsilon)$ of opening $\pi - \epsilon$ contains zeros of φ clustering at the origin.*

It will be clear from the proof that if $n > 1$, then the assumptions of (iii) can be relaxed to those of (i) (instead of (ii)) and even to those mentioned just after (i). Whether or not this is the case when $n = 1$ will remain undecided. It will also remain undecided whether or not the normalization

$$(11) \quad A(0, 0) = 1, \quad B(0, 0) = 0, \quad C(0, 0) = 1$$

implies that the wedge opening $\pi - \epsilon$ in (iii) can be replaced by $\pi/(n + 1) + \epsilon$ for arbitrary $\epsilon > 0$, when n is the integer in (4); cf. the next section.

4. 0-level curves of φ . Although (i)–(iii) suffice for the proofs of (I)–(III), it seems of interest to carry farther some of the notions suggested by (iii).

If the coefficients $A, \dots, F$ of (8) are uniformly Hölder continuous, then the asymptotic formulæ (4), (5), which are on a logarithmic scale, can be replaced by sharp asymptotic formulæ for φ, p, q, r, s, t ; see [2] and the references there to earlier results. In fact, if the normalization (11) holds, then the asymptotic formula for φ is of the form $\varphi = P(x, y) + o(\rho^{n+1})$ as $\rho \rightarrow 0$, where P is a homogeneous, harmonic polynomial of degree $n + 1$. Asymptotic formulæ for p, q, r, s, t are obtained by formal differentiation.

These formulæ imply, for example, that the index of the mapping $(x, y) \rightarrow (p, q)$ is $-n$. Furthermore, if R is sufficiently small, then the set $\varphi = 0$ in $|z| \leq R$ consists of $2n + 2$ Jordan arcs $\gamma_1, \dots, \gamma_{2n+2}$ of class C^1 joining the origin to the circle $|z| = R$; any pair γ_i, γ_k have only the origin common; φ changes signs along γ_i , the angle between tangents of neighboring arcs at the origin is $\pi/(n + 1)$.

There arises the question as to whether these assertions remain valid under the conditions of (i)–(iii). This question will be answered partially in the affirmative in (iv) and partially in the negative in (v).

(iv) *Let $A, \dots, F$ be bounded, measurable functions and let (9) hold in a vicinity of $(x, y) = (0, 0)$. Let $\varphi(x, y) \equiv 0$ be of class C^1 and possess strong L^2 second order partials satisfying (8) almost everywhere. Finally, let (2) hold. Then $p^2 + q^2 > 0$ in a punctured vicinity of the origin and there exists a positive integer n such that the index of the mapping $(x, y) \rightarrow (p, q)$ at the origin is $-n$; furthermore, there exists a Jordan domain $\mathfrak{D}$ containing $(x, y) = (0, 0)$ such that the set $\varphi(x, y) = 0$ in $\mathfrak{D}$ consists of $2n + 2$ Jordan arcs $\gamma_1, \dots, \gamma_{2n+2}$ with the properties that γ_i joins the boundary of $\mathfrak{D}$ to the origin; that any pair γ_i, γ_k have only the point $(x, y) = (0, 0)$ in common; φ changes signs along γ_i ; finally, that γ_i is of class C^1 in a vicinity of any of its points except, possibly, the origin.*

The assertions of (iv) concerning $p^2 + q^2 > 0$ and the index of the mapping $(x, y) \rightarrow (p, q)$ are due to BERS & NIRENBERG [3].

In contrast to (iv), there is the following:

(v) *For any integer $m \geq 2$, there exist continuous functions A, B, C of (x, y) in a vicinity of $(x, y) = (0, 0)$ satisfying (9) with the property that*

$$(12) \quad A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy} = 0$$

has a solution $\varphi = \varphi(x, y)$ of class C^m in a vicinity of the origin satisfying (2) and such that the 0-level curves, $\varphi = 0$, terminating at the origin are spirals.

It will be clear from the example below (with $m = 2$) and from a device of MORREY ([15], p. 154) that there exist Beltrami systems $q_x = -(Fp_x + Gp_y)$, $q_y = Ep_x + Fp_y$ with continuous coefficients on $|z| \leq 1$ satisfying $EG - F^2 \equiv 1$ and having a C^1 -solution $w = p - iq$ such that $w(0) = 0$, $w = w(z)$ maps $|z| \leq 1$ topologically onto $|w| \leq 1$, but $w = w(z)$ maps every radius of every circle $|z| \leq \epsilon$ onto a spiral.

5. Proof of (i). *Ad $p^2 + q^2$.* On a sufficiently small circle $|z| \leq R$, there exists a positive function $\varphi_0(x, y)$ of class C^1 possessing strong L^2 second order derivatives satisfying (8) almost everywhere; [4]. Furthermore, the continuity of A, B, C at the origin implies that if $0 < \epsilon < 1$ and if $R = R(\epsilon)$ is sufficiently small, then the partial derivatives $\varphi_{0x}, \varphi_{0y}$ satisfy uniform Hölder conditions of order $1 - \epsilon$; cf., e.g., [7], Part I.

The function $\psi = \varphi/\varphi_0$ is a solution (almost everywhere) of an equation of the form

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D'\psi_x + E'\psi_y = 0,$$

where D', E' are bounded when $A, B, \dots, F$ are bounded.

It follows from the representation theorem of [3] that, on $|z| < R$, there exist (complex-valued) functions $s(z), w(z), f(z)$ such that $\psi_x - i\psi_y = e^{s(z)}f(w(z))$; $f(z)$ is regular analytic on $|z| < R$; $w = w(z)$ is a topological mapping of $|z| < R$ onto $|w| < R$ satisfying $w(0) = 0$; finally, $s(z), w(z)$ and the inverse mapping $z^{-1}(w)$ satisfy uniform Hölder conditions on $|z| < R, |z| < R$ and $|w| < R$, respectively.

The proof of the representation theorem in [3] and theorem (II) in [7] show that, by virtue of the continuity of A, B, C at $(0, 0)$, there exists for every $\epsilon > 0$, a positive number $S(\epsilon)$ ($< R$) such that the Hölder order of continuity of $s(z), w(z), z^{-1}(w)$ on $|z| \leq S, |z| \leq S, |w| \leq S$, respectively, is at least $1 - \epsilon$. (The part of this assertion concerning $w(z)$ and $z^{-1}(w)$ follows also from theorems on quasi-conformal mappings; MORREY, AHLFORS.)

From $\psi = \varphi/\varphi_0$ and $\psi_x - i\psi_y = e^s f(w)$, it follows that

$$\varphi_x - i\varphi_y = p - iq = \varphi_0 e^s f(w) + \varphi_0^{-1}(\varphi_{0x} - i\varphi_{0y})\varphi.$$

Let this be written in the form

$$(13) \quad p - iq = s_0 f(w) + s_1 \varphi, \quad \text{where } s_0 \neq 0.$$

The functions s_0, s_1 satisfy uniform Hölder conditions of order $1 - \epsilon$ on $|z| \leq S$ and

$$(14) \quad c |z|^{1+\epsilon} \leq |w(z)| \leq C |z|^{1-\epsilon}$$

if $|z| \leq S$ and $c = c(\epsilon), C = C(\epsilon), S = S(\epsilon)$ are suitably chosen. Since $\varphi = p = q = w = 0$ at $z = 0$ imply that $f(0) = 0$, there exists a positive integer n such that

$$(15) \quad f(w) = a_0 w^n + a_1 w^{n+1} + \dots, \quad \text{where } a_0 \neq 0.$$

If ϵ in (14) is replaced by ϵ/n , it follows that $p - iq = O(|z|^{n-\epsilon}) + O(|\varphi|)$ as $z \rightarrow 0$.

Since $p - iq = 0$ at $z = 0$, the Hölder continuity of $p - iq$ shows that $p - iq = O(|z|^{1-\epsilon})$. Assume that $p - iq = O(|z|^{j-\epsilon})$ for some integer j , where $1 \leq j < n$. Then $\varphi = O(|z|^{j+1-\epsilon})$, and so

$$p - iq = O(|z|^{n-\epsilon}) + O(|z|^{j+1-\epsilon}) = O(|z|^{j+1-\epsilon}).$$

Thus, by induction, $p - iq = O(|z|^{n-\epsilon})$ and, consequently, $\varphi = O(|z|^{n+1-\epsilon})$ for every $\epsilon > 0$.

From (13), (15) and the first inequality in (14), it is seen that $|p - iq| \geq \text{const. } |z|^{n+\epsilon}$ for small $|z|$ and some $\text{const.} > 0$. This shows that $p^2 + q^2 \neq 0$ in some punctured vicinity of $z = 0$ and that (4) holds.

6. Proof of (i). *Ad* φ . In order to prove (3), it remains to show that

$$(16) \quad \varphi = O(|z|^{n+1+\mu}) \quad \text{as } z \rightarrow 0$$

cannot hold for any $\mu > 0$. Suppose, on the contrary, that (16) does hold for some $\mu > 0$. The following consequence

$$(17) \quad \iint_{|z| < \rho} |\Delta \varphi|^2 dx dy = O(\rho^{2n-\epsilon}) \quad \text{for every } \epsilon,$$

where $\Delta \varphi = r + t$, of the, as yet, unproved assertion (10), will be used to obtain a contradiction.

Consider Green's relation

$$\rho \int_{|z|=\rho} \varphi \varphi_\rho d\theta = \iint_{|z| < \rho} (|\nabla \varphi|^2 + \varphi \Delta \varphi) dx dy,$$

where $|\nabla \varphi|^2 = p^2 + q^2$. In view of (16) and $\varphi_\rho = O(|z|^{n+\frac{1}{2}\mu})$, where $\mu > 0$, the line integral on the left is $O(\rho^{2n+2+\frac{1}{2}\mu})$. By virtue of Schwarz's inequality, (16) and (17), the integral of $\varphi \Delta \varphi$ is also $O(\rho^{2n+2+\frac{1}{2}\mu})$. Consequently, the integral of $|\nabla \varphi|^2 = p^2 + q^2$ is $O(\rho^{2n+2+\frac{1}{2}\mu})$. But this contradicts the lower limitation $p^2 + q^2 \geq$

const. $|z|^{2n+\frac{1}{2}\mu}$ for small $|z|$ and some const. > 0 . Thus the assertion (3) concerning φ in (i) is proved, under the assumption (17).

7. Proof of (i). *Ad* r, s, t . Since $p(0, 0) = 0$ implies that

$$p(\rho e^{i\theta}) = \int_0^\rho (r(\rho e^{i\theta}) \cos \theta + s(\rho e^{i\theta}) \sin \theta) d\rho,$$

it follows from Schwarz's inequality that

$$p^2(\rho e^{i\theta}) \leq \rho \int_0^\rho [r^2(\rho e^{i\theta}) + s^2(\rho e^{i\theta})] d\rho.$$

If this inequality is multiplied by $dx dy = \rho d\rho d\theta$, an integration gives

$$\iint_{|z| < \rho} p^2 dx dy \leq \rho^3 \iint_{|z| < \rho} \rho^{-1}(r^2 + s^2) dx dy.$$

A similar inequality holds if p, r, s are replaced by q, s, t , respectively. Hence

$$\iint_{|z| < \rho} (p^2 + q^2) dx dy \leq \rho^3 L(\rho),$$

where

$$(18) \quad L(\rho) = \iint_{|z| < \rho} \rho^{-1}(r^2 + 2s^2 + t^2) dx dy.$$

Consequently, (4) shows that $\rho^{-(2n-1)-\epsilon} L(\rho) \rightarrow \infty$ as $\rho \rightarrow 0$ for every $\epsilon > 0$.

In order to estimate $L(\rho)$ from above, note that the ellipticity of (8) and the boundedness of the coefficients of (8) imply the existence of positive constants K, k satisfying

$$(19) \quad r^2 + 2s^2 + t^2 \leq K(s^2 - rt) + k(p^2 + q^2 + \varphi^2).$$

Multiplying this inequality by $d\rho d\theta = \rho^{-1} dx dy$ and integrating over the circle $|z| < \rho$ gives

$$(20) \quad L(\rho) = I(\rho) + O(\rho^{2n+1-\epsilon}) \quad \text{for every } \epsilon > 0,$$

where $I(\rho)$ is the integral of $K(s^2 - rt)\rho^{-1}$ over $|z| < \rho$. By Green's relation

$$(21) \quad I(\rho) = K \int_{|z|=\rho} \rho^{-1} q(r dx + s dy) + K \int_{|z| < \rho} \rho^{-3} q(sx - ry) dx dy.$$

Since $|sx - ry| \leq \rho(r^2 + s^2)^{\frac{1}{2}}$ and

$$K\rho^{-2} |q| (r^2 + s^2)^{\frac{1}{2}} \leq \frac{1}{2}\rho^{-1}(r^2 + s^2) + \frac{1}{2}K^2\rho^{-3}q^2,$$

it follows that the last term of (21) is at most $\frac{1}{2}L(\rho) + O(\rho^{2n-1-\epsilon})$. The line integral in (21) is majorized by

$$O(\rho^{n-\epsilon}) \left(\int_{|z|=\rho} (r^2 + s^2) d\theta \right)^{\frac{1}{2}} = O(\rho^{n-\epsilon}) L_\rho^{\frac{1}{2}},$$

where $L_\rho = dL/d\rho$. Thus, $L = O(\rho^{n-\epsilon}) L_\rho^{\frac{1}{2}} + O(\rho^{2n-1-\epsilon})$ by (20), (21).

For any constant $c > 0$,

$$\rho^{n-\epsilon} L_\rho^{\frac{1}{2}} \leq c^{-1} \rho L_\rho + \frac{1}{4} c \rho^{2n-1-2\epsilon}$$

and so

$$L_\rho - c \rho^{-1} L + O(\rho^{2n-2-\epsilon}) \geq 0;$$

that is,

$$(\rho^{-\epsilon} L)_\rho + O(\rho^{2n-2-\epsilon-\epsilon}) \geq 0.$$

Choose $c = 2n - 1 - 2\epsilon$ and integrate this differential inequality over an interval (ρ, R) to obtain

$$L(\rho) \leq L(R)(\rho/R)^{2n-1-2\epsilon} + O(\rho^{2n-1-2\epsilon}).$$

This estimate, together with the lower estimate mentioned after (16), proves the relation in (10) for $\Phi = r^2 + 2s^2 + t^2$. The relation involving $\Phi = s^2 - rt$ follows from (20). This completes the proof of (i).

8. Proof of (ii). *Ad* Φ . Let λ , $0 < \lambda < 1$, denote the order of Hölder continuity of r, s, t on a circle $|z| \leq R$. Let δ denote the difference operator $\delta g = g(x + \delta x, y) - g(x, y)$. Then, by the mean value theorem of differential calculus,

$$|\delta(p - iq) - (r - is) \delta x| \leq K |\delta x|^{1+\lambda}$$

for a suitable constant K . Since s_0 and s_1 in (13) satisfy Hölder conditions of order $1 - \epsilon$ and $|w(z_1) - w(z_2)| \geq c |z_1 - z_2|^{1+\epsilon}$ for a suitable choice of $c > 0$ it follows from (13), (14) and (15) that, for a sufficiently large C and $1/c$,

$$|\delta(p - iq)| \geq c |z|^{n-1+\epsilon} |\delta x|^{1+\epsilon} - C |z|^{n-\epsilon} |\delta x|^{1-\epsilon}$$

if $|z|$ and $|\delta x|/|z|$ are small. Thus the last two displays give, for $\delta x \neq 0$,

$$|r - is| \geq c |z|^{n-1+\epsilon} |\delta x|^\epsilon - C |z|^{n-\epsilon} |\delta x|^{-\epsilon} - K |\delta x|^\lambda.$$

In this inequality, let $z \neq 0$, $|\delta x| = |z|^{(n-\frac{1}{2})/\lambda}$ and $\epsilon > 0$ so small that $\mu < \frac{1}{2}$, where $\mu = \epsilon + \epsilon(n - \frac{1}{2})/\lambda$. (It can also be supposed that $\lambda < \frac{1}{2}$, so that $\delta x = o(|z|)$ as $z \rightarrow 0$.) Then

$$\begin{aligned} |z|^{n-1+\epsilon} |\delta x|^\epsilon &= |z|^{n-1+\mu}, & |z|^{n-\epsilon} |\delta x|^{-\epsilon} &= |z|^{n-\mu} = o(|z|^{n-1+\mu}), \\ |\delta x|^\lambda &= |z|^{n-\frac{1}{2}} = o(|z|^{n-1+\mu}). \end{aligned}$$

Consequently, for arbitrary $\mu > 0$, $|r - is| \geq \frac{1}{2} c |z|^{n-1+\mu}$ for small $|z|$. In particular, $r^2 + s^2 > 0$ for small $|z| > 0$.

A similar argument shows that $r - is = O(|z|^{n-1-\mu})$ as $z \rightarrow 0$ for every $\mu > 0$. Hence (5) holds for $\Phi = r^2 + s^2$.

By differentiating $p - iq$ in an arbitrary direction (instead of the x -direction), it follows that

$$\Phi = (r \cos \tau + s \sin \tau)^2 + (s \cos \tau + t \sin \tau)^2 > 0$$

for small $|z|$ and that (5) holds as $z \rightarrow 0$ uniformly in τ .

That $\Phi = s^2 - rt$ satisfies $\Phi > 0$ for small $x^2 + y^2 > 0$ and (5) follows from (19) and

$$s^2 - rt \leq \frac{1}{2}(r^2 + s^2) + \frac{1}{2}(s^2 + t^2).$$

Clearly, $s^2 - rt = 0$ at the origin if and only if $r = s = t = 0$ there. In this case, the Hölder continuity of r, s, t implies that $r^2 + s^2 + t^2 = O(|z|^{2\lambda})$ as $z \rightarrow 0$. Hence, by (5), $n - 1 \geq \lambda > 0$, that is, $n > 1$.

This proves all of the assertions of (ii) except that concerning the net (7).

9. Proof of (ii). *Ad* (7). Since $rt - s^2 \neq 0$ in a punctured vicinity of the origin, the index of the net (7) is one-half of the index of the field

$$(22) \quad r \, dx + s \, dy = dp;$$

[14], p. 755.

Consider the system of ordinary differential equations

$$(23) \quad \frac{dx}{dt} = -s(x, y), \quad \frac{dy}{dt} = r(x, y).$$

The solution paths of (23) are the level curves of $p = p(x, y)$. Since $\text{grad } p = (r, s) \neq 0$, except possibly at the origin, a point $(x_0, y_0) \neq (0, 0)$ determines a unique solution path of (23) through it.

From (3) and (13)–(15), it follows that, as $z \rightarrow 0$,

$$(24) \quad p - iq = a_0 w^n + o(|w|^{n+1-\epsilon}), \quad \text{where } a_0 \neq 0,$$

and $\epsilon > 0$ is (arbitrarily) fixed. This makes it clear that (in the w -plane, hence in the z -plane) there is no closed level curve, $p = \text{const.}$, in the vicinity of the origin.

It follows from [17], pp. 245–248, that there exists a Jordan domain $\mathfrak{D}$ containing the origin in its interior and an integer $m > 0$ such that the set $p = 0$ in $\mathfrak{D}$ consists of $2m$ Jordan arcs with the properties that each reaches from the origin to the boundary of $\mathfrak{D}$ and any pair have only the origin in common. (In [17], it is assumed that $\text{grad } p = (r, s)$ is of class C^1 but only the (local) uniqueness of a solution of (23) through a point $(x_0, y_0) \neq (0, 0)$ is needed.)

On the one hand, it follows from BENDIXSON's theory [1] that the index of the map $(x, y) \rightarrow (r, s)$ is $-(m - 1)$. On the other hand, (24) implies that the index of the map $(x, y) \rightarrow (p, -q)$ is n ([3], p. 122); hence $m = n$ ([17], p. 251). Thus, the index of the map $(x, y) \rightarrow (r, s)$ is $1 - n$. This implies that the index of the field (22) is $1 - n$ and completes the proof of (ii).

10. Proof of (iii). Consider first the case $n = 1$. By (ii), $r^2 + s^2 + t^2 > 0$ at $z = 0$. It follows from (2), (8) and (9) that $s^2 - rt > 0$ at $z = 0$. Hence the set $\varphi = 0$ near $z = 0$ consists of two intersecting arcs of class C^1 which have asymptotic directions (on the surface $\varphi = \varphi(x, y)$) at $z = 0$; cf., e.g., [8], p. 617. The angle between these directions depends only on $A(0, 0)$, $B(0, 0)$, $C(0, 0)$ (for example, if $A(0, 0) = C(0, 0)$ and $B(0, 0) = 0$, then these arcs intersect orthogonally at $z = 0$). Clearly, the assertion (iii) holds in the case $n = 1$.

Let $n > 1$. The variation of constants $\psi = \varphi/\varphi_0$ used at the beginning of Section 5 shows that there is no loss of generality in supposing that $F \equiv 0$ in (8). Thus, in (8), let A, B, C be continuous functions on $|z| \leq R$ satisfying (9); D, E be bounded and measurable; and $F \equiv 0$. Then the Harnack inequality proved in [4], pp. 161–164, implies that there exists a positive, continuous, increasing function $h(\rho)$ on $0 \leq \rho < R$ with the properties that h depends only on R and the bounds of $A, \dots, E, (AC - B^2)^{-1}$ and that if $\varphi(z)$ is positive on a circle $|z - z_0| < \rho_0$ contained in $|z| < R$, then

$$(25) \quad \varphi(z_0 + \rho e^{i\theta}) \geq \frac{\varphi(z_0)}{h(\rho R/\rho_0)}$$

for $0 \leq \rho < \rho_0$.

The proof in [4] also implies that there exists a constant μ , $0 < \mu < 1$, such that $(1 - \rho/R)^{2/\mu} h(\rho) = O(1)$ as $\rho \rightarrow R$. Actually, from [4], it follows that, for $R = 1$,

$$(26) \quad \begin{aligned} \varphi(z_0 + \rho e^{i\theta}) &\leq \text{Const. } \varphi(z_0) \left(1 + \int_0^\rho |1 - s/\rho_0|^{-2/\mu} dx \right) \\ &= \varphi(z_0) O(|1 - \rho/\rho_0|^{1-2/\mu}) \end{aligned}$$

as $\rho \rightarrow \rho_0$ and it is stated that the inequality (25) follows from the result (26) and a conformal mapping. On the other hand, if z_0 and $z_0 + \rho e^{i\theta}$ are interchanged in the argument in [4], p. 163, leading to (26), it follows directly that

$$\varphi(z_0) \leq \text{Const. } \varphi(z_0 + \rho e^{i\theta}) |1 - \rho/\rho_0|^{-1} \left(1 + \int_0^\rho |1 - s/\rho_0|^{-2/\mu} ds \right).$$

In this relation and in (26), μ , Const. and the constants implicit in O depend only on the bounds for $A, \dots, E$ and $(AC - B^2)^{-1}$.

If theorem (II) of [7] is combined with the proof of (25), (26), it follows from the continuity of A, B, C that μ can be chosen arbitrarily near to 1 if $R > 0$ is sufficiently small. In what follows, let R be so small that $2/\mu < \frac{5}{2}$. Hence $(1 - \tau)^{5/2} h(\tau R) < 1$ if $\tau < 1$ is fixed sufficiently near to 1. With τ fixed, let $\epsilon > 0$ be chosen so small that if $\sigma = \sin \frac{1}{2}(\pi - \epsilon)$, then $1 > \sigma > \tau$ and $(1 - \tau)^{5/2} h(\tau R/\sigma) < 1$. It will be shown that this choice of ϵ satisfies the assertion of (iii).

If this were not the case, there would exist a sector $|\theta_0 - \arg z| \leq \frac{1}{2}(\pi - \epsilon)$ of a small circle $|z| \leq 2\alpha$ containing no zeros of φ . It can be supposed that $\theta_0 = 0$

and that $\varphi > 0$ on this sector. In particular, $\varphi > 0$ on the circle $|z - x_0| \leq \rho_0$, when $x_0 = \alpha$ and $\rho_0 = x_0\sigma$. By (25),

$$\varphi(x_0(1 - \tau)) \geq \frac{\varphi(x_0)}{h(x_0\tau R/x_0\sigma)} = \frac{\varphi(x_0)}{h(\tau R/\sigma)}.$$

A corresponding inequality holds if x_0 is replaced by $x_0(1 - \tau)$, $x_0(1 - \tau)^2$, $\dots$. Hence

$$\varphi(x_0(1 - \tau)^m) \geq \frac{\varphi(x_0)}{h^m(\tau R/\sigma)} \quad \text{for } m = 1, 2, \dots.$$

On the other hand, (i) implies that $|\varphi(z)| \leq c|z|^{n+\frac{1}{2}}$ for small $|z|$ (and suitably chosen $c > 0$). Consequently,

$$cx_0^{n+\frac{1}{2}}(1 - \tau)^{m(n+\frac{1}{2})} \geq \frac{\varphi(x_0)}{h^m(\tau R/\sigma)} \quad \text{for large } m.$$

Since $\varphi(x_0) \neq 0$, it follows that $(1 - \tau)^{n+\frac{1}{2}}h(\tau R/\sigma) \geq 1$. But $n > 1$ means that

$$(1 - \tau)^{n+\frac{1}{2}}h(\tau R/\sigma) \leq (1 - \tau)^{5/2}h(\tau R/\sigma) < 1.$$

Thus the choice of ϵ shows that the hypothesis that $\varphi \neq 0$ for $|\arg z| \leq \frac{1}{2}(\pi - \epsilon)$ and $0 < |z| \leq 2\alpha$ is untenable. This proves (iii).

11. On the proof of (III). As indicated in [13], pp. 114–115, in order to prove (III), it is sufficient to verify the following two assertions:

(α) Let $u = u_1(x, y)$ and $u = u_2(x, y)$ be two solutions for small $x^2 + y^2$ of the partial differential equation $F(H, K) = 0$,

$$H = \frac{(1 + u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1 + u_x^2)u_{yy}}{2(1 + u_x^2 + u_y^2)^{\frac{3}{2}}}, \quad K = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2}$$

and let $\varphi = u_2 - u_1 \equiv 0$. Then it is impossible that u_1, u_2 have a “semi-proper contact” at the origin, that is, it is impossible that (2) holds and that there exists an arc J of class C^1 through the origin such that φ does not vanish on one side of J for small $x^2 + y^2$.

(β) Let S be a surface satisfying the conditions of (III). Then the set of “non-exceptional” directions U of S , considered as a set on the unit sphere, has an interior point. A unit vector or direction U is called an exceptional direction of S if there exists a point P on S with the properties that $K(P) = 0$ and that U is an asymptotic direction at P . (The conditions of (III) imply that there is no flat point P , $K(P) = H(P) = 0$, on S ; [13], p. 115.)

Ad (α). The conditions of (III) show that $G(x, y, u, \dots) = F(H, K)$ satisfies the conditions of (*). Hence (α) follows from (iii).

Ad (β). A point P satisfying the conditions specified in the definition of an exceptional direction U will be said to belong to U . Suppose that $U_0 = (1, 0, 0)$ is an exceptional direction of S at $P_0 : (x, y, u) = (0, 0, 0)$ belongs to U_0 .

Let $(0, 0, 1)$ be a normal vector to S and P_0 and let S have a local parametrization $u = u(x, y)$ of class C^2 in a vicinity of P_0 .

Since $K = 0$ and U_0 is in an asymptotic direction at P_0 , it follows that $u = \frac{1}{2}by^2 + o(x^2 + y^2)$ as $x^2 + y^2 \rightarrow 0$. Then $u_{yy}(0, 0) = b \neq 0$, since P_0 is not a flat point. The mean curvature at $(0, 0)$ is $H_0 = \frac{1}{2}b$ and $F(H, K) = 0$ for $H = H_0$ and $K = 0$.

There exists a right circular cylinder, say $u = u_0(x, y) = (\alpha^2 - y^2)^{\frac{1}{2}} - \alpha$ such that $\varphi = u - u_0$ satisfies $\varphi = \varphi_x = \varphi_y = \varphi_{xx} = \varphi_{xy} = \varphi_{yy} = 0$ at $(x, y) = (0, 0)$. Since the cylinder $u = u_0(x, y)$ has $K \equiv 0$ and a constant mean curvature H_0 , it follows that $u = u_0(x, y)$ is a solution of $F(H, K) = 0$. Hence (ii) is applicable to $\varphi = u - u_0$. In particular, (5) holds uniformly in τ for the function (6_2) and an integer $n \geq 2$.

Since $u_0(x, y)$ does not depend on x , it follows that $\varphi_x = u_x$, $\varphi_{xx} = u_{xx}$, $\varphi_{xy} = u_{xy}$, $\varphi_{yy} = u_{yy} - u_{0yy}$. Hence

$$(27) \quad (u_{xx} \cos \tau + u_{xy} \sin \tau)^2 + (u_{yy} \cos \tau + (u_{yy} - u_{0yy}) \sin \tau)^2$$

has majorants and minorants (uniform with respect to τ) of the form $c^2 |z|^{2(n-1-\epsilon)}$, $C^2 |z|^{2(n-1+\epsilon)}$, respectively; also

$$(28) \quad u_y = O(|z|) \quad \text{and} \quad u_x = O(|z|^{n-\epsilon}) \quad \text{as} \quad z \rightarrow 0$$

for every $\epsilon > 0$.

Suppose, if possible, that U_0 is a cluster point of exceptional directions $U = (\cos \alpha, \cos \beta, \cos \gamma)$ on the unit sphere. Suppose also that there is a point $P = (x, y, u)$ of S belonging to U and that $P \rightarrow P_0$, $U \rightarrow U_0$, where U and P depend on some index. At P ,

$$(29) \quad u_x \cos \alpha + u_y \cos \beta - \cos \gamma = 0$$

since U is tangent to S , and

$$(30) \quad u_{xx} \cos \alpha + u_{xy} \cos \beta = u_{xy} \cos \alpha + u_{yy} \cos \beta = 0$$

since $K(P) = 0$ and U is in an asymptotic direction at P .

From (28) and (29),

$$(31) \quad \cos \gamma = O(|z|^{n-\epsilon}) + O(|z|) \cos \beta$$

as $P \rightarrow P_0$. Since

$$\sin^2 \alpha = \cos^2 \beta + \cos^2 \gamma = \cos^2 \beta + O(|z|^2),$$

it is seen that

$$\sin \alpha = \pm \cos \beta + O(|z|).$$

Thus, (30) and the remark concerning (27), with $\tau = \pm \alpha$, show that $|u_{0yy} \cos \beta|$ has a minorant of the form $C |z|^{n-1+\epsilon}$ for every $\epsilon > 0$ as $P \rightarrow P_0$. Since $u_{0yy}(0, 0) = b \neq 0$, the same holds for $|\cos \beta|$.

In particular, $\cos \beta \neq 0$ (hence $U \neq U_0$) if $P (\neq P_0)$ is near to P_0 . Conse-

quently, the points P belonging to a fixed exceptional direction, say U_0 , have no cluster points on S . Since S is compact, there are at most a finite number of points, say $P_0, P_1, \dots, P_N$, of S belonging to U_0 . The compactness of S also implies that if $\{U\}$ is a sequence of exceptional directions satisfying $U \rightarrow U_0$ and if P belongs to U , then the limit points of $\{P\}$ are contained in the finite set $P_0, P_1, \dots, P_N$.

It follows from (31) and the minorant obtained for $\cos \beta$ that $\cos \gamma / \cos \beta \rightarrow 0$ as $P \rightarrow P_0$. In other words, if P is sufficiently near to P_0 , then the exceptional direction U , considered as a point of the unit sphere, is in one of two arbitrarily narrow wedges $|\cos \gamma / \cos \beta| \leq \epsilon$ with vertex at U_0 . Thus after a deletion of wedges belonging to $P_0, P_1, \dots, P_N$ from a sufficiently small neighborhood of U_0 , there remains in this neighborhood of U_0 only points of the sphere which are not exceptional. This proves (β) , hence (III).

12. Proof of (iv). As remarked after (iv), the assertions of (iv) concerning $p^2 + q^2 > 0$ for small $|z| > 0$ and the degree of the mapping $(x, y) \rightarrow (p, q)$ are contained in [3], p. 122.

As to the assertions concerning the set $\varphi = 0$, note that the maximum principle (which is valid under the conditions of (iv); [4], p. 154) shows that the set of zeros of φ cannot contain a Jordan curve in $|z| < R$. Since $p^2 + q^2 \geq 0$ according as $x^2 + y^2 \geq 0$, it follows from [17], pp. 244–248, by a consideration of the systems of ordinary differential equations

$$(32) \quad \frac{dx}{dt} = -q(x, y), \quad \frac{dy}{dt} = p(x, y),$$

that there exists a Jordan domain $\mathfrak{D}$ containing $(0, 0)$ and that the set $\varphi = 0$ in $\mathfrak{D}$ consists of an even number, $2m$, of Jordan arcs $\gamma_1, \dots, \gamma_{2m}$ with the properties enumerated above. This is deduced from results of BENDIXSON which also imply ([1], p. 39) that $m = n + 1$ if $-n$ is the index of the mapping $(x, y) \rightarrow (p, q)$ at the origin.

Cf. Section 9 as to remarks about the smoothness of p, q needed for these results.

13. Proof of (v). In order to verify (v), let $m \geq 2$ be an integer,

$$(33) \quad g(\rho) = \log \log (1/\rho) \quad \text{for} \quad 0 < \rho < 1$$

and, according as $0 < \rho < 1$ or $\rho = 0$,

$$(34) \quad \varphi(x, y) = \frac{\rho^m}{\log \rho} \cos(m\theta + g(\rho)) \quad \text{or} \quad \varphi(0, 0) = 0,$$

where $x + iy = \rho e^{i\theta}$. The function (34) is of class C^∞ on the punctured disk $0 < \rho < 1$. It is easily verified that if $0 < |z| \rightarrow 0$,

$$\begin{aligned}
 (35) \quad r &= m(m-1) \frac{\rho^{m-2}}{\log \rho} \left[\cos((m-2)\theta + g) + O\left(\frac{1}{|\log \rho|}\right) \right], \\
 s &= m(m-1) \frac{\rho^{m-2}}{\log \rho} \left[-\sin((m-2)\theta + g) + O\left(\frac{1}{|\log \rho|}\right) \right], \\
 t &= m(m-1) \frac{\rho^{m-2}}{\log \rho} \left[-\cos((m-2)\theta + g) + O\left(\frac{1}{|\log \rho|}\right) \right].
 \end{aligned}$$

Hence φ is of class C^2 if $m \geq 2$. The calculations leading to the formula for r, s, t show that φ is of class C^m where $m \geq 2$.

The 0-level curves of φ are the $2m$ spirals $m\theta + g(\rho) = \frac{1}{2}(2j+1)\pi$, where $j = 1, \dots, 2m$. It will be shown that, for small $|z|$, φ satisfies a linear equation of the type (26) with continuous coefficients A, B, C .

By (35), as $\rho \rightarrow 0$,

$$(36) \quad \frac{\rho^{m-2}}{(\log \rho)^2} \Psi \rightarrow 0 \quad \text{if} \quad \Psi = \frac{r}{r^2 + s^2} \quad \text{or} \quad \Psi = \frac{s}{r^2 + s^2}$$

and

$$(37) \quad r + t = O \frac{\rho^{m-2}}{(\log \rho)^2}.$$

The function (34) satisfies an equation (26), for small $|z|$, with

$$A = 1 - \frac{(r+t)r}{(r^2+s^2)}, \quad B = -\frac{\frac{1}{2}(r+t)s}{r^2+s^2}, \quad C = 1.$$

The relations (36), (37) show that A, B, C are continuous at $\rho = 0$ (as well as for small $\rho > 0$). This proves (v).

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