On Singular Integral Equations with Logarithmic or Cauchy Kernels

R. C. MAC CAMY

Communicated by H. LEWY

1. Introduction. Many of the boundary problems for elliptic partial differential equations in two variables can be reduced to one-dimensional integral equations with logarithmic or Cauchy kernels. Unless the geometry be such that the latter are over infinite or semi-infinite ranges an explicit solution is in general not possible and one is led to the question of numerical approximations.

The standard device for solving integral equations numerically is to replace all functions occurring by polynomial approximations. If the kernels are analytic this is satisfactory and even for singular kernels is still feasible [1]. Polynomial approximations conceal, however, the singular nature of the solutions at the ends of the interval of integration. Since it is precisely this singular behavior which is often of interest in the original boundary problem we shall investigate it here for certain classes of integral equations. We leave aside entirely questions of existence of solutions of these equations since such questions are best discussed within the framework of the differential equation problem they represent.*

Our general method is the following. We shall translate the integral equations into functional relations for multi-valued analytic functions of a complex variable. Without being able to solve these relations we can still obtain information about the structure of the analytic function, which information we translate back into facts concerning the solutions of the integral equations. The idea of introducing complex variables dates back to CARLEMAN [3] and [4] who was able, in this manner, to obtain explicit solutions of some very elementary equations. The necessary tools for a systematic investigation of a large class of integral equations were presented by LEWY [5] and [6] in his study of difference-differential equations.

We study equations of the following three types, all for $0 < x < 2$:†

* A discussion of such singular integral equations is given in [2].

† All integrals which appear should be interpreted in the Cauchy principal value sense. We make the upper limit 2 in all our integrals as a numerical convenience for Section 7. This can of course always be achieved by a change in scale. With analytic $A(x, t)$ it is clear that we could replace $A(x, t)$ by $A(x, x)$ and merely change $C(x, t)$.

Type I

$$(I) \quad \int_0^2 [A(x, t)(x - t)^{-1} + B(x, t) \log |x - t| + C(x, t)]f(t) dt = g(x).$$

Type II

$$(II) \quad f(x) + \frac{1}{\pi} \int_0^2 [A(x, t)(x - t)^{-1} + B(x, t) \log |x - t| + C(x, t)]f(t) dt = g(x).$$

Type III

$$(III) \quad f(x) + \frac{1}{\pi} \int_0^2 [A(x, t)(x - t)^{-1} + B(x, t) \log |x - t| + C(x, t)]f'(t) dt = g(x).$$

In each equation $g(x)$ is a given function analytic for complex x in $0 \leq |x| \leq 2$ and the functions A , B and C are analytic for complex x and t in $0 \leq |x|, |t| \leq 2$. Certain special values of the functions A and B lead to considerable technical difficulty in the exposition of our ideas. We suppress these difficulties in this paper by placing restrictions on A and B . In particular, we consider two cases:

Case (1)

$$\begin{aligned} A(z, z) &\neq 0, & 0 \leq |z| \leq 2, \\ A(z, z) &\neq \pm i, & 0 \leq |z| \leq 2. \end{aligned}$$

Case (2)

$$\begin{aligned} A(z, z) &\equiv 0, & B(z, z) &\neq 0, & 0 \leq |z| \leq 2, \\ B(z, z) &\neq \pm i, & 0 \leq |z| \leq 2. \end{aligned}$$

We mention here prototypes of each equation. For (I) the equation

$$(1.1) \quad \int_0^2 H_0(k|x - t|)f(t) dt = g(x), \quad 0 < x < 2,$$

where H_0 is the Hankel function of first type governs the diffraction of two-dimensional sound waves by a strip of finite width [7]. For (II) we consider

$$(1.2) \quad f(x) + \lambda \int_0^2 \log |x - t| f(t) dt = g(x), \quad 0 < x < 2.$$

It is easy to see that solution of (1.2) is tantamount to solving the following problem in heat flow: Find a function $u(x, y)$ harmonic in $y < 0$, with $u(x, y) - k \log(x^2 + y^2)$ bounded as $(x^2 + y^2) \rightarrow \infty$ for some k , and such that $u_y(x, 0) = 0$ on $-\infty < x < 0$ and $2 < x < +\infty$ and $u_y(x, 0) + \lambda u(x, 0) = g(x)$ on $0 < x < 2$. Finally, for (III) the equation

$$(1.3) \quad f(x) - \int_0^2 f'(t)(t-x)^{-1} dt = g(x), \quad 0 < x < 2,$$

occurs in the theory of the Cauchy process [8] and is a special case of the Prandtl integro-differential equation of airfoil theory [9].

We summarize our results for the three types of equations. Let $\bar{F}$ denote the set of all functions $f(x)$ continuous and satisfying a Holder condition on $0 < x < 2$ and such that $[x(2-x)]^\alpha f(x)$ is continuous at $x = 0$ and $x = 2$ for some α , $0 < \alpha < 1$. Suppose $f \in \bar{F}$ to be a solution of Eqn. (I), (II) or (III). Then:

(a) For Eqn. (I)

$$f(x) = \sum_{m=0}^{\infty} a_m x^{m-\frac{1}{2}},$$

the series converging absolutely and uniformly (except for the first term) in $0 \leq x \leq \rho$ for any ρ , $0 < \rho < 2$.

(b) For Eqn. (II), Case (2)

$$f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0.$$

(c) For Eqn. (II), Case (1) let

$$a = \frac{1}{2\pi i} \log \left\{ \frac{(1 - iA(0, 0))}{(1 + iA(0, 0))} \right\},$$

choosing the branch of the logarithm so that $-\frac{1}{2} < a < +\frac{1}{2}$. Then

$$x^{-a} f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0, \quad \text{if } a \leq 0,$$

$$x^{1-a} f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0, \quad \text{if } a > 0.$$

(d) For Eqn. (III), Case (1)

$$x^{-\frac{1}{2}} f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0.$$

(e) For Eqn. (III), Case (2) let

$$b = \frac{1}{2\pi i} \log \left\{ \frac{B(0, 0) + i}{B(0, 0) - i} \right\} \quad \text{as } x \rightarrow 0,$$

choosing the logarithm so that $0 < b < 1$. Then

$$x^{-b-\frac{1}{2}} f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0,$$

$$x^{-b+\frac{1}{2}} f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0.$$

The symbol $\sim$ denotes "asymptotically equal to as $x \rightarrow 0$ ", a terminology discussed in Section 3 but with fairly obvious meaning when one observes that the power products $x^m(x \log x)^n$ can be ordered according to their vanishing at $x = 0$.

Remarks:

(1) Observe that Eqn. (III) with Case (2) becomes identical with Eqn. (II), Case (1) provided we integrate by parts and replace A in (II) by B . Then

$$a = \frac{1}{2\pi i} \log \left\{ \frac{1 - iB(0, 0)}{1 + iB(0, 0)} \right\} = \frac{1}{2\pi i} \log \left\{ \frac{B(0, 0) + i}{B(0, 0) - i} \right\} + \frac{1}{2} = b + \frac{1}{2}.$$

Thus Case (2) for (III) follows once we prove the result for (II), Case (1).

(2) The asymptotic developments are intended as $x \rightarrow 0$. However, on making the substitutions $\hat{x} = 2 - x$, $\hat{t} = 2 - t$, $\hat{f}(\hat{t}) = f(2 - t)$ in (I), (II), (III) we see that analogous developments hold as $x \rightarrow 2$, it being necessary only to replace x by $(2 - x)$ in the various series.

(3) It is to be observed that the series for (I) converges while those for (II) and (III) are only asymptotic. It seems plausible, but difficult to prove, that the series in (II) and (III) also converge. We shall prove this fact for the special equations (1.2) and (1.3).

(4) Among the coefficients a_{mn} there exist certain relations which are independent of the choice of g . We shall see, in fact, that all the a_{mn} , for $n \geq 1$, are determined recursively by the a_{m0} . This fact makes possible a procedure for approximating the solutions of (I), (II) or (III), a procedure we outline in Section 7.

2. Preliminaries. We define a linear transformation $\bar{U}(x; f)$ over $\bar{F}$ by

$$(2.1) \quad \bar{U}(x, f) = \frac{1}{\pi} \int_0^2 [A(x, t)(x - t)^{-1} + B(x, t) \log |x - t| + C(x, t)] f(t) dt.$$

Here we require that $A(z, w)$, $B(z, w)$, $C(z, w)$ be single-valued analytic functions of z and w in $0 \leq |z|, |w| \leq 2$. Further, we require either

Case (1)

$$A(z, z) \neq 0, \pm i \quad \text{for } 0 \leq z \leq 2$$

or

Case (2)

$$A(z, z) \equiv 0, \quad B(z, z) \neq 0, \pm i \quad \text{for } 0 \leq z \leq 2.$$

We let R denote the domain $0 < |z| < 2$ of a Riemann surface with the origin as logarithmic branch point. On this surface we denote by z^+ the image point of z after a counterclockwise circuit of the origin, i.e. z^+ is such that $\log z^+ =$

$\log z + 2\pi i$. We write F for the class of functions $f(z)$ analytic on R . The notation is suggested, as we shall see, by the fact that if $f \in \bar{F}$ is a solution of any of the equations we study, then replacing x by $z = x + iy$ yields a function $f(z) \in F$.

For $f \in F$ and $|z| > 0$, $0 \leq \arg z \leq 2\pi$ we define two more linear transformations $U(z; f)$ and $I(z; f)$ by

$$(2.2) \quad U(z; f) = \frac{1}{\pi} \int_0^2 [A(z, t)(z - t)^{-1} + B(z, t) \log(t - z) + C(z, t)]f(t) dt,$$

$$(2.3) \quad I(z; f) = 2iA(z, z)f(z) + 2i \int_0^z B(z, t)f(t) dt.$$

In (2.2) the logarithm is intended as that branch which is real for real positive t and real negative z .

Lemma 1. For $f \in \bar{F}$, $U(z; f)$ and $I(z; f)$ are analytic functions on the sector $0 < \arg z < 2\pi$ of R and satisfy, for $\arg z = 0$,

$$(2.4) \quad U(z^+; f) - U(z; f) = I(z; f),$$

$$(2.5) \quad U(z^+; f) + U(z; f) = 2\bar{U}(z; f).$$

Finally, $U(z; f) = O(z^{-\alpha})$ as $z \rightarrow 0$ in $0 \leq \arg z \leq 2\pi$.

Proof. The first part of the lemma is obvious. For the second part observe that as $z \rightarrow z_0$, z_0 real, $0 < z_0 < 2$, from $\text{Im } z \leq 0$

$$\log(t - z) \rightarrow \log|t - z_0| \pm \pi i \quad \text{if } t < z_0,$$

$$\log(t - z) \rightarrow \log|t - z_0| \quad \text{if } t > z_0.$$

Hence if we write

$$\Phi_B(z) = \frac{1}{\pi} \int_0^2 B(z, t) \log(t - z)f(t) dt$$

we have, for $\arg z = 0$,

$$\Phi_B(z^+) - \Phi_B(z) = 2i \int_0^z B(z, t)f(t) dt,$$

$$\Phi_B(z^+) + \Phi_B(z) = \frac{2}{\pi} \int_0^2 B(z, t) \log|t - z| f(t) dt.$$

Next write

$$\Psi_A(z) = \frac{1}{\pi} \int_0^2 A(z, t)(z - t)^{-1}f(t) dt$$

and observe that

$$\Psi_A = \Phi'_A - \Phi_A,$$

so that

$$\begin{aligned}\Psi_A(z^+) - \Psi_A(z) &= 2iA(z, z)f(z), \\ \Psi_A(z^+) + \Psi_A(z) &= \frac{2}{\pi} \int_0^2 A(z, t)(z - t)^{-1}f(t) dt.\end{aligned}$$

Finally, note that

$$U(z; f) = \Psi_A + \Phi_B + \int_0^2 C(z, t)f(t) dt,$$

the last integral having the same value as z approaches a point z_0 , z_0 real, $0 < z_0 < 2$, from $\text{Im } z > 0$ or $\text{Im } z < 0$. (2.4) and (2.5) follow. The last statement of the lemma is proved in [2] [pages 84–85].

We shall have need of the inverse of the operator $I(z; f)$. In Case (1) the equation

$$(2.6) \quad I(z; f) = g$$

can be written*

$$f(z) + \int_0^z K(z, t)f(t) dt = \frac{g(z)}{2iA(z, z)} = h(z)$$

where $K(z, t)$ is analytic. Thus, by the usual theory of Volterra equations,

$$(2.7) \quad f(z) \equiv I^{-1}(z; g) = h(z) + \int_0^z \Gamma(z, t)h(t) dt$$

where $\Gamma(z, t)$ is analytic in z and t . In Case (2) one differentiation of (2.6) with respect to z yields

$$2iB(z, z)f(z) + 2i \int_0^z \frac{\partial B}{\partial z}(z, t)f(t) dt = g'(z),$$

and hence

$$(2.8) \quad f(z) \equiv I^{-1}(z; g) = h^*(z) + \int_0^z \Gamma^*(z, t)h^*(t) dt$$

where $h^*(z) = g'(z)/2i B(z, z)$ and $\Gamma^*(z, t)$ is again analytic.

3. Asymptotic developments. The power products $z^m(z \log z)^n$, m, n integers, may be arranged according to increasing order of vanishing at zero. We call (m, n) the degree of $z^m(z \log z)^n$ and say $(m, n) < (m', n')$ if

$$\lim_{z \rightarrow 0} \frac{z^{m'}(z \log z)^{n'}}{z^m(z \log z)^n} = 0,$$

* One sees here the significance of the restrictions on A and B . For the vanishing of $A(z, z)$ or $B(z, z)$ at a single point can produce singularities in the solution of (2.6) at that point. For example $(1 - x)^{-2}$ is a solution of the equation $(1 - x)f(x) - \int_0^x f(t) dt = 1$.

i.e. $m' + n' > m + n$ or $m' + n' = m + n$, $m > n'$. We say a function $f(z)$ has an estimate of degree (m, n) if

$$f(z) - a_{00} - a_{01}z \log z - a_{10}z - \cdots - a_{mn}z^m(z \log z)^n = o(z^m(z \log z)^n)$$

as $z \rightarrow 0$. If $f(z)$ has estimates of degree (m, n) for all integers (m, n) we say it has an infinite asymptotic estimate and write

$$f(z) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z^m (z \log z)^n.$$

The following result from [5] forms the basis of our computations:

Lemma 2. Suppose that, for positive t , $F(t)$ has an estimate of degree (m, n) . Let $A > 0$ be a constant and $F(t)$ bounded and measurable in $0 < t < A$ and set

$$F_1(z) = \int_0^A F(t) \log(t - x) dt,$$

the logarithm intended as the principal value. Then $F_1(z)$ has an estimate of degree $(m, n + 1)$ on $0 \leq \arg z \leq 2\pi$.

One may ask under what conditions the asymptotic developments for a function actually converge to yield that function. For some functions of the class F this question can be answered, the requirement being that certain functional equations are satisfied. The simplest example is the case of a function $f(z) \in F$ satisfying $f(z^+) - f(z) = 0$. This is merely a statement of the single-valuedness of $f(z)$ and hence, poles being excluded, $f(z)$ is a convergent power series in z , which is of course a special case of the infinite development in z and $z \log z$. Almost as simple is an $f \in F$ satisfying

$$(3.1) \quad f(z^+)e^{2\pi i\theta} - f(z) = 0,$$

$-1 \leq \theta \leq +1$. For consider the function $z^{-\theta}$ which is analytic on R with $(ze^{2\pi i})^{-\theta} = z^{-\theta}e^{-2\pi i\theta}$, so that if $g(z) = z^{-\theta}f(z)$, (3.1) becomes $g(z^+) - g(z) = 0$. Thus if $z^{-\theta}f(z) = O(z^{-1})$ as $z \rightarrow 0$ we find

$$f(z) = z^{\theta} \sum_{m=0}^{\infty} a_m z^m.$$

A deeper result is the following lemma, proved in [5], [6].

Lemma 3. Let $f(z) \in F$ be a solution of

$$(3.2) \quad f'(z^+) - f'(z) + \lambda(f(z^+) + f(z)) = 0, \quad \lambda \text{ constant},$$

which is continuous at $z = 0$. Then

$$(3.3) \quad f(z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z^m (z \log z)^n,$$

the series uniformly and absolutely convergent for $0 \leq |z| \leq \rho < 2$ with any ρ , $0 < \rho < 2$.

We shall need also the following variation of Lemma 3.

Lemma 4. *Let $f(z) \in F$ be a solution of*

$$(3.4) \quad f'(z^+) + f'(z) + \lambda(f(z^+) - f(z)) = 0, \quad \lambda \text{ constant},$$

which is $O(z^{-1+\epsilon'})$, $0 < \epsilon' < 1$, as $|z| \rightarrow 0$ for $|\arg z|$ bounded. Then

$$(3.5) \quad f(z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z^{m-\frac{1}{2}} (z \log z)^n + c, \quad c \text{ constant},$$

the series (after the first term) converging absolutely and uniformly in $0 \leq |z| \leq \rho$, for any ρ , $0 < \rho < 2$.

We observe first that, for some constant c ,

$$(3.6) \quad \lim_{z \rightarrow 0} [(f(z^+) - c) + (f(z) - c)] = 0.$$

For integrating (3.4) from ϵ , with $|\epsilon| > 0$, to z , we find

$$(3.7) \quad f(z^+) + f(z) - (f(\epsilon^+) + f(\epsilon)) = -\lambda \int_{\epsilon}^z (f(t^+) - f(t)) dt.$$

Since $f(z) = O(z^{-1+\epsilon})$ as $z \rightarrow 0$ the right side of (3.7) has a limit as $\epsilon \rightarrow 0$, hence so does the left side. Write

$$\lim_{\epsilon \rightarrow 0} (f(\epsilon^+) + f(\epsilon)) = 2c$$

and observe that the integral on the right side of (3.7) vanishes as $\epsilon \rightarrow 0$, $z \rightarrow 0$. (3.6) then follows from (3.7). Note that $f(z) - c$ is still a solution of (3.4). Set $\tilde{f}(z) = f(z) - c$ so that

$$(3.8) \quad \lim_{z \rightarrow 0} (\tilde{f}(z^+) + \tilde{f}(z)) = 0.$$

Now consider the function $F(z)$ defined by

$$(3.9) \quad F(z) = \int_0^z (z - t)^{\frac{1}{2}} \tilde{f}(t) dt.$$

Since $f(z) \in F$ and is $O(z^{-1+\epsilon})$ as $z \rightarrow 0$ we see that $F(z) \in F$ and $F(z)$ is continuous at $z = 0$. Moreover,

$$F'(z^+) - F'(z) = -\frac{1}{2} \int_0^z (z - t)^{-\frac{1}{2}} (\tilde{f}(t^+) + \tilde{f}(t)) dt$$

or, integrating by parts and using (3.8),

$$\begin{aligned} F'(z^+) - F'(z) &= -\int_0^z (z - t)^{\frac{1}{2}} (\tilde{f}'(t^+) + \tilde{f}'(t)) dt \\ &= \lambda \int_0^z (z - t)^{\frac{1}{2}} (\tilde{f}(t^+) - \tilde{f}(t)) dt = -\lambda(F(z^+) + F(z)). \end{aligned}$$

Thus Lemma 3 is applicable to $F(z)$ and yields

$$(3.10) \quad F(z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} z^m (z \log z)^n,$$

convergent for $0 \leq z \leq \rho < 2$.

The equation (3.9) can be inverted; for multiplying by $(\tau - z)^{\frac{1}{2}}$ and integrating yields

$$\int_0^{\tau} (\tau - z)^{\frac{1}{2}} F(z) dz = \int_0^{\tau} \tilde{f}(t) \left\{ \int_t^{\tau} (\tau - z)^{\frac{1}{2}} (z - t)^{\frac{1}{2}} dz \right\} dt = \frac{\pi}{8} \int_0^{\tau} (\tau - t)^{\frac{3}{2}} \tilde{f}(t) dt$$

or

$$(3.11) \quad \tilde{f}(\tau) = \frac{2}{\pi} \frac{d^2}{d\tau^2} \int_0^{\tau} (\tau - z)^{-\frac{1}{2}} F(z) dz.$$

For all τ with $0 \leq |\tau| \leq \rho$, $0 < \rho < 2$, we can substitute the series (3.10) in (3.11) and integrate termwise. Thus

$$\tilde{f}(\tau) = \frac{2}{\pi} \frac{d^2}{d\tau^2} \left\{ \tau^{\frac{1}{2}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \tau^{m+n} \int_0^1 (1 - \mu)^{-\frac{1}{2}} (\log \mu + \log \tau)^n \mu^{m+n} d\mu \right\}.$$

Since $f(\tau) = 0(\tau^{-1+\epsilon})$ as $\tau \rightarrow 0$, A_{00} and A_{01} must be zero (a fact which follows directly from (3.9)) and the series (3.5) follows.

4. Functional relations. In this section we discuss the analytic continuation of the solution $f(x)$ of (I), (II) or (III) to the Riemann surface R . This we do by rewriting the equations as functional relations on R . For any function $f(z) \in F$ we write

$$\sigma f(z) = f(z^+) + f(z); \quad \Delta f(z) = f(z^+) - f(z).$$

Now let f be a solution of (I) or (II) and form $U(z; f)$. By (2.4) and (2.5) we have, for $\arg z = 0$,

$$(4.1) \quad f(z) = I^{-1}(z; \Delta U),$$

$$(4.2) \quad U(z; f) = \frac{1}{2} \sigma U.$$

Thus (I) and (II) can be rewritten as

$$(A) \quad U(z; f) = 2g(z),$$

$$(B) \quad \Delta U(z; f) + \frac{1}{2} I(z; \sigma U) = I(z; g).$$

If f is a solution of (III) form $U(z; f')$ and note that (III) becomes, on differentiation,

$$(4.3) \quad f'(z) + \bar{U}'(x; f) = g'(x),$$

so that (2.4) and (2.5) yield

$$(C) \quad \sigma U'(z; f') + 2I^{-1}(z; \Delta U) = 2g'(z),$$

while we have also

$$(4.4) \quad f'(z) = I^{-1}(z; \Delta U).$$

The relations (A), (B), (C) all hold initially on $\arg z = 0$. Suppose, however, that $g(z)$ is a single-valued analytic function of z . Then (A), (B), (C) can be used to continue the functions $U(z; f)$ or $U(z; f')$ (defined for $0 \leq \arg z \leq 2\pi$ by the integral (2.2)) as analytic functions to the whole of R . Indeed, (A) yields $U(z^+, f)$ immediately in terms of $U(z^+; f)$ while (B) is a Volterra integral equation, with analytic kernel,* for $U(z^+; f)$ in terms of $U(z; f)$. Finally (C) is a Volterra integro-differential equation which yields $U(z^+; f')$ in terms of $U(z; f)$ except for a constant.

Knowing that $U(z; f)$ in (I) and (II) and $U(z; f')$ in (III) belong to F , we substitute in (4.1), recall that $I^{-1}(z; h)$ involves only multiplication by single-valued analytic functions and integrations and conclude that the function $f(z)$ defined by (4.1) for complex z belongs to F . In the same way we conclude that $f'(z)$ and hence $f(z)$ defined by (4.4) are in F . Thus we have proved

Theorem 1. *If $f(x) \in \bar{F}$ is a solution of (I), (II) or (III) and $g(z)$ is a single-valued analytic function, then $f(z) \in F$, $f(z)$ being obtained by replacing x by $z = x + iy$.*

5. Developments for the solutions. We are now in position to establish the developments outlined in Section 1.

Theorem 2. *Let $f \in \bar{F}$ be a solution of (I). Then*

$$f(z) = \sum_{m=0}^{\infty} a_m z^{m-\frac{1}{2}},$$

the series convergent absolutely and uniformly in $0 \leq |z| \leq \rho$ for any ρ , $0 < \rho < 2$.

Let $U^*(z; f) = U(z; f) - g(z)$ so that (A) becomes

$$(5.1) \quad \sigma U^*(z; f) = 0.$$

This is Eqn. (3.1) with $\theta = -\frac{1}{2}$. By Lemma 1 $U(z; f) = O(z^{-\alpha})$, $\alpha < 1$, hence $z^{\frac{1}{2}}U(z; f) = O(z^{-1+\epsilon})$ and consequently

$$U^*(z; f) = \sum_{m=0}^{\infty} a_m z^{m-\frac{1}{2}}.$$

But then we have

$$\Delta U(z; f) = -2 \sum_{m=0}^{\infty} a_m z^{m-\frac{1}{2}}.$$

Now by (4.1) and (2.7) we have, in Case (1),

$$f(z) = \frac{\Delta U(z; f)}{2iA(z, z)} + \int_0^z \frac{\Gamma(z, t)\Delta U(t; f) dt}{2iA(t, t)},$$

* Recall $A(z, z) \neq \pm i$ in Case (1) and $B(z, z) \neq \pm i$ in Case (2).

and the conclusion of the theorem follows. In Case (2) we have, by (4.1) and (2.8),

$$f(z) = \frac{\Delta U'(z; f)}{2iB(z, z)} + \int_0^z \frac{\Gamma^*(z, t)\Delta U'(t) dt}{2iB(t, t)}.$$

Since we require $f(x)$ to be in $\bar{F}$ we conclude that a_0 must be zero, and the conclusion of the theorem again follows.

Theorem 3. (Case (2).) Let $f \in \bar{F}$ be a solution of (II) with $A(z, z) \equiv 0$, $B(z, z) \neq 0$, $B(z, z) \neq \pm i$ for $0 \leq |z| \leq 2$. Then

$$f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0.$$

Again set $U^* = U - g$; then (B) becomes

$$(5.2) \quad \Delta U^* = -\frac{1}{2}I(z; \sigma U^*).$$

Eqn. (5.2) does not characterize U^* so simply as did Eqn. (4.1) but is readily adapted to an iteration procedure based on Lemma 2. Define an analytic function $\psi(z)$ by setting

$$\psi(z) = \frac{1}{4\pi i} \int_0^2 \frac{I(t; \sigma U^*)}{t - z} dt$$

for $0 \leq \arg z \leq 2\pi$ and continuing $\psi(z)$ analytically to all values of $\arg z$ by the relation

$$\psi(z^+) - \psi(z) = -\frac{1}{2}I(z; \sigma U^*),$$

which holds initially on $\arg z = 0$. We see that the difference $U^* - \psi$ is single-valued and we estimate this difference near $z = 0$. By Lemma 1, $U^*(z; f) = O(z^{-\alpha})$. On the other hand, since $f(x)$ is a solution of (II) (with $A \equiv 0$) and $f = O(x^{-\alpha})$ as $x \rightarrow 0$, we conclude easily that $f(x)$ is continuous at $x = 0$, i.e. has an asymptotic estimate of degree $(0, 0)$. But, for $A \equiv 0$,

$$I(z; \sigma U^*) = 2i \int_0^z B(z, t)\sigma U^*(t) dt.$$

Hence $I(0; \sigma U^*)$ is zero and $I'(z; \sigma U)$ is $O(z^{-\alpha})$ at $z = 0$. The expression for $\psi(z)$ may accordingly be integrated by parts to give

$$(5.3) \quad \psi(z) = -\frac{1}{4\pi i} \int_0^2 I'(t; \sigma U^*) \log(t - z) dt + \frac{1}{4\pi i} I(2; \sigma U^*) \log(2 - z),$$

from which follows the continuity, at $z = 0$, of $\psi(z)$. Thus $U^*(z; f) - \psi(z)$ is a single-valued function which is $o(z^{-1})$ near $z = 0$, i.e. is a power series. Hence, by (5.3), U^* is continuous at $z = 0$ and

$$(5.4) \quad U^*(z) = -\frac{1}{4\pi i} \int_0^2 I'(t; \sigma U^*) \log(t - z) dt + P(z)$$

where $P(z)$ is a power series. Also $I'(t, \sigma U^*)$ is continuous at $z = 0$.

We can now begin an iteration process based on Lemma 2. The continuity of $I'(t; \sigma U^*)$ at $t = 0$ implies by Lemma 2 that the integral on right side of (5.4) has an estimate of degree $(0, 1)$. But the power series $P(z)$ has estimates of any degree; thus by Lemma 2 U has an estimate of degree $(0, 1)$, *i.e.*

$$U^* = a_{00}^* + a_{01}^* z \log z + o(z \log z).$$

Then σU^* has an estimate of degree $(0, 1)$ and, as one readily checks, so also does $I'(z; \sigma U^*)$. Reentering in (5.4), we see that the integral and hence U^* has an estimate of degree $(0, 2)$ by Lemma 2. This process may be repeated indefinitely, it being observed that an estimate of degree $(0, n)$ for U^* implies an estimate of degree $(0, n)$ for $I'(t; \sigma U^*)$ and hence, by (5.4) and Lemma 2, an estimate of degree $(0, n+1)$ for U^* . U^* thus has an estimate of degree $(0, n)$ for arbitrary n , which means

$$U^* \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} z^m (z \log z)^n.$$

Since $\Delta U = \Delta U^*$ it follows from (2.8) and (4.1) that

$$(5.5) \quad f(z) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z^m (z \log z)^n.$$

Theorem 4. (Case (1).) Let $f \in \bar{F}$ be a solution of (II) with $A(z, z) \neq 0$, $A(z, z) \neq \pm i$ for $0 \leq z \leq 2$. Let

$$a = \frac{1}{2\pi i} \log \left\{ \frac{1 - iA(0, 0)}{1 + iA(0, 0)} \right\},$$

choosing the branch of the logarithm so that $-\frac{1}{2} < a < +\frac{1}{2}$. Then

$$\begin{aligned} x^{-a} f(x) &\sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0 \quad \text{if } a \leq 0, \\ x^{1-a} f(x) &\sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0 \quad \text{if } a > 0. \end{aligned}$$

With $U^* = U - g$, Eqn. (5.2) still holds but now reads

$$(5.6) \quad \begin{aligned} (1 + iA(z, z))U^*(z^+) - (1 - iA(z, z))U^*(z) \\ = -i \int_0^+ B(z, t) \sigma U^* dt \equiv \tilde{I}(z; \sigma U^*). \end{aligned}$$

We can reduce Eqn. (5.6) to the calculation of Theorem 3 in the same manner we reduced Eqn. (3.1) to $\Delta f = 0$. Consider first the function $e^{\theta(z)}$ where

$$\theta(z) = \frac{1}{2\pi i} \log \left\{ \frac{1 - iA(z, z)}{1 + iA(z, z)} \right\} \log z,$$

with the branch of logarithm chosen so that

$$-\frac{\pi}{2} < -i \log \frac{1 - iA(0, 0)}{1 + iA(0, 0)} < +\frac{\pi}{2}.$$

Since $A(z, z) \neq \pm i$,

$$\log \frac{1 - iA(z, z)}{1 + iA(z, z)}$$

is single-valued and we have

$$(5.7) \quad \begin{aligned} \theta(z^+) - \theta(z) &= \log \frac{1 - iA(z, z)}{1 + iA(z, z)}, \\ e^{\theta(z)} &= \frac{1 + iA(z, z)}{1 - iA(z, z)} e^{\theta(z^+)}. \end{aligned}$$

Now

$$\theta(z) = a \log z + \sum_{n=1}^{\infty} a_n z^n \log z, \quad a = \frac{1}{2\pi i} \log \frac{1 - iA(0, 0)}{1 + iA(0, 0)}$$

and

$$e^{\theta(z)} = z^a e^{\zeta(z)}, \quad \zeta(z) = \sum_{n=1}^{\infty} a_n z^n \log z.$$

It follows therefore that

$$(5.8) \quad e^{\theta(z)} = z^a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e_{mn} z^m (z \log z)^n.$$

Observe also that $-\frac{1}{2} < a < +\frac{1}{2}$.

We return now to (5.6). If we divide (5.5) by $(1 - iA)e^{\theta(z)}$ we can write, by (5.7),

$$(5.9) \quad \Delta[e^{-\theta(z)} U^*(z)] = e^{-\theta(z)} \tilde{I}(z; \sigma U^*).$$

Consider, as before, the function $\psi(z)$ defined for $0 \leq \arg z \leq 2\pi$ by

$$\psi(z) = \frac{1}{2\pi i} \int_0^2 e^{-\theta(it)} \tilde{I}(t; \sigma U^*)(z - t)^{-1} dt$$

and observe by (5.9) that $e^{-\theta(z)} U^*(z) - \psi(z)$ is single-valued. We have now essentially returned to the situation in Theorem 3, except for the fact that when $U = O(z^{-\alpha})$ and a is positive $e^{-\theta(z)} U^*(z)$ may have a pole at $z = 0$.

Suppose first $a \leq 0$; then, by (5.8) and the estimate $U = O(z^{-\alpha})$, we conclude that $e^{-\theta(z)} U^*(z) = o(z^{-1})$ near $z = 0$. Also

$$e^{-\theta(z)} \tilde{I}(z; \sigma U^*) = O(z^{1-\alpha-a}).$$

It follows (see again [2]) that ψ is continuous at $z = 0$. Thus

$$e^{-\theta(z)} U^*(z) - \psi(z) = \text{power series in } z.$$

With $\psi(z)$ continuous it follows that $e^{-\theta(z)}\tilde{I}(z; \sigma U^*)$ has a continuous derivative at $z = 0$ and, again integrating by parts,

$$(5.10) \quad e^{-\theta(z)}U^*(z) = \frac{1}{2\pi i} \int_0^2 e^{-\theta(t)}\tilde{I}(t; \sigma U^*)' \log(t-z) dt + P(z).$$

(5.10) can clearly be used just as was (5.4) for an iteration procedure based on Lemma 2. One finds

$$e^{-\theta(z)}U^*(z) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}^* z^m (z \log z)^n$$

or, recalling that $U^* = U - g$ and using (5.8) once more,

$$z^{-a}\Delta U(z; f) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} z^m (z \log z)^n.$$

Entering this result in (4.1) yields

$$z^{-a}f(z) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z^m (z \log z)^n.$$

The case $a > 0$ requires a slight modification for then $e^{-\theta(z)}U^*(z)$, being $O(z^{-a-\alpha})$, may possess a pole of first order at $z = 0$. In this case we replace (5.9) by

$$(5.11) \quad \Delta(ze^{-\theta(t)}U^*(z)) = ze^{-\theta(z)}\tilde{I}(z; \sigma U^*).$$

Set

$$\psi(z) = \frac{1}{2\pi i} \int_0^2 te^{-\theta(t)}\tilde{I}(t; \sigma U^*)(z-t)^{-1} dt,$$

observe that both $ze^{-\theta(z)}U^*(z)$ and $\psi(z)$ are $o(z^{-1})$ and conclude

$$(5.12) \quad ze^{-\theta(z)}U^*(z) = \psi(z) + P(z), \quad P \text{ a power series.}$$

The remainder of the computation parallels $a \leq 0$. For $U^* = O(z^{-\alpha})$ implies

$$te^{-\theta(t)}\tilde{I}(t; \sigma U^*) = O(t^{2-a-\alpha}),$$

whence [again by [2]] $\psi(z)$ is continuous. Thus, by (5.12),

$$U^*(z) = cz^{a-1} + o(z^{a-1})$$

and, accordingly, $t e^{-\theta(t)}\tilde{I}(t; \sigma U^*)$ has a continuous derivative at $t = 0$. (5.12) can then be integrated by parts and yields

$$ze^{-\theta(z)}U^*(z) = \frac{1}{2\pi i} \int_0^2 [te^{-\theta(t)}\tilde{I}(t; \sigma U^*)]' \log(t-z) dt + \text{power series.}$$

Lemma 2 now applies to yield

$$ze^{-\theta(z)}U^*(z) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}^* z^m (z \log z)^n$$

or

$$z^{1-a} \Delta U \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} z^m (z \log z)^n.$$

Thus, from (4.1) and (2.7),

$$z^{1-a} f(z) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z^m (z \log z)^n.$$

Theorem 5. Let f & $\bar{F}$ be a solution of (III) with $A(z, z) \neq 0$, $A(z, z) \neq \pm i$ for $0 \leq |z| \leq 2$, then

$$z^{\frac{1}{2}} f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m (x \log x)^n \quad \text{as } x \rightarrow 0.$$

The proof is identical with that of Theorem 3 except for one minor modification. Again set $U^* = U - g$ so that (C) becomes

$$(5.13) \quad \sigma U^{*'} + 2I^{-1}(z; \Delta U^*) = 0.$$

From $U = O(z^{-a})$ we conclude $I^{-1}(z; \Delta U^*)$ and hence $\sigma U^{*'}$ are $O(z^{-a})$. Thus $\sigma U^* = O(z^{1-a})$ and (5.12) can be integrated to yield

$$(5.14) \quad \sigma U^* = -M(z; \Delta U^*), \quad M(z; \Delta U^*) = 2 \int_0^z I^{-1}(t; \Delta U^*) dt.$$

Multiplication of (5.13) by $z^{\frac{1}{2}}$ yields

$$\Delta(z^{\frac{1}{2}} U^*) = z^{\frac{1}{2}} M(z; \Delta U^*).$$

As before, we conclude

$$z^{\frac{1}{2}} U^* = \frac{1}{2\pi i} \int_0^2 (t^{\frac{1}{2}} M(t; \Delta U^*))' \log(t - z) dt + P(z).$$

Again iteration can be performed, using this equation and Lemma 2, to show

$$z^{\frac{1}{2}} U(z; f) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}^* z^m (z \log z)^n.$$

Then (4.4) yields

$$z^{\frac{1}{2}} f'(z) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a'_{mn} z^m (z \log z)^n$$

or

$$z^{-\frac{1}{2}} f(z) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z^m (z \log z)^n.$$

If one substitutes the asymptotic series for U into equations (5.1), (5.9) or (5.14) and then equates coefficients of terms of like degree, it is clear that certain relations can be found among the coefficients. Since the functions $g(z)$ do not

appear in these equations we see that these relations must be independent of the choice of g . We shall return to these relations in Section 7.

6. Special cases. We examine the particular equations (1.2), (1.3) which illustrate types (II), (III) respectively. For (1.2) $A = 0$, $B \equiv 1$, $C = 0$; hence Eqn. (B) becomes

$$\Delta U(z; f) + i \int_0^z U(t) dt = i \int_0^z g(t) dt$$

or, setting $U^* = U - g$,

$$\Delta U^{*'} + i\sigma U^* = 0.$$

This is Eqn. (3.2) of Lemma 4; hence in this case we infer the convergence of the asymptotic development. For (1.3) $A \equiv 1$, $B = 0$, $C = 0$. In this case we have

$$I^{-1}(z; \Delta U) = -\frac{1}{2}i\Delta U$$

and, with $U^* = U - g$, (C) becomes

$$\sigma U^{*'} - i\Delta U^* = 0.$$

This is Eqn. (3.4) of Lemma 3, and we can again infer the convergence of the asymptotic series.

7. Numerical integration. In this section we make some remarks concerning the mechanics of numerical solutions of Eqns. (I), (II) and (III). We restrict ourselves to type (II) with

$$a = \frac{1}{2\pi i} \log \left\{ \frac{1 - iA(0, 0)}{1 + iA(0, 0)} \right\} \neq 0.$$

Let us approximate the solution of (II) by a finite sum

$$(7.1) \quad f(x) = \begin{cases} \sum_{r=0}^N \sum_{s=0}^{N-r} a_{rs} x^{a+r+s} (\log x)^s, & 0 \leq x \leq 1, \\ \sum_{r=0}^N \sum_{s=0}^{N-r} b_{rs} (2-x)^{a+r+s} [\log(2-x)]^s, & 1 \leq x \leq 2, \end{cases}$$

i.e., in the terminology of Section 3, we assume an approximation of degree $(N, 0)$ for $f(x)$. Suppose $0 \leq x < 1$, then clearly

$$(7.2) \quad \int_0^2 C(x, t) f(t) dt + \int_1^2 A(x, t) |x - t|^{-1} \\ + B(x, t) \log |x - t| f(t) dt = \sum_{m=0}^{\infty} C_m x^m$$

where

$$C_m = \sum_{r=0}^N \sum_{s=0}^{N-r} (C_{mrs} a_{rs} + D_{mrs} b_{rs}).$$

We consider next

$$W(A; f) = \frac{1}{\pi} \int_0^1 A(x, t) \log |x - t| f(t) dt.$$

Setting

$$A(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} x^m t^n,$$

we have

$$(7.3) \quad W(A; f) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^N \sum_{s=0}^{N-r} A_{mn} a_{rs} x^m \int_0^1 t^{a+r+s} (\log t)^s \log |x - t| dt.$$

To evaluate the integrals in (7.3) we use the series

$$(7.4) \quad \log |x - t| = \begin{cases} \log x - \sum_{e=1}^{\infty} \frac{1}{e} x^{-e} t^e, & 0 \leq t < x, \\ \log t - \sum_{e=1}^{\infty} \frac{1}{e} x^e t^{-e}, & x < t \leq 1, \end{cases}$$

integrate termwise (a procedure which can be justified by first performing an integration by parts) and employ

$$(7.5) \quad \begin{aligned} \int_0^x t^{a+p+q} (\log t)^q dt &= x^{a+p+q+1} \sum_{k=0}^q [k, p, q] (\log x)^{q-k}, \\ \int_x^1 t^{a+p+q} (\log t)^q dt &= -x^{a+p+q+1} \sum_{k=0}^q [k, p, q] (\log x)^{q-k} - [q, p, q], \\ [k, p, q] &= \frac{(-1)^k q!}{(q-k)!(a+p+q+1)^{k+1}}. \end{aligned}$$

(7.3) then yields

$$(7.6) \quad W(A; f) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} B_{ij}(A) x^{a+i+j+1} (\log x)^j + \sum_{m=0}^{\infty} B_m(A) x^m.$$

By differentiation follows

$$(7.7) \quad W'(A; f) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_{ij}(A) x^{a+i+j} (\log x)^j + \sum_{m=0}^{\infty} A_m(A) x^m.$$

Here

$$B_m(A) = \sum_{r=0}^n \sum_{s=0}^{n-r} B_{mrs} a_{rs}, \quad A_m(A) = \sum_{r=0}^n \sum_{s=0}^{n-r} A_{mrs} a_{rs},$$

while the B_{ij} and A_{ij} are finite linear combinations of the a_{rs} . We show in the appendix, in fact, that, for $i \geq 1$,

$$(7.8) \quad B_{ij}(A) = \beta a_{i-1,i} + \cdots, \quad A_{ij}(A) = -\pi a_{ij} + \alpha a_{i-1,i+1} + \cdots,$$

the dots indicating terms in a_{rs} with degree $(r, s) < (i-1, j)$ or degree $(r, s) < (i-1, j+1)$, respectively, and with $\alpha \neq 0$.

Consider now the left side of Eqn. (II), which by (7.2), (7.6), (7.7) assumes the form

$$(7.9) \quad \begin{aligned} f + W'(A; f) - W\left(\frac{\partial A}{\partial x}; f\right) + W(B) + \sum_{m=0}^{\infty} c_m x^m \\ = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left\{ a_{ij} + A_{ij}(A) - B_{ij}\left(\frac{\partial A}{\partial z}\right) + B_{ij}(B) \right\} x^{a+i+j} (\log x)^i \\ + \sum_{m=0}^{\infty} \left\{ A_m(A) - B_m\left(\frac{\partial A}{\partial z}\right) + B_m(B) + c_m \right\} x^m \text{ on } 0 \leq x < 1. \end{aligned}$$

We can determine the a_{ij} so that

$$(7.10) \quad a_{ij} + A_{ij}(A) - B_{ij}\left(\frac{A}{z}\right) + B_{ij}(B) = 0 \quad \text{for degree } (i, j) \leq (n, 0).$$

For, α being unequal to 0, Eqn. (7.8) shows that (7.10) yields $a_{i-1,i+1}$ for $i \geq 1$ in terms of a_{rs} with degree $(r, s) < \text{degree } (i-1, j+1)$, the constants a_{i0} remaining undetermined. Thus the conditions (7.10) determine a_{ij} , for $j \geq 1$, recursively by relations which are independent of $g(x)$.

It is apparent that consideration of the range $1 < x \leq 2$ will yield an equation of the form

$$(7.11) \quad \begin{aligned} f + \bar{U}(x; f) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left\{ b_{ij} + \tilde{A}_{ij}(A) - \tilde{B}_{ij}\left(\frac{\partial A}{\partial z}\right) \right. \\ \left. + \tilde{B}_{ij}(B) \right\} (2-x)^{a+i+j} (\log 2-x)^i \\ + \sum_{m=0}^{\infty} \left\{ \tilde{A}_m(A) - \tilde{B}_m\left(\frac{A}{z}\right) + \tilde{B}_m(B) + \tilde{C}_m \right\} (2-x)^m \end{aligned}$$

in which A_{ij} , B_{ij} are linear combinations of the b_{rs} , and A_m , B_m , C_m are linear combinations of the a_{rs} and b_{rs} . Just as in Eqns. (7.10), we can make all terms of degree $\leq (N, 0)$ on the right side of (7.11) vanish by choosing the b_{ij} recursively in terms of b_{i0} .

It remains to choose the $2N+2$ constants a_{i0} , b_{i0} , $i = 0, \dots, N$. Suppose

$$g(x) = \begin{cases} \sum_{m=0}^{\infty} g_m x^m, & 0 \leq x \leq 1, \\ \sum_{m=0}^{\infty} h_m (2-x)^m, & 1 \leq x \leq 2. \end{cases}$$

Then we set

$$(7.12) \quad \begin{aligned} A_m(A) - B_m\left(\frac{\partial A}{\partial z}\right) + B_m(B) + C_m &= g_m, \\ A_m(A) - B_m\left(\frac{\partial A}{\partial z}\right) + B_m(B) + C_m &= h_m, \end{aligned} \quad m = 0, 1, \dots, N,$$

a system of $(2N + 2)$ linear equations in the a_{i0} , b_{i0} .

Now let f_0 denote the finite sum (7.1) with the a_{ij} , b_{ij} chosen so as to satisfy (7.12) and the above described recursive relations. Then, by (7.9),

$$\begin{aligned} f_0(x) + \bar{U}(x; f_0) - g(x) &= \sum_{(i,j) > (N,0)} \sum K_{ij} x^{a+i+j} (\log x)^j \\ &\quad + \sum_{m=n+1} (K_m - g_m) x^m \quad \text{on } 0 \leq x < 1, \end{aligned}$$

the K_{ij} and K_m being linear combinations of the a_{ij} , b_{ij} . Thus f_0 "satisfies" Eqn. (II) up to terms of order x^{N+1} on $0 \leq x < 1$. Similarly (7.11) shows that f_0 satisfies (II) up to terms of order $(2-x)^{N+1}$ on $1 < x \leq 2$. Thus an approximation of only a few terms might be expected to yield fairly accurate results near $x = 0$ and $x = 2$.

The numerical method outlined in this paper was used by the author in [11] in obtaining an approximate solution of Eqn. (1.2) and some further numerical details may be found there.

APPENDIX

We wish to verify here some statements made in the preceding section. Using (7.5) and (7.6), Eqn. (7.3) becomes

$$\begin{aligned} (A.1) \quad W(B) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^N \sum_{s=0}^{N-r} \beta_{mn} a_{rs} x^{a+r+s+m+n+1} \left\{ \sum_{k=1}^s ([k, r+n, s] \right. \\ &\quad - [k, r+n-1, s+1])(\log x)^{s+1-k} - [s+1, r+n-1, s+1] \\ &\quad \left. + \sum_{e=1}^{\infty} \frac{1}{e} \sum_{k=0}^s ([k, r+n-e, s] - [k, r+n+e, s](\log x)^{s-k}) \right\} \\ &\quad + \sum_{m=0}^{\infty} B_m x^m = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} B_{ij} x^{a+i+j+1} (\log x)^i + \sum_{m=0}^{\infty} B_m x^m. \end{aligned}$$

Thus a_{rs} appears in $B_{ij}(B)$ only if $r+s+m+n+1 = i+j$ and either $s+1-k = j$ with $k \geq 1$ or $s-k = j$ with $k \geq 0$. It follows that we must have $r \leq i-1$, $m+n \leq i+j$ and $s = i+j-1-m-n-r$, so that the largest degree (r, s) which can appear in B_{ij} is $(i-1, j)$.

Differentiation of (A.1) yields, for $i \geq 1$,

$$\begin{aligned}
 A_{ii}(B) = & B(0, 0)(a + i + j + 1) \left\{ [1, i, j] - [1, i - 1, j + 1] \right. \\
 & + \sum_{e=1}^{\infty} \frac{1}{e} ([0, i - e, j] - [0, i + e, j]) \left. \right\} a_{ii} \\
 & + B(0, 0) \left\{ (a + i + j + 1)([2, i - 1, j + 1] - [1, i - 2, j + 2]) \right. \\
 (A.2) \quad & + \sum_{e=1}^{\infty} \frac{1}{e} ([1, i - e - 1, j + 1] - [1, i + e - 1, j + 1]) \\
 & + (j + 1)([1, i - 1, j + 1] - [1, i - 2, j + 2]) \\
 & + \sum_{e=1}^{\infty} \frac{1}{e} ([0, i - 1 - e, j + 1] \\
 & \left. - [0, i - 1 - e, j + 1]) \right\} a_{i-1, j+1} + \cdots,
 \end{aligned}$$

the dots indicating terms in a_{rs} with degree $(r, s) < (i - 1, j + 1)$.

Now we can use the formula

$$\frac{\pi \operatorname{ctn} \pi z}{z} = \frac{1}{z^2} + \sum_{e=1}^{\infty} \frac{1}{e} \left[\frac{1}{z - e} - \frac{1}{z + e} \right]$$

and its derivative

$$\frac{-\pi \operatorname{ctn} \pi z}{z^2} - \frac{\pi^2 \operatorname{csc}^2 \pi z}{z} = \sum_{e=1}^{\infty} \frac{1}{e} \left[\frac{1}{(z + e)^2} - \frac{1}{(z - e)^2} \right] - \frac{2}{z^3}$$

to evaluate the various coefficients in (A.2). We find

$$\begin{aligned}
 A_{ii}(B) = & \pi B(0, 0) \operatorname{ctn} \pi a a_{ii} - \pi B(0, 0)(j + 1) \\
 (A.3) \quad & \cdot \left\{ \frac{\operatorname{ctn} \pi a}{a + i + j + 1} + \pi \operatorname{csc}^2 \pi a - \frac{\operatorname{ctn} \pi a}{a + i + j + 1} \right\} a_{i-1, j+1} + \cdots.
 \end{aligned}$$

Now, recalling that

$$2\pi i a = \log \left\{ \frac{1 - iA(0, 0)}{1 + iA(0, 0)} \right\},$$

one finds readily that $A(0, 0) = -\tan \pi a$ and (A.3) becomes

$$(A.4) \quad A_{ii}(B) = -\pi a_{ii} + \frac{2\pi^2(j + 1)}{\sin 2\pi a} a_{i-1, j+1} + \cdots.$$

This is formula (7.9), it being noted that for $-\frac{1}{2} < a < \frac{1}{2}$, $a \neq 0$ the coefficient of $a_{i-1, j+1}$ is never zero.

This work was supported by Mathematics Division, Air Force Office of Scientific Research, Contract No. AF 18(600)-1138.

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Carnegie Institute of Technology
Pittsburgh, Pennsylvania