*Basic Principles of the Unified Field Theory of the Second Kind**

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GENERAL OUTLINE

The connection Γ_{λ}^{μ} of the Einstein unified field theory in a four-dimensional space-time X_4 is defined as follows†

$$(A) \quad \partial_{\omega} g_{\lambda\mu} - \Gamma_{\lambda}^{\alpha}{}_{\omega} g_{\alpha\mu} - \Gamma_{\omega}^{\alpha}{}_{\mu} g_{\lambda\alpha} = 0, \quad \left(\partial_{\omega} \stackrel{\text{def}}{=} \frac{\partial}{\partial x^{\omega}} \right).$$

Here $g_{\lambda\mu}$ is a general real tensor. This connection and the unified field theory have been dealt with in a series of papers,‡ as well as in the book mentioned in the previous footnote. The solution of (A) exists and is unique if

$$(Ba) \quad g(g - 2\mathfrak{h}) \neq 0 \quad \text{for} \quad \mathfrak{h} < 0.$$

Here g ($\mathfrak{h}$) is the determinant of $g_{\lambda\mu}$ (of $h_{\lambda\mu} \stackrel{\text{def}}{=} g_{(\lambda\mu)}$). We also investigated the case

$$(Bb) \quad g(g - 2\mathfrak{h}) = 0$$

and found necessary and sufficient conditions for the existence of at least one solution of (A). For all cases in which conditions (Bb) were satisfied we displayed all sets of solutions in explicit form.

In this paper we shall deal with another degenerate case, namely

$$(C) \quad \mathfrak{h} = 0, \quad g \neq 0.$$

In this case the procedure for solving (A) needs quite different tools and requires another approach.

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†EINSTEIN, A., *The meaning of Relativity*, Princeton Univ. Press, 1950.

‡For the bibliography see HLAVATÝ, V. *Geometry of Einstein's unified theory* (Noordhoff, Gröningen; in print).

This paper is divided into two* parts. The first part contains two chapters. The second part (to be published) contains three chapters. The first part exposes the necessary prerequisites of the tensorial algebra and analysis. The second part deals with the solution $\Gamma_{\lambda}^{\nu}{}_{\mu}$ of (A).

In the first chapter we investigate the algebraic structure imposed on X_4 by the tensor $g_{\lambda\mu}$ which satisfies the condition (C). In the second chapter we shall find an auxiliary connection $\overset{0}{\Delta}_{\lambda}^{\nu}{}_{\mu}$ defined in terms of the given data.

The second part starts with the third chapter. In it we first use $\overset{0}{\Delta}_{\lambda}^{\nu}{}_{\mu}$ in order to get necessary information about the connection $\Gamma_{\lambda}^{\nu}{}_{\mu}$. The requirement of the existence of $\Gamma_{\lambda}^{\nu}{}_{\mu}$ satisfying (A) imposes some conditions on the structure of X_4 . These conditions are studied in the fourth chapter. Finally, the fifth chapter displays the explicit form of the solution of (A).

Greek indices denote components with respect to an arbitrary holonomic coordinate system. The range of these indices is I, II, III, IV.

Roman indices denote components with respect to a non-holonomic coordinate system (non-holonomic vector frame). Their range is in general 1, 2, 3, 4. However, there is an exception to this rule. The indices

$$(D) \quad x, y, \quad x^* \neq x, \quad y^* \neq y$$

run from 1 to 2 while the indices

$$(E) \quad z, t, \quad z^* \neq z, \quad t^* \neq t$$

run from 3 to 4.

Occasionally we shall also use the indices u, v , whose range will be explicitly stated whenever necessary.

With the exception of indices sub (D) and (E), all indices, whether Greek or roman, follow the usual summation convention. *The summation for indices sub (D) and (E) will always be indicated by the symbol $\sum$.*

*Two points of interest are worthwhile mentioning, namely, the local quadratic cone and the local transformation group.

There are two different kinds of non-degenerate quadratic cones in a (linear) four-dimensional space. A cone of the *first* kind has only one singular point (*i.e.* the vertex). A cone of the *second* kind possesses a line of singular points, namely the vertex-line. *Local considerations of relativity have been based on a cone of the first kind.* This corresponds to the assumption $\mathfrak{h} \neq 0$. In this paper we shall deal with the case (C) assuming that the rank of $\mathfrak{h}$ is equal to three. In other words, our considerations are based on a cone of the second kind. Its vertex-line is the (local) preferred time-axis. This fact is connected with the structure of the local transformation group.

It is well known that the local transformation group of general relativity is a Lorentz group. In the unified theory based on the cone of the second kind the corresponding local group is a Galilean group (Eq. (6.2b,d)) although the space-time is still curved. Therefore this theory may be looked upon as a link between the classical mechanics and the general (unified) theory of relativity.

CHAPTER I. ALGEBRA OF $g_{\lambda\mu}$

1. The Tensors $g_{\lambda\mu}$, $E^{\omega\mu\lambda\nu}$ and $e_{\omega\mu\lambda\nu}$. The basic real tensor of X_4 will be denoted by $g_{\lambda\mu}$. It may be decomposed into its symmetric part $h_{\lambda\mu}$ and its skew symmetric part $k_{\lambda\mu}$:

$$(1.1) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

$$h_{\lambda\mu} \stackrel{\text{def}}{=} g_{(\lambda\mu)}, \quad k_{\lambda\mu} = g_{[\lambda\mu]}.$$

Agreement (1.1). Throughout this paper we assume that

(1) the rank of $\mathfrak{g} \stackrel{\text{def}}{=} \text{Det}((g_{\lambda\mu}))$ is 4,

$$(1.2a) \quad \mathfrak{g} \neq 0,$$

(2) the rank of $\mathfrak{h} \stackrel{\text{def}}{=} \text{Det}((h_{\lambda\mu}))$ is 3,

$$(1.2b) \quad \mathfrak{h} = 0,$$

and the signature of $h_{\lambda\mu}$ is $(+++)$,

(3) the rank of $\mathfrak{k} \stackrel{\text{def}}{=} \text{Det}((k_{\lambda\mu}))$ is either 4 or 2^* ,

$$(1.3) \quad \mathfrak{k} \geq 0.$$

Definition (1.1). The tensor $g_{\lambda\mu}$ will be said to be of the $\left. \begin{matrix} \text{first} \\ \text{second} \end{matrix} \right\}$ class if

$$\left\{ \begin{array}{l} \mathfrak{k} \neq 0 \\ \mathfrak{k} = 0 \end{array} \right.$$

Note. If not explicitly stated otherwise, all subsequent results will hold for both classes.

Theorem (1.1). There is only one vector, say, a^ν , known up to an arbitrary non-zero multiplicative factor, satisfying the condition

$$(1.4) \quad a^\lambda h_{\lambda\mu} = 0 \quad (= a^\lambda h_{\mu\lambda}).$$

The arbitrary multiplicative factor may always be chosen in such a way that a^ν is real.

The proof follows at once from the requirement (2) of Agreement (1.1).

Denote by $\mathfrak{E}^{\omega\mu\lambda\nu}$ ($e_{\omega\mu\lambda\nu}$) the contravariant (the covariant) indicator, whose components are $+1$, -1 , 0 in all coordinate systems.† Moreover, denote by $\epsilon \neq 0$ an arbitrary density of weight ± 1 and put

$$(1.5) \quad \sigma \stackrel{\text{def}}{=} \frac{\epsilon}{|\epsilon|}.$$

* $\mathfrak{k}$, as a determinant of a skew symmetric tensor $k_{\lambda\mu} \neq 0$ in X_4 , can be either of rank 4 or of rank 2. In the first case $\mathfrak{k}$ is a square, $\mathfrak{k} > 0$.

† $\mathfrak{E}^{\omega\mu\lambda\nu}$ ($e_{\omega\mu\lambda\nu}$) is skew symmetric in all indices. It is a tensor density of weight 1 (-1).

Theorem (1.2). $E^{\omega\mu\lambda\nu}$ and $e_{\omega\mu\lambda\nu}$, defined by

$$(1.6a) \quad E^{\omega\mu\lambda\nu} \stackrel{\text{def}}{=} \sigma \mathfrak{E}^{\omega\mu\lambda\nu} / \sqrt{|\mathfrak{g}|},$$

$$(1.6b) \quad e_{\omega\mu\lambda\nu} \stackrel{\text{def}}{=} \sigma \mathfrak{e}_{\omega\mu\lambda\nu} \sqrt{|\mathfrak{g}|},$$

are components of tensors which satisfy the following conditions:

$$(1.7) \quad \begin{aligned} (a) \quad & E^{\alpha\beta\gamma\delta} e_{\alpha\beta\gamma\delta} = 4! \\ (b) \quad & E^{\alpha\beta\gamma\nu} e_{\alpha\beta\gamma\lambda} = 3! \delta_\lambda^\nu \\ (c) \quad & E^{\alpha\beta\omega\mu} e_{\alpha\beta\lambda\nu} = 4 \delta_{[\lambda\nu]}^{\omega\mu} \\ (d) \quad & E^{\alpha\mu\lambda\nu} e_{\alpha\xi\eta\xi} = 3! \delta_{[\xi\eta\xi]}^{\mu\lambda\nu} . \end{aligned}$$

Proof. The tensorial character of $E^{\omega\mu\lambda\nu}$ and $e_{\omega\mu\lambda\nu}$ is obvious. An easy inspection shows that (1.7) hold if we replace E and e by $\mathfrak{E}$ and $\mathfrak{e}$. Therefore, according to (1.6), they hold also for E , e .

We shall use $\frac{1}{2}E$ ($\frac{1}{2}e$) for raising (lowering) *both* indices of a covariant (contravariant) skew symmetric tensor. This operation will be symbolized by a dot preceding the kernel letter in parentheses. Thus, for instance, starting with $k_{\lambda\mu}$ we obtain

$$(1.8a) \quad (\cdot k)^{\omega\nu} \stackrel{\text{def}}{=} \frac{1}{2} E^{\omega\nu\lambda\mu} k_{\lambda\mu} .$$

If a contravariant skew symmetric tensor $m^{\lambda\nu}$ is given we define $(\cdot m)_{\omega\nu}$ by

$$(1.8b) \quad (\cdot m)_{\omega\nu} \stackrel{\text{def}}{=} \frac{1}{2} e_{\omega\nu\lambda\mu} m^{\lambda\mu} .$$

Using (1.7c) we easily obtain

$$(1.8c) \quad (\cdot k)_{\lambda\nu} = \frac{1}{2} e_{\lambda\nu\alpha\beta} (\cdot k)^{\alpha\beta} = \frac{1}{4} e_{\lambda\nu\alpha\beta} E^{\alpha\beta\omega\mu} k_{\omega\mu} = k_{\lambda\nu}$$

and

$$(1.8d) \quad (\cdot m)^{\lambda\nu} = m^{\lambda\nu} .$$

The tensor $h_{\lambda\mu}$ will be used for *lowering* of indices in the usual manner. This operation will be symbolized by a bar preceding the kernel letter. Thus, for instance,

$$(1.8e) \quad (\cdot k)_{\alpha\beta} \stackrel{\text{def}}{=} (\cdot k)^{\omega\mu} h_{\omega\alpha} h_{\mu\beta} = \frac{1}{2} E^{\omega\mu\lambda\nu} h_{\omega\alpha} h_{\mu\beta} k_{\lambda\nu} ,$$

$$(1.8f) \quad (\cdot m)_{\alpha\beta} \stackrel{\text{def}}{=} m^{\omega\mu} h_{\omega\alpha} h_{\mu\beta} .$$

In contradistinction to the case where $\mathfrak{h}$ is of rank 4, in our case $(\cdot k)_{\alpha\beta}$ and $(\cdot k)^{\lambda\nu}$ ($(\cdot m)_{\alpha\beta}$ and $m^{\lambda\nu}$) are two *different* tensors, not merely different components of the same tensor.

The tensor $(\cdot k)_{\alpha\beta}$ being skew symmetric, we may raise its indices by means of E :

$$(1.9a) \quad (\cdot \cdot k)^{\lambda\nu} \stackrel{\text{def}}{=} \frac{1}{2} E^{\lambda\nu\alpha\beta} (\cdot k)_{\alpha\beta} = \frac{1}{4} E^{\lambda\nu\alpha\beta} E^{\omega\mu\xi\eta} h_{\omega\alpha} h_{\mu\beta} k_{\xi\eta} .$$

Of course

$$(\cdot \cdot k)^{\lambda\nu} \neq (\cdot k)^{\lambda\nu}.$$

In order to avoid cumbersome notation we shall abbreviate the symbol by underlining the kernel letter:

$$(1.9b) \quad \underline{k}^{\lambda\nu} \stackrel{\text{def}}{=} (\cdot \cdot k)^{\lambda\nu} \cdot *$$

2. The Quadric $\mathcal{C}$ and the Complex $\mathcal{C}$. Denote by $T_4 \stackrel{\text{def}}{=} T_4(O)$ the tangent linear four-dimensional space of X_4 at a point O and by $P_3 \stackrel{\text{def}}{=} P_3(O)$ the ideal projective three-dimensional space of T_4 . Then $h_{\lambda\mu}$ ($k_{\lambda\mu}$) may be looked upon as projective homogeneous coordinates of a quadric $\mathcal{C}$ (of a linear line complex $\mathcal{C}$) in P_3 . The components v^ν (w_λ) of a contravariant (covariant) vector in T_4 may be looked upon as homogeneous projective coordinates of a point (of a plane) in P_3 .

According to statement (2) of Agreement (1.1) $\mathcal{C}$ is an imaginary quadratic cone with the real vertex a^ν . Moreover, according to Definition (1.1), $\mathcal{C}$ is a $\begin{cases} \text{general} \\ \text{special} \end{cases}$ complex if $g_{\lambda\mu}$ is of the $\begin{cases} \text{first} \\ \text{second} \end{cases}$ class. If $\mathcal{C}$ is special, then $k_{\lambda\mu}$ is a line called the axis of $\mathcal{C}$. The *polarity* with respect to $\begin{cases} \mathcal{C} \\ \mathcal{C} \end{cases}$ will be termed $\begin{cases} \mathcal{C} \\ \mathcal{C} \end{cases}$ -polarity.

Raising and/or lowering of both indices (of a covariant [contravariant] quadratic skew symmetric tensor) by means of E and/or e have a geometrical meaning in P_3 . Thus $k_{\lambda\mu}$ are the *Plücker plane* coordinates of $\mathcal{C}$ in P_3 , while $(\cdot k)^{\alpha\nu}$ defined by (1.8a) are *Plücker point* coordinates of the same complex $\mathcal{C}$. Similarly $m^{\lambda\nu}$ and $(\cdot m)_{\lambda\mu}$ are different Plücker coordinates of the same complex, say $\mathfrak{M}$, in P_3 . On the other hand, $\underline{k}^{\lambda\nu}$ defined by (1.9) are point coordinates of a complex, say $\underline{\mathcal{C}}$, in P_3 , whose plane coordinates are $(\cdot \cdot k)_{\lambda\nu}$ defined by (1.8e). In the next theorem we shall use all these components.

Theorem (2.1). $(\cdot \cdot k)_{\lambda\nu}$ and $(\underline{k})^{\lambda\nu}$ are Plücker coordinates of the same line, say A . This line is a ruling of $\mathcal{C}$ if and only if

$$(2.1a) \quad L = 0$$

where

$$(2.1b) \quad 4L \stackrel{\text{def}}{=} (\cdot k)^{\alpha\beta} (\cdot \cdot k)_{\alpha\beta} = \frac{1}{4} E^{\gamma\delta\omega\mu} E^{\alpha\beta\lambda\nu} h_{\gamma\alpha} h_{\delta\beta} k_{\omega\mu} k_{\lambda\nu} = k_{\alpha\beta} (\underline{k})^{\alpha\beta}.$$

Proof. We have from (1.8e) and (1.2b)

$$(2.2) \quad \text{Det} ((\cdot \cdot k)_{\alpha\beta}) = \text{Det} ((\cdot k)^{\alpha\beta}) \mathfrak{h}^2 = 0.$$

Hence the rank of the determinant on the left hand side of (2.2) must be equal to two and consequently $(\cdot \cdot k)_{\alpha\beta}$ are Plücker plane coordinates of a line, say A .

*The operations symbolized by $\cdot$, $\cdot$, and $\underline{\cdot}$ are necessary for the considerations of this chapter. They will not be used in the remaining chapters.

Then, of course, $(\underline{k})^{\lambda\nu}$ defined by (1.9) are Plücker point coordinates of A . The equation (2.1a) follows from the definition (2.1a) and from (1.9b) and (1.8a,e). On the other hand, it is known that

$$k_{\alpha\beta}(\underline{k})^{\alpha\beta} = 0$$

is a necessary and sufficient condition for A to be a ruling of $\mathcal{C}$.

Theorem (2.2). *Assume that the $\mathcal{C}$ -polar plane of a'_4 exists and denote it by*

$$(2.3) \quad \overset{3}{a}_\lambda \stackrel{\text{def}}{=} k_{\lambda\mu} \overset{3}{a}_\mu.$$

Then the following statements hold:

(1) *The line A is incident with a'_4 :*

$$(2.4) \quad a^\lambda({}_1\underline{k})_{\lambda\mu} = 0.$$

(2) *It is $\mathcal{H}$ -polar conjugate to the plane $\overset{3}{a}_\lambda$.*

(3) *Its $\mathcal{C}$ -polar conjugate line B has the coordinates*

$$(2.5) \quad \begin{aligned} \text{(a)} \quad l_{\lambda\nu} &= 8k_{\lambda\nu}L - ({}_1\underline{k})_{\lambda\nu}k_{\alpha\beta}(\underline{k})^{\alpha\beta} \\ \text{(b)} \quad (\underline{l})^{\lambda\nu} &= 8(\underline{k})^{\lambda\nu}L - (\underline{k})^{\lambda\nu}k_{\alpha\beta}(\underline{k})^{\alpha\beta}. \end{aligned}$$

(4) *The lines A, B are skew.*

(5) *They are $\mathcal{H}$ -polar conjugate as well as $\mathcal{C}$ -polar conjugate.*

(6) *We have*

$$(2.6) \quad L \neq 0.$$

Proof. Using (1.8e) we obtain, by virtue of (1.4),

$$({}_1\underline{k})_{\alpha\beta} \overset{3}{a}_4^\beta = (\underline{k})^{\lambda\nu} h_{\lambda\alpha} h_{\nu\beta} \overset{3}{a}_4^\beta = 0.$$

In order to prove the statement (2), denote by e_μ an arbitrary plane not incident with a'_4 :

$$\epsilon \stackrel{\text{def}}{=} \overset{3}{a}_4^\mu e_\mu \neq 0,$$

and consider the line P , the intersection of $\overset{3}{a}_\mu$ and e_μ . Its Plücker plane coordinates are

$$(2.7a) \quad p_{\lambda\mu} = \overset{3}{a}_{[\lambda e_\mu]} = e_{[\mu} k_{\lambda] \alpha} \overset{3}{a}_\alpha$$

and its Plücker point coordinates are given by

$$(2.7b) \quad (\underline{p})^{\alpha\beta} = \frac{1}{2} E^{\alpha\beta\lambda\mu} \overset{3}{a}_\lambda e_\mu = \frac{1}{2} E^{\alpha\beta\lambda\mu} e_\mu \overset{3}{a}_4^\lambda k_{\lambda\delta}.$$

The $\mathcal{H}$ -polar conjugate line Q to P has the Plücker plane coordinates

$$(2.8a) \quad q_{\omega\nu} = ({}^{\cdot}p)^{\alpha\beta} h_{\alpha\omega} h_{\beta\nu} = \tfrac{1}{2} E^{\alpha\beta\gamma\delta} h_{\alpha\omega} h_{\beta\nu} e_{\delta} k_{\gamma\lambda} a_{\lambda}^{\lambda}.$$

Substituting

$$k_{\gamma\lambda} = \tfrac{1}{2} e_{\gamma\lambda\xi\eta} ({}^{\cdot}k)^{\xi\eta}$$

into the right hand side and making use of (1.7d) one obtains

$$(2.8b) \quad 2q_{\omega\nu} = ({}^{\cdot}k)_{\omega\nu} \epsilon.$$

Hence A is $\mathcal{H}$ -polar conjugate to any line P which is incident with $\overset{3}{a}_{\mu}$ without being incident with a'' . This proves the statement (2).

The Plücker plane coordinates of the line B , $\mathcal{C}$ -polar conjugate to A , are given by

$$(2.9) \quad \rho l_{\lambda\nu} = 2k_{\lambda\nu} \mathfrak{E}^{\omega\mu\alpha\beta} k_{\omega\mu} ({}^{\cdot}k)_{\alpha\beta} - ({}^{\cdot}k)_{\lambda\nu} \mathfrak{E}^{\omega\mu\alpha\beta} k_{\omega\mu} k_{\alpha\beta},$$

where ρ is an arbitrary factor of proportionality.*

Using (1.8a) we see that, by virtue of (1.6a),

$$(2.10a) \quad \mathfrak{E}^{\omega\mu\lambda\nu} k_{\omega\mu} k_{\lambda\nu} = 2\sigma(\sqrt{|\mathfrak{g}|} ({}^{\cdot}k)^{\alpha\beta} k_{\alpha\beta})$$

and according to (2.1b)

$$(2.10b) \quad 2\mathfrak{E}^{\omega\mu\lambda\nu} k_{\omega\mu} ({}^{\cdot}k)_{\lambda\nu} = 4\sigma\sqrt{|\mathfrak{g}|} ({}^{\cdot}k)^{\lambda\nu} ({}^{\cdot}k)_{\lambda\nu} = 16\sigma\sqrt{|\mathfrak{g}|} L.$$

Substituting from (2.10) into (2.9) and setting

$$\rho = 2\sigma\sqrt{|\mathfrak{g}|}$$

we obtain (2.5a), and from it, in the usual way, (2.5b).

The lines A , B are $\mathcal{C}$ -polar conjugate and A is incident with a'' . Therefore B must be incident with the $\mathcal{C}$ -polar plane $\overset{3}{a}_{\mu}$ of a'' . Hence the $\mathcal{C}$ -polar conjugate lines A , B can meet if and only if they are both incident with the $\mathcal{C}$ -polar plane $\overset{3}{a}_{\mu}$ of a'' . In particular, A being incident with a'' must be incident with $\overset{3}{a}_{\mu}$ and therefore it must also be a ruling of $\mathcal{C}$. In other words, (2.1a) must hold. On the other hand, A , incident with a'' and $\overset{3}{a}_{\mu}$, is also $\mathcal{H}$ -polar conjugate to $\overset{3}{a}_{\mu}$. Hence $\overset{3}{a}_{\mu}$ must be a tangent plane of $\mathcal{H}$ and A must be the common ruling of $\overset{3}{a}_{\mu}$ and $\mathcal{H}$. This is impossible because, according to the requirement (2) of Agreement (1.1), $\mathcal{H}$ is an imaginary cone with only one real point, namely a'' , and $\overset{3}{a}_{\mu}$ as well as A are real. Hence A and B cannot meet and (2.1a) cannot hold. This proves the statements (4) and (6).

*See, for instance, HLAVATÝ, V., *loc. cit.*, Appendix III, in a slightly different form.

In order to prove the $\mathcal{K}$ -polar conjugacy of A and B , let us observe that one of the obvious consequences of requirement (2) of Agreement (1.1) is

$$(2.11a) \quad E^{\alpha\beta\gamma\delta} h_{\alpha\lambda} h_{\beta\nu} h_{\gamma\omega} h_{\delta\mu} = 0$$

so that

$$(2.11b) \quad (\underline{k})^{\alpha\beta} h_{\alpha\lambda} h_{\beta\nu} = \frac{1}{4} E^{\alpha\beta\gamma\delta} E^{\omega\mu\xi\eta} h_{\alpha\lambda} h_{\beta\nu} h_{\gamma\omega} h_{\delta\mu} k_{\xi\eta} = 0$$

and consequently according to (2.5b)

$$(2.12) \quad (l)_{\lambda\nu} = 8L(l)_{\lambda\nu}.$$

This equation shows that the $\mathcal{K}$ -polar conjugate line to B is A .^{*} The $\mathcal{C}$ -polar conjugacy of A , B is obvious from the statement (3).

The following theorem completes the statements of the previous theorem.

Theorem (2.3). *A necessary and sufficient condition for the existence of $\overset{3}{a}_\mu$ is (2.6).*

Proof. According to Theorem (2.2) we need to prove only the sufficiency. Assume that (2.6) holds, i.e. that A is not a ruling of $\mathcal{C}$. If $\mathcal{C}$ is a special complex and A , incident with $\overset{4}{a}''$, is not a ruling of $\mathcal{C}$ then the axis $k_{\lambda\mu}$ of $\mathcal{C}$ cannot be incident with $\overset{4}{a}''$. Therefore the $\mathcal{C}$ -polar plane $\overset{3}{a}_\mu$ of $\overset{4}{a}''$, which is defined by $\overset{4}{a}''$ and the axis $k_{\lambda\mu}$, exists. If the complex $\mathcal{C}$ is general, then $\overset{3}{a}_\mu$ always exists. Hence it remains to be shown that (2.1a) cannot be satisfied for general $\mathcal{C}$. The incompatibility of these two requirements follows at once from the proof of statement (4) of Theorem (2.2).

3. The Determinant $g(\lambda)$. The following two lemmas will be proved simultaneously.

Lemma (3.1). *We have*

$$(3.1) \quad \begin{aligned} (a) \quad & E^{\omega\mu\lambda\nu} E^{\alpha\beta\gamma\delta} k_{\omega\alpha} h_{\mu\beta} h_{\lambda\gamma} h_{\nu\delta} = 0, \\ (b) \quad & E^{\omega\mu\lambda\nu} E^{\alpha\beta\gamma\delta} h_{\omega\alpha} k_{\mu\beta} k_{\lambda\gamma} k_{\nu\delta} = 0, \\ (c) \quad & E^{\omega\mu\lambda\nu} E^{\alpha\beta\gamma\delta} h_{\omega\alpha} h_{\mu\beta} k_{\lambda\gamma} k_{\nu\delta} = 8L. \end{aligned}$$

Lemma (3.2). *If $g_{\lambda\mu}$ is of the first class, then there is a unique tensor $\overset{-1}{k}^{\lambda\nu}$ with the following properties:*

$$(3.2) \quad \begin{aligned} (a) \quad & \text{It satisfies the condition} \\ & \overset{-1}{k}^{\lambda\nu} k_{\lambda\mu} = \overset{-1}{k}^{\nu\lambda} k_{\mu\lambda} = \delta_\mu^\nu. \end{aligned}$$

^{*}The $\mathcal{K}$ -polar conjugate line to A is any line incident with $\overset{3}{a}_\mu$, so that B , incident with $\overset{3}{a}_\mu$, may be looked upon as $\mathcal{K}$ -polar conjugate to A .

(b) It is given by

$$(3.3a) \quad \overset{-1}{k}{}^{\lambda\nu} \stackrel{\text{def}}{=} \kappa (\cdot k)^{\lambda\nu} \sqrt{\frac{|\mathfrak{g}|}{\mathfrak{f}}} ,$$

$$(3.3b) \quad \kappa \stackrel{\text{def}}{=} \text{sgn } E^{\omega\mu\lambda\nu} k_{\omega\mu} k_{\lambda\nu} .$$

(c) If we put

$$(3.4a) \quad \overset{-1}{K} \stackrel{\text{def}}{=} \frac{1}{4} \overset{-1}{k}{}^{\lambda\nu} (\cdot k)_{\lambda\nu} = \frac{1}{4} \overset{-1}{k}{}^{\lambda\nu} \overset{-1}{k}{}^{\alpha\beta} h_{\alpha\lambda} h_{\beta\nu}$$

then

$$(3.4b) \quad L = \overset{-1}{K} \frac{\mathfrak{f}}{|\mathfrak{g}|} .$$

Proof. A simple inspection shows that

$$E^{\omega\mu\lambda\nu} E^{\alpha\beta\gamma\delta} h_{\mu\beta} h_{\lambda\gamma} h_{\nu\delta} \quad (E^{\omega\mu\lambda\nu} E^{\alpha\beta\gamma\delta} k_{\mu\beta} k_{\lambda\gamma} k_{\nu\delta})$$

is symmetric (skew symmetric) in $\omega\alpha$ while $k_{\omega\alpha}$ ($h_{\omega\alpha}$) is skew symmetric (symmetric) in $\omega\alpha$. Therefore (3.1a, b) must hold. On the other hand, using (1.7) and (1.8) we obtain, by virtue of (2.1b),

$$\begin{aligned} E^{\omega\mu\lambda\nu} E^{\alpha\beta\gamma\delta} h_{\omega\alpha} h_{\mu\beta} k_{\lambda\gamma} k_{\nu\delta} &= \frac{1}{4} E^{\omega\mu\lambda\nu} E^{\alpha\beta\gamma\delta} h_{\omega\alpha} h_{\mu\beta} e_{\lambda\gamma\rho\sigma} e_{\nu\delta\xi\eta} (\cdot k)^{\rho\sigma} (\cdot k)^{\xi\eta} \\ &= 2 (\cdot k)^{\alpha\beta} (\cdot k)_{\alpha\beta} = 8L . \end{aligned}$$

A simple computation shows that

$$(3.5) \quad (\cdot k)^{\lambda\nu} k_{\lambda\mu} = \frac{1}{2} E^{\lambda\nu\alpha\beta} k_{\alpha\beta} k_{\lambda\mu} = \frac{\kappa}{\sqrt{|\mathfrak{g}|}} \sqrt{\mathfrak{f}} \delta_{\mu}^{\nu} ,$$

where κ is given by (3.3b). Hence if we define $\overset{-1}{k}{}^{\lambda\nu}$ by (3.3a) we obtain (3.2).

It is obvious that there is only one tensor $\overset{-1}{k}{}^{\lambda\nu}$ satisfying (3.2). Using (2.1b) again we get, by virtue of (3.3a) and (3.4a),

$$4L = (\cdot k)^{\alpha\beta} (\cdot k)_{\alpha\beta} = \frac{\mathfrak{f}}{|\mathfrak{g}|} \overset{-1}{k}{}^{\alpha\beta} (\cdot k)_{\alpha\beta} = 4\overset{-1}{K} \frac{\mathfrak{f}}{|\mathfrak{g}|} .$$

We shall use these results in the proofs of the following three theorems.

Theorem (3.1). Put

$$(3.6) \quad \begin{aligned} (a) \quad G_{\alpha\beta}(\lambda) &\stackrel{\text{def}}{=} \lambda h_{\alpha\beta} - k_{\alpha\beta} = \lambda h_{\beta\alpha} + k_{\beta\alpha} , \\ (b) \quad \mathfrak{G}(\lambda) &\stackrel{\text{def}}{=} \text{Det } ((G_{\alpha\beta}(\lambda)) . \end{aligned}$$

Then

$$(3.7) \quad \mathfrak{G}(\lambda) = 2 |\mathfrak{g}| L \lambda^2 + \mathfrak{f} .$$

Theorem (3.2). *If $g_{\lambda\mu}$ is of the first class then*

$$(3.8) \quad \mathfrak{G}(\lambda) = \mathfrak{k}(2\lambda^2 K + 1)$$

and

$$(3.9) \quad \mathfrak{g} = \mathfrak{G}(1) = \mathfrak{k}(2K + 1).$$

Theorem (3.3). *If $g_{\lambda\mu}$ is of the second class then*

$$(3.10) \quad \mathfrak{G}(\lambda) = \mathfrak{g}\lambda^2$$

and

$$(3.11a) \quad 2L = \gamma,$$

where

$$(3.11b) \quad \gamma = \text{sgn } \mathfrak{g}. *$$

Proof. Our starting point is the following set of obvious equations

$$(3.12) \quad \begin{aligned} (a) \quad \mathfrak{G}(\lambda) &= \frac{1}{4!} \mathfrak{E}^{\omega\mu\lambda\nu} \mathfrak{E}^{\alpha\beta\gamma\delta} G_{\omega\alpha} G_{\mu\beta} G_{\lambda\gamma} G_{\nu\delta} \\ &= \frac{|\mathfrak{g}|}{4!} E^{\omega\mu\lambda\nu} E^{\alpha\beta\gamma\delta} G_{\omega\alpha} G_{\mu\beta} G_{\lambda\gamma} G_{\nu\delta}, \\ (b) \quad \mathfrak{h} &= \frac{1}{4!} \mathfrak{E}^{\omega\mu\lambda\nu} \mathfrak{E}^{\alpha\beta\gamma\delta} h_{\omega\alpha} h_{\mu\beta} h_{\lambda\gamma} h_{\nu\delta} = 0 \\ (c) \quad \mathfrak{k} &= \frac{1}{4!} \mathfrak{E}^{\omega\mu\lambda\nu} \mathfrak{E}^{\alpha\beta\gamma\delta} k_{\omega\alpha} k_{\mu\beta} k_{\lambda\gamma} k_{\nu\delta}. \end{aligned}$$

If we substitute into (3.12a) from (3.6a) and make use of (3.1) and (3.12b, c) we obtain

$$\begin{aligned} 4!\mathfrak{G}(\lambda) &= 3! |\mathfrak{g}| \lambda^2 E^{\omega\mu\lambda\nu} E^{\alpha\beta\gamma\delta} k_{\omega\alpha} k_{\mu\beta} h_{\lambda\gamma} h_{\nu\delta} + \mathfrak{E}^{\omega\mu\lambda\nu} \mathfrak{E}^{\alpha\beta\gamma\delta} k_{\omega\alpha} k_{\mu\beta} k_{\lambda\gamma} k_{\nu\delta} \\ &= 3! |\mathfrak{g}| \lambda^2 8L + 4!\mathfrak{k}, \end{aligned}$$

and this equation is equivalent to (3.7). If $g_{\lambda\mu}$ is of the first class, then we may use (3.4b) and in this way get (3.8) from (3.7). The equation (3.9) follows at once from (3.8) and (3.6).

If $g_{\lambda\mu}$ belongs to the second class ($\mathfrak{k} = 0$), the equation (3.7) reduces to

$$(3.13a) \quad \mathfrak{G}(\lambda) = 2 |\mathfrak{g}| L\lambda^2,$$

so that

$$(3.13b) \quad \mathfrak{G}(1) = \mathfrak{g} = 2 |\mathfrak{g}| L$$

* $\mathfrak{g}$ is a scalar density of weight 2 and therefore its sign is independent of the choice of coordinate system.

and accordingly

$$(3.13c) \quad 2L = \gamma, \quad \gamma = \operatorname{sgn} g.$$

Substituting from (3.13c) into (3.13a) we obtain (3.10), while the equations (3.13c) are identical with (3.11a,b).

Corollary (3.1). *We always have*

$$(3.14a) \quad L \neq 0.$$

Proof. For the second class (3.14a) is an immediate consequence of (3.11a, b). On the other hand, we see from (3.4a) that for the first class $\bar{K}^{-1}$ may be thought of as a sum of three real squares and therefore

$$(3.14b) \quad \bar{K}^{-1} > 0.$$

Using (3.4b) and (3.14b) we get (3.14a).

Corollary (3.2). *The $\mathbb{C}$ -polar plane $\bar{a}_\mu^3$ of a^ν * always exists.*

The proof follows from Theorem (2.3) and Corollary (3.1).

4. The Point Frame and Plane Frame in P_3 . In the next Theorem we shall use the line B involved in Theorem (2.2).

Theorem (4.1). *The intersection points a_1^ν, a_2^ν of B and $3C$ have the following properties:*

(1) *They are complex conjugate and satisfy the requirement*

$$(4.1) \quad h_{xy} \stackrel{\text{def}}{=} h_{\lambda\mu} a_x^\lambda a_y^\mu = \alpha \delta_{xy}, \quad \alpha \neq 0.$$

(2) *They satisfy the equations*

$$(4.2) \quad a_x^\lambda k_{\lambda\mu} = \lambda a_x^\lambda h_{\lambda\mu},$$

where for the first class

$$(4.3) \quad \lambda \stackrel{\text{def}}{=} \lambda_1 = -\lambda_2 = i / \sqrt{2K}$$

and for the second class

$$(4.4) \quad \lambda \stackrel{\text{def}}{=} \lambda_1 = \lambda_2 = 0.$$

Proof. According to statement (5) of Theorem (2.2), the line B is incident with the $\mathbb{C}$ -polar plane $\bar{a}_\mu^3$ of a^ν . According to statements (1) and (4) of the

*See (2.3).

same theorem, B is not incident with a'' . From the requirement (2) of Agreement (1.1) we see that the real line B , incident with a'' , cannot be tangent to the imaginary cone $\mathcal{H}$. Therefore B intersects $\mathcal{H}$ in two different points a'' , a'' which are null points of $\mathcal{H}$ and cannot be $\mathcal{H}$ -polar conjugate. Therefore they satisfy (4.1). The complex conjugacy of the points a'' and a'' follows from elementary analytic geometry.

Assume now that $g_{\lambda\mu}$ belongs to the first class. Then the $\mathcal{C}$ -polar plane of a'' exists and is incident with the line A , which is $\mathcal{C}$ -polar conjugate to B . Because $a'' \neq a''$, its $\mathcal{H}$ -polar plane (*i.e.* the tangent plane of $\mathcal{H}$ in a'') exists and is incident with the $\mathcal{H}$ -polar line of B , *i.e.* with A . Therefore the $\mathcal{C}$ -polar plane of a'' is identical with its $\mathcal{H}$ -polar plane. This fact is analytically expressed by

$$(4.5) \quad G_{\alpha\beta}(\lambda)a'' = 0$$

where $G_{\alpha\beta}(\lambda)$ is given by (3.6) and a'' stands for either one of the points a'' , a'' . The system (4.5) has a non-trivial solution only for a λ which is a solution of $g(\lambda) = 0$. According to (3.8) these solutions are given by (4.3) and the system (4.5) is equivalent to (4.2).

Assume now that $g_{\lambda\mu}$ is of the second class. Then $k_{\lambda\mu}$ is a line,

$$(4.6) \quad k_{\lambda\mu}(\cdot k)^{\lambda\mu} = 0,$$

so that (2.5a) reduces to

$$(4.7) \quad l_{\lambda\nu} = 8Lk_{\lambda\nu}.$$

Hence the line B is the axis $k_{\lambda\mu}$ of $\mathcal{C}$. Therefore *any* point a'' incident with B has the property that its $\mathcal{C}$ -polar plane does not exist:

$$(4.8) \quad a''k_{\lambda\mu} = 0.$$

On the other hand, the equation $g(\lambda) = 0$ yields, according to (3.10), the values (4.4). Substituting them into (4.2) we obtain (4.8). Although *all* points a'' of the axis $k_{\lambda\mu}$ satisfy (4.8), only two points a'' also satisfy (4.1), namely, the intersection points of B and $\mathcal{H}$.

Theorem (4.2). *The factors of proportionality of the components a'' may always be chosen in such a way that*

- (1) *the corresponding components a'' and a'' are complex conjugate*

$$(4.9) \quad \bar{a}'' = a'', \quad \bar{a}'' = a'';$$

- (2) *the equations (4.1) reduce to*

$$(4.10a) \quad h_{xy} = h_{\lambda\mu}a''a'' = \delta_{xy}.$$

Proof. The first statement follows from the complex conjugacy of the points in question, *i.e.* from

$$\bar{a}_1^{\text{I}} : \bar{a}_1^{\text{II}} : \bar{a}_1^{\text{III}} : \bar{a}_1^{\text{IV}} = a_2^{\text{I}} : a_2^{\text{II}} : a_2^{\text{III}} : a_2^{\text{IV}}.$$

If the factors of proportionality are chosen according to statement (1) then

$$a_{1\ 2}^\lambda a^\mu h_{\lambda\mu} = \alpha, \quad \alpha > 0 \quad \text{and real.}$$

This fact yields at once the proof of the statement (2).

Theorem (4.3). *Let $a_3^\nu \neq a_4^\nu$ be an arbitrary real point incident with the line A . Its factor of proportionality may always be chosen in such a way that*

$$(4.10b) \quad h_{33} \stackrel{\text{def}}{=} a_{3\ 3}^\lambda a^\mu h_{\lambda\mu} = 1.$$

Proof. The point a_3^λ being real, the scalar h_{33} may be thought of as a sum of three squares. The proof follows easily from this fact.

Theorem (4.4). *The factor of proportionality of a_4^ν may always be chosen in such a way that*

$$(4.11) \quad k_{34} \stackrel{\text{def}}{=} a_{3\ 4}^\lambda a^\mu k_{\lambda\mu} = a_{3\ 3}^\lambda a_\lambda^3 = 1$$

and

$$(4.12) \quad a_\lambda^3 = h_{\lambda\mu} a^\mu.$$

Proof. The scalar k_{34} contains only one arbitrary factor, namely the factor of proportionality of a_4^ν . The proof of (4.11) follows easily from this fact and from (2.3). On the other hand, we see from the statement (2) of Theorem (2.2) that the $\mathcal{H}$ -polar plane of every point $a_3^\nu \neq a_4^\nu$ on A is identical with a_μ^3 ,

$$a_\lambda^3 = a_{4\ \lambda}^\mu k_{\lambda\mu} = {}^* \sigma a_{3\ \lambda}^\mu h_{\lambda\mu}.$$

Making use of (4.11) and (4.10b) we obtain ${}^* \sigma = 1$, *i.e.* (4.12).

We have chosen factors of proportionality of all four points a_i^ν . Later on we shall see that the above introduced conditions may be satisfied even if we change in a certain way these proportionality factors and even the point a_3^ν itself.

5. Continuation.

Theorem (5.1). *There is a unique set of four linearly independent planes a_λ^i satisfying conditions*

$$(5.1) \quad (a) \quad a_i^\nu a_\nu^i = \delta_i^i, \quad (b) \quad a_i^\nu a_\lambda^i = \delta_\lambda^i.$$

The proof follows from the fact that the points a_i^ν are linearly independent.

Theorem (4.2). Put

$$(5.2) \quad \begin{aligned} (a) \quad h_{ij} &\stackrel{\text{def}}{=} a_{ij}^\lambda a_{ij}^\mu h_{\lambda\mu}, \\ (b) \quad k_{ij} &\stackrel{\text{def}}{=} a_{ij}^\lambda a_{ij}^\mu k_{\lambda\mu}, \\ (c) \quad g_{ij} &\stackrel{\text{def}}{=} a_{ij}^\lambda a_{ij}^\mu g_{\lambda\mu}. \end{aligned}$$

Then

$$(5.3) \quad \begin{aligned} (a) \quad ((h_{ij})) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ (b) \quad ((k_{ij})) &= \begin{bmatrix} 0 & \lambda_1 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ (c) \quad ((g_{ij})) &= \begin{bmatrix} 0 & 1 + \lambda_1 & 0 & 0 \\ 1 + \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \end{aligned}$$

so that

$$(5.4) \quad \begin{aligned} (a) \quad h &\stackrel{\text{def}}{=} -\text{Det}((h_{ij})) = 0, \\ (b) \quad k &\stackrel{\text{def}}{=} -\text{Det}((k_{ij})) = -\lambda^2, \\ (c) \quad g &\stackrel{\text{def}}{=} -\text{Det}((g_{ij})) = 1 - \lambda^2.* \end{aligned}$$

Proof. Among the points $a_1'', \dots, a_4''$ there are only two which are not $\mathcal{H}$ -polar conjugate, namely a_1'', a_2'' . This fact, together with (4.10) and (1.4), leads to (5.3a). Making use of (4.2) we get

$$(5.5a) \quad k_{xi} = \lambda_{xi} h_{xi}.$$

On the other hand, recalling that a_4'' is incident with a_λ^3 and that (4.11) holds, we obtain

* $-\lambda^2$ is non-negative as follows from (3.14b), (4.3) and (4.4).

$$(5.5b) \quad k_{z4} = \begin{cases} 0 & \text{for } z = 4 \\ 1 & \text{for } z = 3 \end{cases}.$$

These equations, together with the skew-symmetry of $k_{\lambda\mu}$, yield (5.3b). The proof of the remaining part of our theorem is obvious.

Note. The equations (5.3) and (5.4) will play an important role in subsequent considerations.

6. The Vector Frame in T_4 . Previous results may be interpreted in T_4 . Then a_i^{ν} are contravariant vectors in T_4 whose ideal points in P_3 are a_i^{ν} ; $\dot{a}_\lambda$ are covariant vectors in T_4 whose ideal planes in P_3 are $\dot{a}_\lambda$. The $\mathcal{H}$ -polarity in P_3 is equivalent to orthogonality in X_4 with respect to the degenerate tensor $h_{\lambda\mu}$. The points a_1^{ν}, a_2^{ν} on $\mathcal{H}$ in P_3 are null vectors on $h_{\lambda\mu}$ in X_4 while, according to (4.10b), a_3^{ν} is a unit vector (with respect to $h_{\lambda\mu}$):

$$(6.1) \quad h_{\lambda\mu} a_x^\lambda a_x^\mu = 0, \quad h_{\lambda\mu} a_3^\lambda a_3^\mu = 1.$$

Definition (6.1). The vector set consisting of a_i^{ν} and $\dot{a}_\lambda$ will be termed the frame F .

In subsequent theorems we shall find all possible frames with the properties of F .

Theorem (6.1). The most general linear transformation of F which preserves the position of the vectors $a_1^{\nu}, a_2^{\nu}, a_4^{\nu}$ and leaves $h_{i,i}, k_{i,i}$ invariant is of the form

$$(6.2) \quad (a) \quad \dot{a}_\lambda' = A_i^{i'} \dot{a}_\lambda, \quad (b) \quad a_{i'}^{\nu} = A_i^{i'} a_i^{\nu}$$

where

$$(6.2) \quad (c) \quad ((A_i^{i'})) = \begin{bmatrix} e^{i\varphi} & 0 & 0 & 0 \\ 0 & e^{-i\varphi} & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & -\rho & \epsilon \end{bmatrix},$$

$$(6.2) \quad (d) \quad ((A_i^{i'})) = \begin{bmatrix} e^{-i\varphi} & 0 & 0 & 0 \\ 0 & e^{i\varphi} & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & \rho & \epsilon \end{bmatrix}. *$$

Here φ and ρ are two arbitrary real scalar functions and $\epsilon = \pm 1$.

*In (6.2c,d) the superscripts (the subscripts) denote the rows (the columns) of the respective matrix.

Proof. The most general linear transformation which preserves the position of the vectors a_1^r, a_2^r, a_4^r and leaves h_{ij} invariant is

$$(6.3) \quad a_{1'}^r = e^{-i\varphi} a_1^r, \quad a_{2'}^r = e^{i\varphi} a_2^r, \quad a_{3'}^r = \epsilon_3 a_3^r + \rho a_4^r, \quad a_{4'}^r = \epsilon_4 a_4^r,$$

where φ, ρ are arbitrary scalar functions and $\epsilon_3, \epsilon_4 = \pm 1$. The invariance of k_{ij} requires $\epsilon_3 = \epsilon_4 = \epsilon$, so that (6.3) reduces to (6.2b, d). The remaining equations (6.2a, c) follow from (6.2b, d) and (5.1).

Note. The frames $a_{i'}^r, a_{\lambda}^{i'}$ will be termed F' -frames.

Definition (6.2). If $T_{\lambda}^{r\cdots}$ is a tensor then

$$(6.4a) \quad T_{i'}^{r\cdots} \stackrel{\text{def}}{=} a_{i'}^i a_{\lambda}^{\lambda} \cdots T_{\lambda}^{r\cdots}$$

are termed non-holonomic components of $T_{\lambda}^{r\cdots}$ or, for short, F -components.

Note. A simple inspection shows that

$$(6.4b) \quad T_{\lambda}^{r\cdots} = a_{i'}^r a_{\lambda}^{i'} \cdots T_{i'}^{r\cdots}.$$

Theorem (6.3). The F -components transform by (6.2) into F' -components according to

$$(6.5) \quad T_{i'}^{r\cdots} = A_a^{i'} A_j^b \cdots T_b^{a\cdots},$$

where the A 's are defined by (6.2c,d). In particular, the F -components h_{ii}, k_{ii} are invariant under (6.2).

The proof is obvious.

Definition (6.3). Any object Ω invariant under (6.2) will be termed an F -invariant object (frame invariant object).

Note. $g_{\lambda\mu}, h_{\lambda\mu}$ and $k_{\lambda\mu}$ are F -invariant. Nevertheless they can be expressed in terms of the vectors $a_{\lambda}^{i'}$.

Theorem (6.4). Put

$$(6.6) \quad \lambda_{\lambda}^{\lambda} = -\lambda_{\lambda}^{\lambda} \stackrel{\text{def}}{=} 1.$$

Then

$$(6.7) \quad \begin{aligned} (a) \quad h_{\lambda\mu} &= 2a_{(\lambda}^1 a_{\mu)}^2 + a_{\lambda}^3 a_{\mu}^3, \\ (b) \quad k_{\lambda\mu} &= \lambda_{\lambda}^1 a_{\mu}^2 + \lambda_{\lambda}^2 a_{\mu}^1 + \lambda_{\lambda}^3 a_{\mu}^4 + \lambda_{\lambda}^4 a_{\mu}^3 = 2\lambda_{[\lambda}^1 a_{\mu]}^2 + 2a_{[\lambda}^3 a_{\mu]}^4, \\ (c) \quad g_{\lambda\mu} &= (1 + \lambda) a_{\lambda}^1 a_{\mu}^2 + (1 + \lambda) a_{\lambda}^2 a_{\mu}^1 + a_{\lambda}^3 a_{\mu}^3 + \lambda_{\lambda}^3 a_{\mu}^4 + \lambda_{\lambda}^4 a_{\mu}^3. \end{aligned}$$

Proof. Making use of (5.1) we obtain

$$g_{\lambda\mu} = g_{\alpha\beta} \delta_{\lambda}^{\alpha} \delta_{\mu}^{\beta} = g_{\alpha\beta} a_{\dot{i}}^{\alpha} a_{\dot{j}}^{\beta} a_{\lambda}^{\dot{i}} a_{\mu}^{\dot{j}} = g_{\dot{i}\dot{j}} a_{\lambda}^{\dot{i}} a_{\mu}^{\dot{j}} .$$

Substituting into this equation from (5.3c) we obtain (6.7c). This equation, together with (1.1), leads to (6.7a,b).

Note. The F -invariance of $h_{\lambda\mu}$ and $k_{\lambda\mu}$ is seen also from the equations resulting from (6.7a,b):

$$(6.8) \quad \begin{aligned} (a) \quad h_{\lambda\mu} &= 2a_{(\lambda}^{1'} a_{\mu)}^{2'} + a_{\lambda}^{3'} a_{\mu}^{3'} , \\ (b) \quad k_{\lambda\mu} &= 2\lambda a_{[\lambda}^{1'} a_{\mu]}^{2'} + 2a_{[\lambda}^{3'} a_{\mu]}^{4'} . \end{aligned}$$

Besides F -invariance we shall need another concept connected with the transformation (6.2). Denote by Φ (Φ') an object associated with F (with F') and symbolize by Φ_A an F -component or a set of F -components of Φ and by Φ'_A the corresponding set of F' -components of Φ' . The sets Φ_A and Φ'_A lead to the following

Definition (6.4). *If*

$$(6.9) \quad (a) \quad \Phi_A = 0 \quad \text{and} \quad (b) \quad \Phi'_A = 0 ,$$

then either one of the sets of equations (6.9a) or (6.9b) will be termed a set of re-productive equations.

The importance of this notion will be seen later.

7. The Tensors $a^{\lambda\nu}$ and $a_{\lambda\mu}$.

Theorem (7.1). *Put*

$$(7.1) \quad \begin{aligned} (a) \quad a^{\lambda\nu} &\stackrel{\text{def}}{=} 2a_{\underset{1}{\lambda}}^{\underset{2}{\lambda}} a_{\underset{3}{\nu}}^{\underset{3}{\lambda}} + a_{\underset{3}{\lambda}}^{\underset{3}{\lambda}} a_{\underset{4}{\nu}}^{\underset{4}{\lambda}} = a^{\nu\lambda} , \\ (b) \quad a_{\lambda\mu} &\stackrel{\text{def}}{=} h_{\lambda\mu} + a_{\lambda}^{\underset{4}{\lambda}} a_{\mu}^{\underset{4}{\lambda}} = a_{\mu\lambda} . \end{aligned}$$

Then

$$(7.2) \quad a^{\lambda\nu} a_{\lambda\mu} = \delta_{\mu}^{\nu} .$$

The proof follows from (7.1) and (5.1).

Theorem (7.3). *The vectors $a_{\dot{i}}^{\nu}$ and $a_{\lambda}^{\dot{i}}$ are related by*

$$(7.4) \quad \begin{aligned} (a) \quad a_{\lambda}^{\dot{i}} &= a^{ij} a_{\dot{i}}^{\nu} a_{\nu\lambda} , \\ (b) \quad a_{\dot{i}}^{\nu} &= a_{ij} a_{\lambda}^{\dot{i}} a^{\lambda\nu} . \end{aligned}$$

Proof. Put

$$b_{\lambda}^{\dot{i}} \stackrel{\text{def}}{=} a^{ij} a_{\dot{i}}^{\nu} a_{\nu\lambda} .$$

Then

$$(7.5) \quad b_{\lambda}^i a^{\lambda} = a^{ij} a^{\prime} a^{\lambda} a_{\nu\lambda} = a^{\prime\prime} a_{i\lambda} = \delta_k^i.$$

Comparing (7.5) and (5.1) we get at once $b_{\lambda}^i = \overset{i}{a}_{\lambda}^i$. The equation (7.4b) may be proved in a similar way.

Theorem (7.4). *Put*

$$(7.6) \quad \begin{aligned} (a) \quad a^{\prime\prime\lambda} &\stackrel{\text{def}}{=} 2a_{1'}^{\prime} a_{2'}^{\lambda} + a_{3'}^{\prime} a_{3'}^{\lambda} + a_{4'}^{\prime} a_{4'}^{\lambda}, \\ (b) \quad a'_{\lambda\mu} &\stackrel{\text{def}}{=} h_{\lambda\mu} + \overset{4'}{a}_{\lambda}^{\prime} \overset{4'}{a}_{\mu}^{\prime}. \end{aligned}$$

Then

$$(7.7) \quad \begin{aligned} (a) \quad a^{\prime\prime\lambda} &= a^{\nu\lambda} + 2\epsilon\rho a_3^{\prime} a_4^{\lambda} + \rho^2 a_4^{\prime} a_4^{\lambda}, \\ (b) \quad a'_{\lambda\mu} &= a_{\lambda\mu} - 2\epsilon\rho \overset{3}{a}_{(\lambda} \overset{4}{a}_{\mu)} + \rho^2 \overset{3}{a}_{\lambda} \overset{3}{a}_{\mu}, \\ (c) \quad a^{\prime\prime\lambda} a'_{\mu\lambda} &= \delta_{\mu}^{\nu}, \\ (d) \quad ((a^{\prime\prime}{}^{i'j'})) &= ((a'_{i'j'})) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Proof. The equations (7.7a,b) follow from (7.6) and (6.2). They lead at once to (7.7c). The last equation is obvious.

Note. The equations (7.7a,b) show that $a^{\nu\lambda}$ ($a_{\lambda\mu}$) is not F -invariant. The equations (7.7d) and (7.3) show that, for instance,

$$(7.8) \quad (a) \quad a_{13} = 0, \quad (b) \quad a'_{1'3'} = 0.$$

Therefore either one of these equations is reproductive.

We shall use $a^{\nu\lambda}$ and $a_{\lambda\mu}$ for raising and lowering indices. In order to stress that this operation changes the geometrical meaning of the object under consideration, we shall symbolize this kind of raising (lowering) of indices by changing the kernel letter from capital to lower case or *vice versa*. Thus, for instance, if v_{λ} (W^{ν}) is a given vector, we write

$$(7.9) \quad V^{\nu} \stackrel{\text{def}}{=} v_{\lambda} a^{\lambda\nu}, \quad w_{\lambda} = W^{\nu} a_{\nu\lambda}.$$

If v_{λ} (W^{ν}) is frame invariant we may define

$$(7.10) \quad \begin{aligned} V^{\prime\prime} &\stackrel{\text{def}}{=} v'_{\lambda} a^{\lambda\nu} = v_{\lambda} a^{\lambda\nu}, \\ w'_{\lambda} &\stackrel{\text{def}}{=} W^{\nu} a'_{\nu\lambda} = W^{\nu} a'_{\nu\lambda}. \end{aligned}$$

If $v_\lambda (W')$ is F -invariant, the vector $V' (w_\lambda)$ is in general not F -invariant. In spite of this fact the following theorem holds:

Theorem (7.5). *We always have*

$$(7.11) \quad v_\lambda W^\lambda = V^\lambda w_\lambda .$$

If v_λ and W' are F -invariant then also

$$(7.12) \quad v_\lambda W^\lambda = V^\lambda w_\lambda = V'^\lambda w'_\lambda .$$

Proof. We have

$$V^\lambda w_\lambda = v_\alpha a^{\alpha\lambda} W^\beta a_{\beta\lambda} = v_\alpha W^\alpha .$$

The remaining part of the proof is obvious.

Note. If W' is a linear combination of a''_1, a''_2, a''_3 , then we have, by virtue of (7.1b),

$$W^\lambda h_{\lambda\mu} = W^\lambda a_{\lambda\mu} = w_\mu .$$

Hence in such cases w_μ is the $\mathcal{H}$ -polar plane to the point W' in P_3 .

In the subsequent chapters we shall need the tensors involved in the next two theorems.

Theorem (7.6). *Put*

$$(7.13) \quad (a) \quad K'_\lambda \stackrel{\text{def}}{=} k_{\lambda\mu} a^{\mu\nu}, \quad (b) \quad K'_{\lambda'} = k_{\lambda\mu} a'^{\mu\nu}.$$

Then

$$(7.14) \quad K'_\lambda = \sum_x \lambda_{\lambda}^x a'_x + \sum_{\lambda'} \lambda_{\lambda}^{\lambda'} a'_{\lambda'}$$

and

$$(7.15) \quad K'_{\lambda'} = K'_\lambda + \epsilon\rho \left[\sum_{\lambda} \lambda a'^{\lambda}_{\lambda'} + \epsilon\rho a^3_{\lambda'} a'_{\lambda'} \right].$$

Theorem (7.7). *The following two equations hold and are reproductive:*

$$(7.16a) \quad K_x^i - \lambda \delta_x^i = 0,$$

$$(7.16b) \quad K_{\lambda'}^i - \lambda \delta_{\lambda'}^i = 0.*$$

Proof. Multiplying (6.7b) by $a^{\mu\nu}$ one obtains (7.14) by virtue of (7.4). Multiplying (6.7b) by $a'^{\mu\nu}$ and making use of (7.7a) one obtains (7.15). Multiplying (7.14) by $a^{\lambda}_{\lambda'}$, one obtains (7.16a) by virtue of (5.1). By the same token one obtains (7.16b) by multiplying (7.14) by $a^{\lambda}_{\lambda'}$. Using

$$(7.17) \quad K'_{\lambda'} = \sum_x \lambda_{\lambda'}^x a'_x + \sum_{\lambda} \lambda_{\lambda'}^{\lambda} a'_{\lambda}$$

*See conditions (E) of the general outline.

instead of (7.14), one gets

$$(7.18) \quad \begin{aligned} (a) \quad K'_{x'}{}^{i'} - \lambda'_{x'} \delta_{x'}^{i'} &= 0, \\ (b) \quad K'_{x'}{}^{i'} - \lambda'_{x'} \delta_{x'}^{i'} &= 0 \end{aligned}$$

with

$$\lambda'_{i'} \stackrel{\text{def}}{=} \lambda_i.$$

Hence equations (7.16) are reproductive.

CHAPTER II. THE CONNECTION $\overset{0}{\Lambda}_\lambda{}^\nu{}_\mu$

8. The Tensor $C_{\lambda\mu}$. Let A_μ , $B_{\alpha\lambda}$ be components of two objects. We shall need very often their linear combination which we symbolize by

$$(8.1) \quad A_{(\mu} B_{\alpha\lambda)} \stackrel{\text{def}}{=} \frac{1}{2} [A_\mu B_{\alpha\lambda} + A_\lambda B_{\alpha\mu} - A_\alpha B_{\mu\lambda}].$$

Thus the covariant Christoffel symbols $\gamma_{\alpha\mu\lambda}$ of $h_{\lambda\mu}$ may be written as follows

$$(8.2) \quad \gamma_{\alpha\mu\lambda} = \partial_{(\mu} h_{\alpha\lambda)}.$$

They transform according to

$$(8.3) \quad \gamma_{\alpha'\mu'\lambda'} = A_{\alpha'}^\alpha [A_{\mu'}^\mu A_{\lambda'}^\lambda \gamma_{\alpha\lambda\mu} + h_{\alpha\beta} \partial_{\lambda'} A_{\mu'}^\beta]. *$$

Besides $\gamma_{\alpha\mu\lambda}$ we also have the Christoffel symbols $\left\{ \begin{smallmatrix} \nu \\ \lambda \mu \end{smallmatrix} \right\}$ of $a_{\lambda\mu}$. We denote by $\overset{\alpha}{\nabla}_\mu$ the symbol of the covariant derivative with respect to $\left\{ \begin{smallmatrix} \nu \\ \lambda \mu \end{smallmatrix} \right\}$. Moreover we shall use the abbreviations

$$(8.4) \quad H_\lambda{}^\nu \stackrel{\text{def}}{=} \delta_\lambda^\nu - \underset{4}{a}_\lambda^\nu a^\nu{}_\dagger$$

and

$$(8.5) \quad p_{\lambda\mu} \stackrel{\text{def}}{=} \overset{\alpha}{\nabla}_\mu \underset{4}{a}_\lambda^\alpha.$$

Theorem (8.1). Put

$$(8.6) \quad C_{\lambda\mu} \stackrel{\text{def}}{=} -a^\alpha{}_\dagger \gamma_{\alpha\lambda\mu} (= C_{\mu\lambda}).$$

*We put here

$$A_{\lambda'}^\mu \stackrel{\text{def}}{=} \frac{\partial x^\mu}{\partial x^{\lambda'}}, \quad A_{\lambda'}^{\nu'} \stackrel{\text{def}}{=} \frac{\partial x^{\nu'}}{\partial x^{\lambda'}}.$$

†Of course we have

$$H_{\lambda'}{}^{\nu'} = h_{\lambda\mu} a^{\lambda\nu}.$$

Then $C_{\lambda\mu}$ is a tensor and

$$(8.7) \quad C_{\lambda\mu} = H_{(\lambda}{}^{\alpha} p_{\mu)\alpha}.$$

The tensor $C_{\lambda\mu}$ transforms by (6.2) according to

$$(8.8) \quad C'_{\lambda\mu} = \epsilon C_{\lambda\mu}.$$

Proof. Using (1.4) one easily obtains from (8.3) and (8.6)

$$C_{\lambda'\mu'} = A_{\lambda'}^{\lambda} A_{\mu'}^{\mu} C_{\lambda\mu}.$$

Hence $C_{\lambda\mu}$ is a tensor. The equation (8.8) follows easily from (6.2) and (8.6).

The equation (8.6) may be rewritten by means of (7.1b) as

$$(8.9a) \quad C_{\lambda\mu} = -a_{\lambda}^{\alpha} [\partial_{(\lambda} a_{\alpha\mu)} - \partial_{[\lambda} \overset{4}{a}_{\alpha} \overset{4}{a}_{\mu]}].$$

Using normal coordinates (normal with respect to $\left\{ \begin{smallmatrix} \nu \\ \lambda \mu \end{smallmatrix} \right\}$) we see at once that (8.9a) is equivalent to

$$(8.9b) \quad C_{\lambda\mu} = a_{\lambda}^{\alpha} \overset{a}{\nabla}_{[\lambda} \overset{4}{a}_{\alpha} \overset{4}{a}_{\mu]}.$$

In order to evaluate the right hand term let us observe that (7.4a) yields for $j = 4$

$$\overset{4}{a}_{\lambda} = a_{\lambda}^{\nu} a_{\nu}^4$$

and that, by virtue of (5.1) for $i = j = 4$,

$$(8.10) \quad p_{\lambda\mu} a_{\lambda}^{\lambda} = a_{\lambda}^{\lambda} \overset{a}{\nabla}_{\mu} \overset{4}{a}_{\lambda} = 0.$$

Using (8.10) we easily obtain

$$(8.11) \quad \begin{aligned} a_{\lambda}^{\alpha} \overset{a}{\nabla}_{[\lambda} \overset{4}{a}_{\alpha} \overset{4}{a}_{\mu]} &= p_{(\lambda\mu)} - \frac{1}{2} (a_{\lambda}^{\alpha} \overset{4}{a}_{\alpha} p_{\mu} + a_{\mu}^{\alpha} \overset{4}{a}_{\alpha} p_{\lambda}) \\ &= \delta_{(\mu}^{\alpha} p_{\lambda)\alpha} - a_{\mu}^{\alpha} \overset{4}{a}_{\alpha} p_{\lambda} = H_{(\mu}{}^{\alpha} p_{\lambda)\alpha}. \end{aligned}$$

The equations (8.9b) and (8.11) yield (8.7).

Corollary (8.1). We have

$$(8.12) \quad C_{\lambda\mu} a_{\lambda}^{\lambda} = 0.$$

Proof. Using (8.7) we obtain, by virtue of (8.10) and (8.4),

$$(8.13) \quad 2C_{\lambda\mu} a_{\lambda}^{\lambda} = (H_{\lambda}{}^{\alpha} a_{\lambda}^{\lambda}) p_{\mu\alpha} + H_{\mu}{}^{\alpha} p_{\lambda\alpha} a_{\lambda}^{\lambda} = 0.$$

Note. It is well known that there exists a coordinate system in which the

components of a given vector field have the values δ_{iV}^r . In particular, every coordinate system y^r for which

$$(8.14) \quad a^r_4 = \delta_{iV}^r$$

will be termed a *preferred y -system*.

Theorem (8.2). *We have in every preferred y -system*

$$(8.15) \quad (a) \quad h_{iV\lambda} = 0, \quad (b) \quad a^4_{iV} = 1$$

and

$$(8.16) \quad C_{\lambda\mu} = \frac{1}{2} \partial_{iV} h_{\lambda\mu}.$$

Proof. The equation (8.15a) follows from (8.14) and (1.4). The equation (8.15b) is a consequence of (8.14) and (5.1a) for $i = j = 4$. The equation (8.16) follows from (8.6), (8.14) and (8.15a).

9. The Connection $\overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu}$.

Theorem (9.1). *Put*

$$(9.1) \quad \overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu} \stackrel{\text{def}}{=} a^{r\alpha} \gamma_{\alpha\lambda\mu} + a^r_4 \partial_{(\lambda} a_{\mu)}^4 (= \overset{0}{\Lambda}_{\mu}{}^r{}_{\lambda}).$$

Then $\overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu}$ is a connection which satisfies the following condition:

$$(9.2) \quad \overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu} h_{r\omega} = \gamma_{\omega\lambda\mu} + a^4_{\omega} C_{\lambda\mu}.$$

Proof. Making use of (1.4) we obtain from (9.1)

$$\begin{aligned} \overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu} h_{r\omega} &= H_{\omega}{}^{\alpha} \gamma_{\alpha\lambda\mu} + h_{r\omega} a^r_4 \partial_{(\lambda} a_{\mu)}^4 \\ &= (\delta_{\omega}^{\alpha} - a^4_{\omega} a^{\alpha}) \gamma_{\alpha\lambda\mu}. \end{aligned}$$

This equation, together with (8.6), leads to (9.2). The equation (8.3) yields

$$a^{r'\alpha'} \gamma_{\alpha'\lambda'\mu'} = A^{r'}_{\lambda'} [A_{\lambda'}^{\lambda} A_{\mu'}^{\mu} a^{r\alpha} \gamma_{\alpha\lambda\mu} + H_{\beta'}^r \partial_{\lambda'} A_{\mu'}^{\beta'}].$$

Moreover,

$$a^{r'}_4 \partial_{(\lambda'} a_{\mu')}^4 = A^{r'}_{\lambda'} [A_{\lambda'}^{\lambda} A_{\mu'}^{\mu} a^r_4 \partial_{\lambda} a_{\mu}^4 + a^r_4 a_{\beta'}^4 \partial_{(\lambda'} A_{\mu')}^{\beta'}].$$

Therefore, by virtue of (9.1),

$$\begin{aligned} (9.3) \quad \overset{0}{\Lambda}_{\lambda'}{}^{r'}{}_{\mu'} &= A^{r'}_{\lambda'} [A_{\lambda'}^{\lambda} A_{\mu'}^{\mu} \overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu} + (H_{\beta'}^r + a^r_4 a_{\beta'}^4) \partial_{\lambda'} A_{\mu'}^{\beta'}] \\ &= A^{r'}_{\lambda'} [A_{\lambda'}^{\lambda} A_{\mu'}^{\mu} \overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu} + \partial_{\lambda'} A_{\mu'}^{r'}]. \end{aligned}$$

Hence $\overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu}$ as defined by (9.1) is a connection.

Theorem (9.2). Denote by ∇_μ the symbol of the covariant derivative with respect to $\overset{0}{\Lambda}_\lambda{}^\mu$. Then

$$(9.4) \quad \overset{0}{\nabla}_\mu \overset{4}{a}_\lambda = \partial_{[\mu} \overset{4}{a}_{\lambda]} + C_{\lambda\mu}.$$

Proof. We have, by virtue of (9.1),

$$\begin{aligned} \overset{0}{\nabla}_\mu \overset{4}{a}_\lambda &= \partial_\mu \overset{4}{a}_\lambda - \overset{4}{a}_\nu [a^\nu{}_\alpha \gamma_{\alpha\lambda\mu} + a^\nu{}_\alpha \partial_{(\lambda} \overset{4}{a}_{\mu)}] \\ &= \partial_\mu \overset{4}{a}_\lambda - \overset{4}{a}^\alpha \gamma_{\alpha\lambda\mu} - \partial_{(\lambda} \overset{4}{a}_{\mu)}. \end{aligned}$$

This equation in conjunction with (8.6) leads to (9.4).

Note. In the next chapter we shall see that our problem of solving the equation (A) revolves around the existence of a symmetric connection $\overset{0}{\Lambda}_\lambda{}^\mu$ which is a solution of

$$(9.5) \quad \overset{0}{\Lambda}_\lambda{}^\mu h_{\nu\omega} = \gamma_{\omega\lambda\mu}.$$

Theorem (9.3). A necessary and sufficient condition for the existence of a symmetric connection $\overset{0}{\Lambda}_\mu{}^\lambda$ which is a solution of (9.5) is

$$(9.6) \quad C_{\lambda\mu} = 0.$$

If this condition is satisfied then $\overset{0}{\Lambda}_\lambda{}^\mu$, defined by (9.1), is a solution of (9.5).

Proof. Multiplying (9.5) by $a^\omega{}_4$ we obtain, by virtue of (1.4) and (8.6), the equation (9.6). The sufficiency of (9.6) follows from (9.2). The remaining part of the proof is obvious.

Agreement (9.1). Throughout this paper we shall assume the validity of (9.6).

Theorem (9.4). In every preferred y -system we have besides (8.15) also

$$(9.7) \quad h_{\lambda\mu} = h_{\lambda\mu}(y^I, y^{II}, y^{III}).$$

The proof follows at once from (8.16) and (9.6).

In the next theorem we shall use the tensor $p_{\lambda\mu}$ defined by (8.5).

Theorem (9.5). We have

$$(9.8) \quad p_{\lambda\mu} a^\lambda{}_u a^\mu{}_v = a^\lambda{}_u a^\mu{}_v \partial_{[\mu} \overset{4}{a}_{\lambda]} \quad (u, v = 1, 2, 3).$$

Proof. Using (8.7) and (9.6) we easily obtain the following non-holonomic components:

$$(9.9) \quad \begin{aligned} (a) \quad C_{u\cdot} &= p_{(u\cdot)} = 0 \\ (b) \quad 2C_{i4} &= a^\lambda{}_i a^\mu{}_4 (H_\lambda{}^\alpha p_{\mu\alpha} + H_\mu{}^\alpha p_{\lambda\alpha}) = 0. \end{aligned}$$

Therefore we find, by virtue of (8.5),

$$p_{u^v} = p_{[u^v]} = a_{\underset{u}{\lambda}}^{\lambda} a_{\underset{v}{\mu}}^{\mu} \overset{0}{\nabla}_{[\mu} \overset{4}{a}_{\lambda]} = a_{\underset{u}{\lambda}}^{\lambda} a_{\underset{v}{\mu}}^{\mu} \partial_{[\mu} \overset{4}{a}_{\lambda]} .$$

10. Continuation.

Theorem (10.1). *The tensor $\overset{0}{\nabla}_{\mu} \overset{4}{a}_{\lambda}$ is skew symmetric:*

$$(10.1) \quad \overset{0}{\nabla}_{\mu} \overset{4}{a}_{\lambda} = \partial_{[\mu} \overset{4}{a}_{\lambda]} .$$

The proof follows from (9.4) and (9.6).

Theorem (10.2). *Put*

$$(10.2a) \quad \overset{0}{w}_{\mu} \stackrel{\text{def}}{=} -a_{\underset{4}{\lambda}}^{\lambda} \overset{0}{\nabla}_{\mu} \overset{4}{a}_{\lambda} = a_{\underset{4}{\lambda}}^{\lambda} \partial_{[\lambda} \overset{4}{a}_{\mu]} .$$

The vector $\overset{0}{w}_{\mu}$ is not frame invariant:

$$(10.3) \quad \overset{0}{w}'_{\mu} = a_{\underset{4'}{\lambda}}^{\lambda} \partial_{[\lambda} \overset{4'}{a}_{\mu]} = \overset{0}{w}_{\mu} - \epsilon a_{\underset{4}{\lambda}}^{\lambda} \partial_{[\lambda} \rho \overset{3}{a}_{\mu]} .$$

Nevertheless the equation

$$(10.2b) \quad \overset{0}{w}_4 = 0$$

is not only reproductive, but also frame invariant.

Proof. Using (6.2) we easily obtain (10.3). Therefore

$$\overset{0}{w}'_4 = \overset{0}{w}'_{\underset{4'}{4}} a^{\mu} = \epsilon a_{\underset{4'}{\lambda}}^{\lambda} [\overset{0}{w}_{\mu} - \epsilon a_{\underset{4}{\lambda}}^{\lambda} \partial_{[\lambda} \rho \overset{3}{a}_{\mu]}] = \epsilon \overset{0}{w}_{\underset{4}{\mu}} a^{\lambda} = \epsilon \overset{0}{w}_4 = \overset{0}{w}_4 = 0 .$$

Hence (10.2b) is not only reproductive, but also frame invariant.

Note. If $v^r = dx^r/dp$ is the tangent vector of a line whose differential equations are

$$(10.4) \quad v^{\lambda} (\overset{0}{\nabla}_{\lambda} v^{|\mu}) v^{\mu} = 0 ,$$

then we say that this curve is a v^r -path (with respect to $\overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu}$).

Theorem (10.3). *We have*

$$(10.5a) \quad \overset{0}{\nabla}_{\mu} a^r_{\underset{4}{\mu}} = \overset{0}{w}_{\underset{4}{\mu}} a^r_{\underset{4}{\mu}} .$$

Therefore the curves defined by

$$(10.5b) \quad a^r_{\underset{4}{\mu}} = dx^r/dp$$

are $a^r_{\underset{4}{\mu}}$ -paths (with respect to $\overset{0}{\Lambda}_{\lambda}{}^r{}_{\mu}$):

$$(10.5c) \quad a^{\mu} \overset{0}{\nabla}_{\mu} a^r_{\underset{4}{\mu}} = 0 .$$

Proof. Using a preferred y -system we get, according to (8.14), (8.15), (9.7) and (10.2a),

$$\begin{aligned}\overset{0}{\nabla}_{\mu} a^{\nu}_{\lambda} &= \partial_{\mu} a^{\nu}_{\lambda} + a^{\nu\alpha}_{\lambda} \gamma_{\alpha\lambda\mu} a^{\lambda}_{\lambda} + a^{\nu}_{\lambda} a^{\lambda}_{\lambda} \partial_{(\lambda} a_{\mu)} \\ &= \frac{1}{2} a^{\nu}_{\lambda} \partial_{IV} a^{\lambda}_{\mu} = a^{\nu}_{\lambda} \partial_{[IV} a_{\mu]} = a^{\nu}_{\lambda} w^0_{\mu}.\end{aligned}$$

The remaining part of the proof is obvious.

Theorem (10.4). *The tensor $h_{\lambda\mu}$ is covariant constant with respect to $\overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu}$,*

$$(10.6) \quad \overset{0}{\nabla}_{\omega} h_{\lambda\mu} = 0.$$

Proof. Starting with (9.5) we obtain

$$\overset{0}{\Lambda}_{\lambda}{}^{\alpha}{}_{\omega} h_{\alpha\mu} + \overset{0}{\Lambda}_{\mu}{}^{\alpha}{}_{\omega} h_{\lambda\alpha} = \gamma_{\mu\lambda\omega} + \gamma_{\lambda\mu\omega} = \partial_{\omega} h_{\lambda\mu},$$

and this equation is equivalent to (10.6).

Corollary (10.1). *We have*

$$(10.7) \quad \begin{aligned}(a) \quad & \overset{0}{(\nabla}_{\omega} a^{\lambda})^{\bar{x}}_{\lambda} = -(\overset{0}{\nabla}_{\omega} \bar{a}^x_{\lambda}) a^{\lambda} = 0, \\ (b) \quad & \overset{0}{(\nabla}_{\omega} a^{\lambda})^3_{\lambda} = -(\overset{0}{\nabla}_{\omega} \bar{a}^3_{\lambda}) a^{\lambda} = 0.\end{aligned}$$

Proof. Starting with

$$0 = a^{\lambda}_{\lambda} a^{\mu}_{\lambda\mu} = a^{\lambda}_{\lambda} \bar{a}^{\bar{x}}_{\lambda} = a^{\lambda}_{\lambda} a^{\mu}_{\lambda\mu}$$

we obtain, according to (10.6),*

$$\overset{0}{(\nabla}_{\omega} a^{\lambda})^{\bar{x}}_{\lambda} + \overset{0}{(\nabla}_{\omega} \bar{a}^x_{\lambda}) a^{\lambda} = 2\overset{0}{(\nabla}_{\omega} a^{\lambda}) a_{\lambda} = 2\overset{0}{(\nabla}_{\omega} a^{\lambda}) \bar{a}^x_{\lambda} = 0,$$

and this equation yields (10.7a). The equation (10.7b) may be proved in a similar manner.

Note I. The frame invariance of (10.1), (10.5), (10.6) will be proved later.

Note II. We have at our disposal two tensors $h_{\lambda\mu}$ and $a_{\lambda\mu}$. The tensor $h_{\lambda\mu}$ gave rise to the connection $\overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu}$, the tensor $a_{\lambda\mu}$ might be used for constructing the connection of its Christoffel symbols $\left\{ \begin{smallmatrix} \nu \\ \lambda \mu \end{smallmatrix} \right\}$. The question arises under what condition these two connections are identical.

*

$$a_{\lambda} = a^{\mu}_{\lambda} h_{\lambda\mu} = a^{\mu}_{\lambda} a_{\lambda\mu} = \bar{a}^{\bar{x}}_{\lambda}.$$

Theorem (10.5). *A necessary and sufficient condition for*

$$(10.8) \quad \overset{0}{\Lambda}_\lambda{}^\nu{}_\mu = \{\lambda{}^\nu{}_\mu\}$$

is: The vector field $\overset{4}{a}_\lambda$ is a gradient. This condition is not frame invariant.

Proof. The equation (10.8) holds if and only if

$$(10.9a) \quad \overset{0}{\nabla}_\omega a_{\lambda\mu} = 0.$$

This equation is equivalent to

$$(10.9b) \quad \overset{0}{\nabla}_\omega h_{\lambda\mu} + \overset{0}{\nabla}_\omega \overset{4}{a}_\lambda \overset{4}{a}_\mu = \overset{0}{\nabla}_\omega \overset{4}{a}_\lambda \overset{4}{a}_\mu = 0.$$

If $\overset{4}{a}_\mu$ is a gradient, then we have, by virtue of (10.1),

$$\overset{0}{\nabla}_\omega \overset{4}{a}_\lambda = 0,$$

and (10.9) are satisfied. If (10.9b) is satisfied then also

$$(\overset{0}{\nabla}_\omega \overset{4}{a}_\lambda) \overset{4}{a}_\mu \overset{4}{a}^\lambda + \overset{0}{\nabla}_\omega \overset{4}{a}_\mu = 0,$$

or according to (10.1) and (10.2)

$$(10.10) \quad \overset{4}{a}_\mu \overset{0}{w}_\omega = \partial_{[\omega} \overset{4}{a}_{\mu]}.$$

The right hand side of (10.10) is skew symmetric so that

$$\overset{4}{a}_{(\mu} \overset{0}{w}_{\omega)} = 0.$$

This equation requires

$$(10.11) \quad \overset{0}{w}_\omega = 0$$

so that (10.10) reduces to

$$(10.12) \quad \partial_{[\omega} \overset{4}{a}_{\mu]} = 0.$$

Hence $\overset{4}{a}_\mu$ is a gradient. The equation (10.12) is not frame invariant, for

$$(10.13) \quad \partial_{[\omega} \overset{4'}{a}_{\mu]} = \epsilon \partial_{[\omega} \overset{4}{a}_{\mu]} - \partial_{[\omega} \rho \overset{3}{a}_{\mu]}.$$

Note. The condition (10.12) imposed on $\overset{4}{a}_\mu$, not being frame invariant, will not be used. In other words, we shall not replace $\overset{0}{\Lambda}_\lambda{}^\nu{}_\mu$ by $\left\{ \overset{\nu}{\lambda}{}_\mu \right\}$ because (10.8) is not frame invariant.

11. The Connection $\overset{0}{\Lambda}_\lambda{}^\nu{}_\mu$ and Changes of Frame. The tensor $h_{\lambda\mu}$ is frame invariant and so also is $\gamma_{\alpha\lambda\mu}$:

$$(11.1) \quad (a) \quad h'_{\lambda\mu} = h_{\lambda\mu}, \quad (b) \quad \gamma'_{\alpha\lambda\mu} = \gamma_{\alpha\lambda\mu}.$$

This is, however, not the case with $\overset{0}{\Lambda}_\lambda{}^\nu{}_\mu$.

Theorem (11.1). Put

$$(11.2) \quad \overset{0}{\Lambda}_{\lambda}{}^{\prime\prime}{}_{\mu} \stackrel{\text{def}}{=} a^{\prime\prime\alpha} \gamma'_{\alpha\lambda\mu} + a^{\prime}{}_{\lambda}{}^{\prime}{}_{\mu} \partial_{(\lambda} \overset{4}{a}_{\mu)}$$

and

$$(11.3) \quad t_{\lambda\mu} \stackrel{\text{def}}{=} \overset{0}{\nabla}_{(\lambda} \overset{3}{a}_{\mu)} \rho.$$

Then

$$(11.4) \quad \overset{0}{\Lambda}_{\lambda}{}^{\prime\prime}{}_{\mu} = \overset{0}{\Lambda}_{\lambda}{}^{\prime}{}_{\mu} - \epsilon a^{\prime}{}_{\lambda}{}^{\prime}{}_{\mu} t_{\lambda\mu},$$

$$(11.5) \quad t'_{\lambda\mu} \stackrel{\text{def}}{=} \overset{0}{\nabla}'_{(\lambda} \overset{3'}{a}_{\mu)} \rho = \epsilon t_{\lambda\mu}^*$$

and

$$(11.6) \quad \overset{0}{\Lambda}_{\lambda}{}^{\prime\prime}{}_{\mu} h_{\nu\omega} = \gamma_{\omega\lambda\mu}.$$

Proof. We have, according to (7.7a), (9.6) and (11.1b),

$$(11.7) \quad a^{\prime\prime\alpha} \gamma'_{\alpha\lambda\mu} = a^{\prime\alpha} \gamma_{\alpha\lambda\mu} + \epsilon \rho a^{\prime}{}_{\lambda}{}^{\alpha}{}_{\mu} \gamma_{\alpha\lambda\mu}.$$

Moreover, if we make use of (6.2), then

$$(11.7b) \quad a^{\prime}{}_{\lambda}{}^{\alpha}{}_{\mu} \partial_{(\lambda} \overset{4}{a}_{\mu)} = \epsilon a^{\prime}{}_{\lambda}{}^{\alpha}{}_{\mu} [\epsilon \overset{4}{a}_{\mu)} - \rho \overset{3}{a}_{\mu)}] = a^{\prime}{}_{\lambda}{}^{\alpha}{}_{\mu} \partial_{(\lambda} \overset{4}{a}_{\mu)} - \epsilon a^{\prime}{}_{\lambda}{}^{\alpha}{}_{\mu} \partial_{(\lambda} \overset{3}{a}_{\mu)} \rho.$$

Therefore according to (11.7), (11.2) and (9.1)

$$(11.8) \quad \overset{0}{\Lambda}_{\lambda}{}^{\prime\prime}{}_{\mu} = \overset{0}{\Lambda}_{\lambda}{}^{\prime}{}_{\mu} + \epsilon a^{\prime}{}_{\lambda}{}^{\alpha}{}_{\mu} [\rho \alpha^{\alpha} \gamma_{\alpha\lambda\mu} - \partial_{(\lambda} \overset{3}{a}_{\mu)} \rho].$$

Using (9.1) again we obtain

$$\rho a^{\alpha}{}_{\lambda}{}^{\alpha}{}_{\mu} \gamma_{\alpha\lambda\mu} - \partial_{(\lambda} \overset{3}{a}_{\mu)} \rho = \rho a^{\alpha}{}_{\lambda}{}^{\alpha}{}_{\mu} (\overset{0}{\Lambda}_{\lambda}{}^{\prime}{}_{\mu} a_{\nu\alpha} - \overset{4}{a}_{\alpha} \partial_{(\lambda} \overset{4}{a}_{\mu)}) - \partial_{(\lambda} \overset{3}{a}_{\mu)} \rho = -\overset{0}{\nabla}_{(\lambda} \overset{3}{a}_{\mu)} \rho.$$

Substituting from this equation into (11.8) we obtain (11.4). Therefore

$$\overset{0}{\nabla}'_{(\lambda} \overset{3'}{a}_{\mu)} \rho = \overset{0}{\nabla}_{(\lambda} \overset{3}{a}_{\mu)} \rho + \epsilon a^{\alpha}{}_{\lambda}{}^{\alpha}{}_{\mu} \rho t_{\mu\lambda} = \epsilon \nabla_{(\lambda} \overset{3}{a}_{\mu)} \rho.$$

This equation is equivalent to (11.5). The equation (11.6) is obvious.

Note. We are now able to check the frame invariance of some of our previous results.

Theorem (11.2). The equations (10.1), (10.5) and (10.6) are frame invariant.

* $\overset{0}{\nabla}'_{\lambda}$ is the symbol of the covariant derivative with respect to $\overset{0}{\Lambda}_{\lambda}{}^{\prime}{}_{\mu}$.

Proof. A simple inspection based on (11.2) and (10.3) shows that

$$\begin{aligned}\overset{0}{\nabla}'_{\mu} a^{\nu}_{\lambda} &= \epsilon \overset{0}{\nabla}_{\mu} a^{\nu}_{\lambda} - a^{\nu}_{\lambda} a^{\lambda} t_{\lambda\mu}, \\ \overset{0}{w}'_{\mu} a^{\nu}_{\lambda} &= \epsilon a^{\nu}_{\lambda} \overset{0}{w}_{\mu} - a^{\nu}_{\lambda} a^{\lambda} \partial_{[\lambda} a_{\mu]}^3 \rho.\end{aligned}$$

Therefore

$$\begin{aligned}(11.9) \quad \overset{0}{\nabla}'_{\mu} a^{\nu}_{\lambda} - \overset{0}{w}'_{\mu} a^{\nu}_{\lambda} &= \epsilon (\overset{0}{\nabla}_{\mu} a^{\nu}_{\lambda} - \overset{0}{w}_{\mu} a^{\nu}_{\lambda}) - a^{\nu}_{\lambda} a^{\lambda} (t_{\lambda\mu} - \partial_{[\lambda} a_{\mu]}^3 \rho) \\ &= \epsilon (\overset{0}{\nabla}_{\mu} a^{\nu}_{\lambda} - \overset{0}{w}_{\mu} a^{\nu}_{\lambda}) - a^{\nu}_{\lambda} (a^{\lambda} \overset{0}{\nabla}_{\mu} a_{\lambda}^3 \rho).\end{aligned}$$

On the other hand, by virtue of (10.5a)

$$a^{\lambda} \overset{0}{\nabla}_{\mu} a_{\lambda}^3 \rho = -\rho a_{\lambda}^3 \overset{0}{\nabla}_{\mu} a^{\lambda} = -\rho a_{\lambda}^3 \overset{0}{w}_{\mu} a^{\lambda} = 0.$$

Hence (11.9) reduces to

$$\overset{0}{\nabla}'_{\mu} a^{\nu}_{\lambda} - \overset{0}{w}'_{\mu} a^{\nu}_{\lambda} = \epsilon (\overset{0}{\nabla}_{\mu} a^{\nu}_{\lambda} - \overset{0}{w}_{\mu} a^{\nu}_{\lambda}).$$

Therefore (10.5a, c) are frame invariant. The frame invariance of (10.1) may be proved in a similar manner. The frame invariance of (10.6) follows from (11.1a), (11.4) and (1.4).

12. F-components of $\overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu}$.

Theorem (12.1). *The F-components $\overset{0}{\Lambda}_i{}^i{}_k$ of $\overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu}$ are*

$$(12.1) \quad \overset{0}{\Lambda}_i{}^i{}_k = -a^{\lambda}_i a^{\mu}_k \overset{0}{\nabla}_{\mu} a_{\lambda}^i = a_{\lambda}^i a^{\mu}_k \overset{0}{\nabla}_{\mu} a^{\lambda}_i.$$

Proof. One obtains (12.1) from the usual transformation rule

$$(12.2) \quad \overset{0}{\Lambda}_i{}^i{}_k = a_{\lambda}^i (a^{\lambda}_i a^{\mu}_k \overset{0}{\Lambda}_{\mu}{}^{\nu}{}_{\nu} + \partial_k a^{\nu}_i) \quad (\partial_k \stackrel{\text{def}}{=} a^{\lambda}_k \partial_{\lambda}).$$

A simple inspection shows the equivalence of (12.2) and (12.1).

Note. If $T_i{}^{i}{}_{\dots}$ are F-components of a tensor, then

$$\begin{aligned}(12.3) \quad \overset{0}{\nabla}_k T_i{}^{i}{}_{\dots} &\stackrel{\text{def}}{=} a^{\mu}_k (a_{\nu}^i a^{\lambda}_i \dots \overset{0}{\nabla}_{\mu} T_{\lambda}{}^{\nu}{}_{\dots}) \\ &= \partial_k T_i{}^{i}{}_{\dots} + T_i{}^{i}{}_{\dots} \overset{0}{\Lambda}_i{}^i{}_k + \dots - T_i{}^{i}{}_{\dots} \overset{0}{\Lambda}_i{}^i{}_k - \dots.\end{aligned}$$

We shall use this formula very often in subsequent considerations.

Theorem (12.2). *The F -components $\overset{0}{\Lambda}_i{}^i{}_k$ satisfy the following conditions:*

$$(12.4) \quad \begin{aligned} (a) \quad & \overset{0}{\Lambda}_{x^*}{}^x{}_i = 0, & (b) \quad & \sum^x \overset{0}{\Lambda}_x{}^x{}_i = 0, \\ (c) \quad & \overset{0}{\Lambda}_x{}^3{}_i + \overset{0}{\Lambda}_3{}^{x*}{}_i = 0, & (d) \quad & \overset{0}{\Lambda}_3{}^3{}_i = 0, \\ (e) \quad & \overset{0}{\Lambda}_4{}^i{}_k = \delta_i^0 w_k, & (f) \quad & \overset{0}{\Lambda}_{(i}{}^4{}_{k)} = 0. \end{aligned}$$

Proof. The equations (12.4a-d) follow directly from the formula

$$(12.5) \quad \begin{aligned} 0 &= \overset{0}{\nabla}_i h_{u^*} = \partial_i h_{u^*} - \overset{0}{\Lambda}_u{}^i h_{i^*} - \overset{0}{\Lambda}_v{}^i h_{u^*} \\ &= -(\overset{0}{\Lambda}_u{}^i h_{i^*} + \overset{0}{\Lambda}_v{}^i h_{u^*}) \quad (u, v = 1, 2, 3). \end{aligned}$$

The equations (12.4e, f) may be obtained from (10.5a) and (10.1) rewritten in F -components. Thus, for instance,

$$\overset{0}{\nabla}_k a^i{}_4 = \partial_k a^i{}_4 + \overset{0}{\Lambda}_i{}^i{}_k a^i{}_4 = \overset{0}{\Lambda}_4{}^i{}_k = \overset{0}{w}_k a^i{}_4 = \overset{0}{w}_k \delta_4^i.$$

The equations (12.4d, e) have an interesting consequence.

Theorem (12.3). *We have*

$$(12.6) \quad \overset{0}{\nabla}_\mu a^3{}_\lambda = \sum^x c_{kx} \overset{k}{a}_\mu{}^x{}_\lambda; \quad c_{kx} \stackrel{\text{def}}{=} -\overset{0}{\Lambda}_x{}^3{}_k.$$

Proof. We may write

$$(12.7) \quad \begin{aligned} (a) \quad & \overset{0}{\nabla}_\mu a^3{}_\lambda = c_{ki} \overset{k}{a}_\mu{}^i{}_j, \\ (b) \quad & a^\mu \overset{0}{\nabla}_\mu a^3{}_\lambda = c_{ki} a^i{}_j. \end{aligned}$$

According to (12.4d, e) we must have

$$c_{kx} = 0,$$

and therefore (12.7a) reduces to (12.6).

Theorem (12.4.) *The equations (12.4) have the following properties:*

- (1) *They are reproductive.**
- (2) *The equations (12.4a-e) hold also with primed indices.†*

Thus, for instance, we have $\overset{0}{\Lambda}'_{x^}{}^{x'}{}_{i'} = 0$, and so on.

†I.e. we have also $\overset{0}{\Lambda}_{x^*}{}^{x'}{}_{i'} = 0$, and so on. However the left-hand terms in (12.4) are not frame invariant. Thus, for instance,

$$\overset{0}{\Lambda}_{x^*}{}^{x'}{}_{i'} = A^i{}_{i'} A^{x*}{}_{x^*} A^{x'}{}_{x^*} \overset{0}{\Lambda}_{x^*}{}^x{}_i.$$

Proof. Both statements follow easily from the manner in which conditions (12.4) were obtained. Thus, for instance, (10.5a) being frame invariant, we may start with

$$\overset{0}{\nabla}'_{\mu} a^{\nu} = \overset{0}{w}'_{\mu} a^{\nu}$$

and express this equation in non-holonomic components with respect to the frame $F'(a^{\nu}_{i'}, \dot{a}_{\lambda})$.

In doing so we obtain

$$(12.8) \quad \overset{0}{\Lambda}'_{i', k'} = \delta_{i', k'}^{\overset{0}{w}'_{k'}}.$$

On the other hand, if we multiply (10.5a) by ϵ we obtain

$$\nabla_{\mu} a^{\nu} = w_{\mu} a^{\nu},$$

and this equation leads to

$$(12.9) \quad \overset{0}{\Lambda}_{i', k'} = \delta_{i', k'}^{\overset{0}{w}_{k'}}.$$

The equations (12.8), (12.9) prove both statements about (12.4e). The remaining equations (12.4a-d) lead in a similar manner to both statements of our theorem.

Note. Theorem (12.4) may also be proved by means of the following equations:

$$(12.10a) \quad \overset{0}{\Lambda}'_{i', k'} = \overset{0}{\Lambda}_{i', k'} - \delta_{i', k'}^{\overset{0}{t}_{i', k'}},$$

$$(12.10b) \quad \overset{0}{\Lambda}_{i', k'} = A_a^{i'} [A_{i'}^b A_k^c \Lambda_b^a + \partial_k A_{i'}^a] \quad (\partial_k \stackrel{\text{def}}{=} A_{k'}^b \partial_b = a^{\mu}_{k'} \partial_{\mu}).$$

The equation (12.10a) follows from (11.4). The equation (12.10b) expresses the known transformation rule for connection coefficients.

13. The Connection Λ_{λ}^{ν} . In subsequent considerations we shall denote by $b_{\lambda\mu}$ an arbitrary symmetric tensor which transforms by (6.2) according to

$$(13.1) \quad b'_{\lambda\mu} = \epsilon b_{\lambda\mu}.$$

Theorem (13.1). *The most general solution of*

$$(13.2) \quad \Lambda_{\lambda}^{\nu} h_{\nu\omega} = \gamma_{\omega\lambda\mu}$$

*We have, by virtue of (10.3),

$$\overset{0}{w}'_{k'} = \overset{0}{w}'_{\mu} a^{\mu}_{k'} = \overset{0}{w}_{k'} - a^{\lambda}_{4' k'} \partial_{[\lambda} \overset{3}{a}_{\mu]} \rho.$$

†It is not difficult to find a tensor with the transformation rule (13.1). For instance, $\overset{0}{\nabla}_{(\mu} \dot{a}_{\lambda)}$ is such a tensor.

is

$$(13.3) \quad \Lambda_{\lambda}{}^{\nu}{}_{\mu} = \overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu} + a^{\nu}{}_{\underset{4}{b}_{\lambda\mu}} ,$$

where $b_{\lambda\mu}$ is an arbitrary symmetric tensor. This connection is not frame invariant. If (13.1) holds then

$$(13.4) \quad \Lambda_{\lambda}{}^{\nu}{}_{\mu} = \overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu} - \epsilon a^{\nu}{}_{\underset{4}{b}_{\lambda\mu}} .$$

Proof. Recalling (1.4) we see that any two solutions Λ and $\overset{0}{\Lambda}$ of (13.2) must be related by

$$\Lambda_{\lambda}{}^{\nu}{}_{\mu} - \overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu} = a^{\nu}{}_{\underset{4}{b}_{\lambda\mu}} ,$$

where $b_{\lambda\mu}$ is an arbitrary symmetric tensor. Assuming (13.1) we obtain (13.4) from (13.3) and (11.4).

Theorem (13.2). *If we denote by ∇_{μ} the symbol of the covariant derivative with respect to $\Lambda_{\lambda}{}^{\nu}{}_{\mu}$ then*

$$(13.5) \quad \begin{aligned} (a) \quad & \nabla_{\mu} \overset{4}{a}_{\lambda} = \partial_{[\mu} \overset{4}{a}_{\lambda]} - b_{\mu\lambda} . \\ (b) \quad & \nabla_{\mu} a^{\nu}{}_{\underset{4}{\lambda}} = w_{\mu} a^{\nu}{}_{\underset{4}{\lambda}} , \\ (c) \quad & w_{\mu} = \overset{0}{w}_{\mu} + b_{\lambda\mu} a^{\lambda}{}_{\underset{4}{\lambda}} . \end{aligned}$$

These equations are frame invariant.

Proof. Using (10.1) and (13.3) one obtains at once (13.5a). On the other hand, (13.3) and (10.5a) lead to

$$\nabla_{\mu} a^{\nu}{}_{\underset{4}{\lambda}} = \overset{0}{\nabla}_{\mu} a^{\nu}{}_{\underset{4}{\lambda}} + a^{\nu}{}_{\underset{4}{b}_{\lambda\mu}} a^{\lambda}{}_{\underset{4}{\lambda}} = (\overset{0}{w}_{\mu} + b_{\lambda\mu} a^{\lambda}{}_{\underset{4}{\lambda}}) a^{\nu}{}_{\underset{4}{\lambda}} ,$$

and this equation yields (13.5b,c). The proof of frame invariance follows the pattern of the proof of Theorem (11.2).

Theorem (13.3). *The tensor $h_{\lambda\mu}$ is covariant constant with respect to the connection $\Lambda_{\lambda}{}^{\nu}{}_{\mu}$:*

$$(13.6) \quad \nabla_{\omega} h_{\lambda\mu} = 0 .$$

The proof is similar to the proof of Theorem (10.4).

In the next theorem we introduce the F -components of

$$(13.7) \quad \Lambda_i{}^{\dot{i}}{}_{\underset{k}{k}} = -a^{\lambda}{}_{\underset{i}{i}} a^{\mu}{}_{\underset{k}{k}} \nabla_{\mu} \overset{\dot{i}}{a}_{\lambda} = \overset{\dot{i}}{a}_{\lambda} a^{\mu}{}_{\underset{k}{k}} \nabla_{\mu} a^{\lambda}{}_{\underset{i}{i}} .$$

Theorem (13.4). *The equations (12.4a-d) hold also for Λ_i^i instead of Λ_i^0 . Moreover,*

$$(13.8) \quad (a) \quad \Lambda_{(i}^4{}_{k)} = b_{ik} , \quad (b) \quad \Lambda_4^i{}_k = \delta_4^i w_k .$$

Proof. The equation (13.8a), ((13.8b)) follows from (13.5a), ((13.5b)). The equations (12.4a-d) for Λ_i^i (instead of Λ_i^0) follow from (13.6).

This concludes the preliminary investigations necessary for the solution of our problem.

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