Plastic Flow and Fracture in Solids

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CONTENTS

1. Basic concepts and main results	292
Fracture along Lüders Bands	
2. Previous results and formulæ	296
3. Wave strength	297
4. Compatibility conditions of the second order	298
5. Further consequences of the compatibility conditions	298
6. Condition for fracture	300
Weak Stationary Discontinuities	
7. Compatibility conditions of the first order. Wave strength	303
8. Application of the condition of plane stress	304
9. Compatibility conditions of the second order	306
10. Stability of the wave	307
Weak Discontinuities whose Velocity of Propagation is Different from Zero	
11. Continuity of the density	309
12. Results obtained from the conditions of compatibility of the first order	309
13. Plane stress	310
14. Simplification by canonical coordinates	311
15. Case I ($\nu_1 = \pm 1, \nu_2 = 0$)	312
16. Case II ($\nu_1 = 0, \nu_2 = \pm 1$)	313
17. Case III ($\nu_1 \nu_2 \neq 0, \sigma_1^* = \sigma_2^*$)	313
18. Case IV ($\nu_1 \nu_2 \neq 0, \sigma_1^* \neq \sigma_2^*$)	313
Transition from Elastic to Plastic State without Fracture	
19. Preliminary remarks. Wave strength	315
20. Compatibility conditions of the second order	315
21. Further consequences of the compatibility conditions. Sta- bility of the wave	316

Fracture in the Direction of Tension

22. Basic formulæ of the problem	319
23. Fracture points. Propagation of cracks	321

1. Basic concepts and main results. In this investigation¹ we shall be concerned with an *ideal solid*² possessing the following properties. *First*, the dynamical behavior of the solid can be described by the ordinary or linear theory of elasticity for stresses satisfying the condition $\sigma_{ij}^* \sigma_{ij}^* < K$ where K is a material constant and the quantities σ_{ij}^* are the components of the stress deviator. *Second*, the solid has a sharply defined *yield point* given by the quadratic yield condition³

$$(1.1) \quad \sigma_{ij}^* \sigma_{ij}^* = K.$$

More explicitly, we understand by this latter requirement that when the stress condition (1.1) is satisfied the solid will enter the plastic state which, in this article, we shall assume to be governed by the ordinary form of the Prandtl-Reuss equations,⁴ namely

$$(1.2) \quad \frac{d\sigma_{ij}^*}{dt} = 2\mu(\epsilon_{ij} - \psi\sigma_{ij}^*),$$

in which μ is the usual elastic constant, ϵ_{ij} are the components of the rate of strain tensor and the quantity ψ is defined by

$$(1.3) \quad \psi = \frac{\epsilon_{ab}\sigma_{ab}^*}{K}.$$

As customary, we suppose the material in the plastic state to be incompressible as expressed by the condition

¹ This investigation was conducted at the request of the Naval Research Laboratory and has been supported by the Office of Naval Research under Contract Nonr-908(09), Indiana University, NR 041 037.

² In selecting the properties which define the *ideal solid* we have been influenced by the fact that a solid having precisely these properties has been chosen as the basis of the discussion in the recent book by W. PRAGER & P. G. HODGE, JR., *Theory of Perfectly Plastic Solids*, J. Wiley and Sons, New York, 1951. See also R. HILL, *The Mathematical Theory of Plasticity*, Oxford, Clarendon Press, 1950.

³ This yield condition was originally proposed by R. VON MISES, *Mechanik der festen Körper im plastisch-deformablen Zustand*, Nach. Ges. Wiss. Göttingen, Math.-Physik. Klasse (1913) pp. 582-592.

⁴ A dynamically correct form of the Prandtl-Reuss equations is obtained by replacing the ordinary time derivatives in the left member of (1.2) by the absolute time derivatives of the components σ_{ij}^* . For a brief discussion of this problem and references to earlier work see T. Y. THOMAS, *On the structure of the stress-strain relations*, Proc. N. A. S. 41 (1955) pp. 716-720; also *Kinematically preferred coordinate systems*, *ibid.* 41 (1955) pp. 762-770. The equations (1.2) have been used in this discussion since the additional terms which are introduced by the absolute time derivatives are either zero or negligible for the problem under consideration.

$$(1.4) \quad v_{i,i} = 0,$$

where the v_i are the components of the flow velocity and the comma denotes partial differentiation with respect to the coordinates of a suitably chosen and dynamically admissible rectangular system to which all formulæ in this article are referred. Because of the condition (1.4) there is no distinction between the components of the rate of strain tensor and its deviator ϵ^* and hence the symbol $*$, used to denote the formation of the deviator, can be omitted as we have done in writing the relations (1.2) and (1.3).

Experimental evidence that the onset of plastic flow can be a factor in the cause of failure in solids is to be found in the fact that plastic deformation, in greater or less degree, can usually be observed after the occurrence of fracture. This implies, from the mathematical standpoint, that fracture in solids is connected with the discontinuities in the velocity and stress components, or their derivatives, which necessarily arise in the transition from the equations of elasticity theory to the equations governing the plastic state.

It is commonly considered that plastic flow is initiated locally, *e.g.* as the result of the occurrence of a minute crack or crevice in the body where a high concentration of stress is attained. The plastic region thus formed then spreads or is propagated so as to include a larger domain within the body. Adopting this point of view we are led to the assumption of a plastic wave front Σ in the solid. On the rear of the surface Σ the material will be in the plastic state while in the immediate front of this surface we have elastically deformed material in the process of becoming plastic as the result of the propagation. *This conception enables us to confine the above mentioned discontinuities in the velocities and stresses to the wave surface Σ .* In particular, the wave surface Σ , or the associated discontinuity, may be *stationary* in the sense that its normal velocity of propagation is everywhere equal to zero, and, indeed, the first application which we shall make of these ideas to the problem of fracture in solids will involve a wave surface of this character.

Defining the strength of the wave Σ in a manner appropriate to the nature of the discontinuity associated with this surface, several situations can arise. In the first instance, the strength of the wave may remain finite at all points during the entire course of its propagation. If Σ is non-stationary the surface Σ will then sweep out a region of the solid which will become plastic. On the other hand, the wave surface Σ may approach a limiting position $\Sigma(T)$ as the time $t \rightarrow T$ such that at points of $\Sigma(T)$ the wave strength will be infinite. This situation must manifest itself physically as a fracture⁵ in the solid over the surface $\Sigma(T)$.

⁵ The analogous situation in the theory of gases is resolved by the formation of a *shock wave* in the absence of the possibility of fracture in a gas. An extension of the usual one dimensional treatment of this phenomenon in gas dynamics has recently been given by T. Y. THOMAS, *The growth and decay of sonic discontinuities in ideal gases*, Jour. Math. and Mech. **6** (1957) pp. 455-470. Cf. JOSEPH B. KELLER, *Geometrical acoustics. I. The Theory of weak shock waves*, Jour. App. Phys. **25** (1954) pp. 938-947.

Under other circumstances we may find that points of infinite wave strength develop on Σ which will generate a surface of fracture Σ^* , distinct from Σ , as a consequence of the wave propagation. In such cases the surface Σ^* will result from a *crack* having a definite velocity of propagation in the solid.

We have applied the above theory to the problem of fracture in thin plates under tension⁶ subject to the assumption that the elastic region is in a state of equilibrium.⁷ First considered is the question of fracture along the surface of separation Σ of a Lüders band and the surrounding elastic material. Now when we supplement the results of our previous discussion of these bands with additional information obtained from the conditions of compatibility of the second order⁸ we find that the strength of the wave Σ must become infinite within a finite time. *Hence the plate will fracture along the Lüders band when the tension, required to produce the band, remains constant as assumed in this discussion.* Because of the manner in which the load is usually applied there will be a small but distinct load drop at the instant of formation of the Lüders band which will be instrumental in preventing fracture. Even so, fracture may occur if the applied load is allowed to increase immediately after the appearance of the band (see §6).

In the Lüders band problem, as here treated, we have an example of a *strong* singularity in that not all of the discontinuities $[\sigma_{ij}]$ and $[v_i]$ vanish over the wave surface Σ . However, these discontinuities must satisfy the general dynamical conditions

$$(1.5) \quad \rho(G - v_n) = \bar{\rho}(G - \bar{v}_n),$$

$$(1.6) \quad [\sigma_{ij}]\nu^i = -\rho(G - v_n)[v_i],$$

where ρ and $\bar{\rho}$ denote the densities on the elastic and plastic sides of Σ respectively while v_n and $\bar{v}_n$ are the corresponding components of the normal velocity. The quantity G denotes the coordinate velocity of Σ in the direction of the unit normal vector ν which we suppose to be directed from the plastic to the elastic side of Σ . The surface Σ will be said to bear a *weak* discontinuity if $[\sigma_{ij}] = 0$ and $[v_i] = 0$ over Σ . More specifically, the order of the singularity is given by the order of the non-vanishing derivatives of the quantities σ_{ij} and v_i with surfaces Σ associated with a singularity of order zero (strong singularity) when not

⁶ The term tension is generally employed. However, all formulæ in this article are valid for compression as well as tension and similar or corresponding results can be stated.

⁷ Propagation of the plastic region into an elastic region in dynamical equilibrium is analogous to the propagation of a wave in gas dynamics into a gas at rest. Without this restriction on the elastic material there is the possibility of determining the influence on fracture of the rapidity with which the load is applied.

⁸ T. Y. THOMAS, *Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics*, Jour. of Math. and Mech. **6** (1957) pp. 311-322; also *A correction to "Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics,"* *ibid.* **6** (1957) pp. 907-908. No correction of any of the equations of the original article on extended compatibility conditions is necessary in case the normal velocity G is constant in space.

all of the quantities $[\sigma_{ij}]$ and $[v_i]$ vanish over the surface. When the singularity over the surface Σ is of weak type the conditions (1.5) and (1.6) are satisfied automatically.

A problem closely connected with the Lüders band problem has been treated for which the surface Σ is stationary and singular of order 1 relative to the stress components σ_{ij} . It is assumed that this surface is singular of order 1 *at least* relative to the velocity components v_i . Such wave surfaces are shown to be identical geometrically with the surfaces of separation of the elastic and plastic regions of the Lüders band problem, but their strength is finite and constant in time so that their formation does not result in fracture. Moreover, it is shown that all first and second derivatives of the velocity components v_i are continuous over these surfaces Σ . *Hence these surfaces are associated essentially with a weak form of discontinuity in the stress.* The occurrence of such weak discontinuities⁹ is the cause of the dulling or *cloud effect* which is observed on the flat sides of the plate when the yield point is reached.

Turning next to weak singular surfaces Σ whose normal velocity G is different from zero we find that four cases can arise when the plastic region is propagated into an elastic medium in dynamical equilibrium. However, only two of these cases are applicable to the problem of the flat plate under consideration. For Case I we find that points of infinite wave strength will not necessarily arise nor will such points develop on Σ during the course of propagation of this wave surface so that fracture should not occur (see §21). However, there is a possibility, although not an absolute necessity, that points of infinite wave strength, called *fracture points* in §23, will appear on the wave surfaces Σ of Case II. These fracture points will generate a crack which will be propagated in the direction of the tension (or compression) in the plate with a velocity given by $\sqrt{\mu/\rho}$. Such cracks have been produced by the compression of hollow cylinders¹⁰ which can be identified with the flat plate provided their radii are sufficiently large.

Strong discontinuities whose velocities of propagation are different from zero have not been treated in this article, and the investigation of such discontinuities in flat plates may lead to interesting and significant results. One can also consider the application of this theory of fracture to various other problems, *e.g.* the problem of fracture in cylindrical bars under tension, compression, or torsion, or a combination of such stress effects, for which experimental data on fracture is available.

FRACTURE ALONG LÜDERS BANDS

As a first application of the basic concepts of §1 we shall investigate the strength of the wave surface Σ separating the plastic region of the Lüders band and the

⁹ The weak discontinuities producing this effect appear to be referred to by some writers as *Lüders lines*. However, these lines are not to be confused with the *Lüders bands* which are associated with a type of strong discontinuity in the flat plate. See A. NADAI, *Theory of Flow and Fracture of Solids*, McGraw-Hill, New York, 1950.

¹⁰ See A. NADAI, *Theory of Flow and Fracture of Solids*, *loc. cit.*, p. 346.

surrounding region of the elastic material. Some of the results and formulæ from our previous articles on Lüders bands¹¹ will be used in this discussion. However, certain of these formulæ will appear in a somewhat simplified form due to the present assumption that the elastic region is in a state of dynamical equilibrium. The reader is referred to the first of the previous articles for the definition of the two coordinate systems employed and for the meaning of the various symbols.

2. Previous results and formulæ. From our foregoing work the surface Σ in the Lüders band problem is a plane which is perpendicular to the x_1, x_2 coordinate plane, *i.e.* the flat sides of the plate, and which has an inclination of $\pm 35^\circ 16'$ relative to the x_1 direction. Relative to the y coordinate system the formulæ for the derivatives $\bar{v}_{i,j}$ are given by

$$(2.1) \quad \begin{cases} \bar{v}_{1,1} = 0; & \bar{v}_{1,2} = \omega_1; & \bar{v}_{1,3} = \omega = 0, \\ \bar{v}_{2,1} = 0; & \bar{v}_{2,2} = \omega_2; & \bar{v}_{2,3} = 0, \\ \bar{v}_{3,1} = 0; & \bar{v}_{3,2} = \omega_3 = 0; & \bar{v}_{3,3} = -\omega_2, \end{cases}$$

where account is taken of the fact that the quantities ω and ω_3 were proved to be equal to zero. *It is explicitly assumed in this discussion that the quantities ω_1 and ω_2 do not vanish at any point of Σ .*

Since the velocity components $v_i = 0$ in the elastic region, by the present equilibrium assumption the condition of *slip* becomes $\bar{v}_i v_i = 0$, and from this it follows that $\bar{v}_2 = 0$ relative to the y system. Hence $\bar{v}_{2,1} = \bar{v}_{2,3} = 0$ as contained in the above relations (2.1).

For definiteness let us choose the y system so that (1) the y_1, y_3 plane is the plane Σ and (2) the origin of the y system is midway between the flat sides of the plate. Then $y_2 = 0$ is the equation of the plane Σ and we may take the y_1, y_3 coordinates as the parametric coordinates of points of Σ in the equations giving the general conditions of compatibility.⁸ Thus we have

$$(2.2) \quad y_1 = y_1; \quad y_2 = 0; \quad y_3 = y_3$$

as the equations of the plane Σ where the y 's in the left members are the coordinates of the y system while the quantities y_1 and y_3 in the right members of these equations are the parametric coordinates of Σ .

Neglecting possible end effects it is clear from symmetry that the quantities σ_{ij} and v_i and their derivatives, when evaluated on Σ , will be independent of the coordinate y_1 . We shall refer to this condition as the *symmetry condition*.

The relations (2.2) will enter into the evaluation of those terms in the general conditions of compatibility of the surface Σ which involve differentiation with respect to the parametric coordinates. Thus the geometric conditions of com-

¹¹ T. Y. THOMAS, *The Lüders band problem*, Jour. Math. and Mech. 7 (1958) pp. 17-27; and *On the velocity of formation of Lüders bands*, *ibid.* 7 (1958) pp. 141-148.

patibility⁸ of order 1 are

$$(2.3) \quad [v_{i,j}] = \bar{v}_{i,j} = \omega_i \nu_j + \bar{v}_{i,\alpha} y_{j,\alpha},$$

in which α has the values 1, 3 and the comma denotes differentiation with respect to the parametric coordinates y_1 and y_3 . Using (2.2) we see that the conditions (2.3) are identical with (2.1) when account is taken of the fact that $\bar{v}_2 = 0$ and $\omega = \omega_3 = 0$, the above symmetry condition, and the condition of incompressibility (1.4) of the plastic material.

Now it has been shown in our second paper on Lüders bands¹¹ that $[\sigma_{ij}] = 0$ over Σ . Since the stress is assumed to be constant in the elastic region it follows that $\bar{\sigma}_{ij} = \text{const.}$ along Σ and the constant values of $\bar{\sigma}_{ij}$ will be independent of the time. Hence the geometrical conditions of compatibility of order 1 for the stress components take the simple form

$$(2.4) \quad [\sigma_{ij,k}] = \bar{\sigma}_{ij,k} = \xi_{ij} \nu_k,$$

in which the quantities ξ_{ij} are functions defined over Σ . Referring (2.4) to the y system and putting $k = 3$ we see that these relations are satisfied identically in view of the condition of plane stress. However, when we take $j = 3$ the equations (2.4) give

$$(2.5) \quad \xi_{i3} = 0.$$

Since $G = 0$ for the plane Σ in this consideration it follows that the δ time derivatives of the components σ_{ij} and v_i are identical with their partial time derivatives. Hence the kinematical conditions of compatibility of order 1 for σ_{ij} and v_i reduce to identities.

3. Wave strength. Since the quantities $[\sigma_{ij}] = 0$ the strength of the wave Σ can be considered to depend on the discontinuities

$$[v_{i,j}] = \bar{v}_{i,j}.$$

But all non-vanishing derivatives $\bar{v}_{i,j}$ are determined by the values of ω_1 and ω_2 in accordance with the relations (2.1). Now it has been shown that¹¹

$$(3.1) \quad \omega_1 = 2\tau\bar{\psi} \sin \theta \cos \theta; \quad 2\omega_2 = \tau\bar{\psi} \cos^2 \theta,$$

where θ gives the inclination of the Lüders band and τ is the yield stress. Or, eliminating the quantity $\tau\bar{\psi}$ between the two relations (3.1), we have

$$(3.2) \quad \omega_1 = 4\omega_2 \tan \theta.$$

From (2.1) and (3.2) we see that all derivatives $\bar{v}_{i,j}$ vanish when $\omega_2 = 0$. Hence we can define the strength Ω of the wave Σ in the Lüders band problem by the equation

$$(3.3) \quad \Omega = |\omega_2|,$$

where $|\omega_2|$ is the absolute value of the quantity ω_2 .

4. Compatibility conditions of the second order. The coefficients $b_{\alpha\beta}$ of the second fundamental form vanish since Σ is a plane. Moreover, since the wave velocity $G = 0$ for the Lüders band problem the relations (28) and (61) of our paper on *extended compatibility conditions*⁸ give

$$(4.1) \quad [v_{i,jk}] = \bar{v}_{i,jk} = \hat{\omega}_i \nu_j \nu_k + \omega_{i,\alpha} (\nu_j y_{k,\alpha} + \nu_k y_{j,\alpha}) + \bar{v}_{i,\alpha\beta} y_{j,\alpha} y_{k,\beta} ,$$

$$(4.2) \quad \left[\frac{\partial^2 v_i}{\partial y_j \partial t} \right] = \frac{\partial^2 \bar{v}_i}{\partial y_j \partial t} = \frac{\partial \omega_i}{\partial t} \nu_j + \frac{\partial \bar{v}_{i,\alpha}}{\partial t} y_{j,\alpha} ,$$

respectively, where we have referred these equations to the y system. Relations (4.1) are the geometrical conditions of compatibility and relations (4.2) are kinematical conditions of compatibility for the velocity components v_i . The kinematical conditions of compatibility which involve the derivatives $\partial^2 v_i / \partial t^2$ reduce to identities since the velocity $G = 0$ on Σ .

The analogous conditions of compatibility for the stress components σ_{ij} are given by

$$(4.3) \quad [\sigma_{ij,km}] = \bar{\sigma}_{ij,km} = \hat{\xi}_{ij} \nu_k \nu_m + \xi_{ij,\alpha} (\nu_k y_{m,\alpha} + \nu_m y_{k,\alpha}) ,$$

$$(4.4) \quad \left[\frac{\partial^2 \sigma_{ij}}{\partial y_k \partial t} \right] = \frac{\partial^2 \bar{\sigma}_{ij}}{\partial y_k \partial t} = \frac{\partial \xi_{ij}}{\partial t} \nu_k .$$

Terms corresponding to the last terms in the relations (4.1) and (4.2) do not appear in (4.3) and (4.4) due to the fact that the components $\bar{\sigma}_{ij}$ have constant values on Σ as observed in §2. As for the case of the velocity, the remaining kinematical conditions of the second order for the σ_{ij} reduce to identities.

Certain simple consequences of the above conditions of compatibility can be observed immediately. Thus from (4.1) and the condition (1.4), expressing the incompressibility of plastic material, we have

$$\bar{v}_{i,ik} \nu_k = \hat{\omega}_i \nu_i + \omega_{i,\alpha} y_{i,\alpha} = \hat{\omega}_2 + \omega_{1,1} = 0 ,$$

when use is made of (2.2) and the fact that $\omega_3 = 0$. But $\omega_{1,1} = 0$ from the symmetry condition (§2) and hence

$$(4.5) \quad \hat{\omega}_2 = 0 .$$

Also, putting $j = 3$ in (4.3) and using (2.5) and the condition of plane stress, the resulting equations are seen to reduce to

$$(4.6) \quad \hat{\xi}_{i3} = 0 .$$

Similarly, when we take $m = 3$ in (4.3) we obtain

$$(4.7) \quad \xi_{ii,3} = 0 .$$

5. Further consequences of the compatibility conditions. Consider the equations of motion in the Lüders band immediately behind the plane Σ , namely

$$(5.1) \quad \bar{\sigma}_{ii,i} = \bar{\rho} \frac{d\bar{v}_i}{dt} = \bar{\rho} \left(\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_{i,i} \bar{v}_i \right) .$$

Then from (2.3) and (2.4) we have

$$(5.2) \quad \xi_{i,j} \nu_j = \bar{\rho} \left\{ \frac{\partial \bar{v}_i}{\partial t} + (\omega_i \nu_j + \bar{v}_{i,\alpha} y_{j,\alpha}) \bar{v}_i \right\}.$$

But in the y system the ν_i have the values 0, 1, 0 and $\bar{v}_2 = 0$. When we make these substitutions and also avail ourselves of (2.1) and (2.2) the equations (5.2) become

$$(5.3) \quad \xi_{i2} = \bar{\rho} \left(\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_{i,3} \bar{v}_3 \right).$$

Taking $i = 1, 2, 3$ in (5.3) in turn we see that

$$(5.4) \quad \xi_{12} = \bar{\rho} \frac{\partial \bar{v}_1}{\partial t},$$

$$(5.5) \quad \xi_{22} = \bar{\rho} \frac{\partial \bar{v}_2}{\partial t} = 0,$$

$$(5.6) \quad \frac{\partial \bar{v}_3}{\partial t} = -\bar{v}_{3,3} \bar{v}_3 = \omega_2 \bar{v}_3.$$

In obtaining (5.5) we have of course used the fact that $\bar{v}_2 = 0$ on Σ . Also the condition (2.5) and the last of the relations (2.1) are used in the derivation of (5.6). We shall find that the equation (5.6) has an important application in the following discussion.

Expanding the left members of the Prandtl-Reuss equations (1.2) we write these equations in the form

$$(5.7) \quad \frac{\partial \sigma_{ij}^*}{\partial t} + \sigma_{ij,m}^* v_m = 2\mu(\epsilon_{ij} - \psi \sigma_{ij}^*).$$

Now differentiate (5.7) with respect to the coordinates y_k and multiply by the vector components ν_k . Then, when we evaluate the resulting equations on the rear of the plane Σ , we obtain

$$(5.8) \quad \frac{\partial^2 \bar{\sigma}_{ij}^*}{\partial y_k \partial t} \nu_k + \bar{\sigma}_{ij,k}^* \nu_k \bar{v}_m + \bar{\sigma}_{ij,m}^* \bar{v}_{m,k} \nu_k = 2\mu(\bar{\epsilon}_{ij,k} \nu_k - \sigma_{ij}^* \bar{\psi}_{,k} \nu_k - \bar{\psi} \bar{\sigma}_{ij,k}^* \nu_k).$$

Application of the compatibility conditions (2.3), (2.4), (4.1) and (4.3) will now lead to various new relations. However, only that relation obtainable from (5.8) by taking $i = 1, j = 3$ will be needed for our purpose. But for these values of i and j we see that the entire left member and also the second and third terms in the right member of (5.8) are equal to zero on account of the condition of plane stress. Hence, when we replace the ν_k by their appropriate values, the equation in question becomes

$$(5.9) \quad \bar{\epsilon}_{13,2} = \frac{1}{2}(\bar{v}_{1,32} + \bar{v}_{3,12}) = 0,$$

relative to the y system. But from (4.1) we find

$$\bar{v}_{1,32} = \omega_{1,3} ; \quad \bar{v}_{3,12} = 0,$$

when use is made of (2.2) and the condition $\omega_3 = 0$. Hence (5.9) gives

$$(5.10) \quad \omega_{1,3} = 0.$$

From (5.10) and the fact that $\omega_{1,1} = 0$, from the symmetry condition of §2, we now have the following results. *At any fixed time the quantity ω_1 is constant on the plane Σ .* A similar statement can be made with regard to the quantity ω_2 on account of the relation (3.2).

6. Condition for fracture. Let us now differentiate (5.6) partially with respect to the parametric coordinate y_3 of the plane Σ . This gives the equation

$$(6.1) \quad \frac{\partial^2 \bar{v}_3}{\partial y_3 \partial t} = \omega_2 \bar{v}_{3,3} ,$$

since ω_2 can depend only on the time by the result at the end of §5. Hence, replacing the quantity $\bar{v}_{3,3}$ by its value $-\omega_2$ in accordance with the last equation (2.1), we have

$$(6.2) \quad \frac{\partial \omega_2}{\partial t} = \omega_2^2 .$$

Now the quantity $\bar{\psi}$ is positive.¹² Also $\tau > 0$ for the tension problem under consideration, and hence it follows from the second equation (3.1) that ω_2 is positive. The equation (3.3) defining the strength of the wave Σ can therefore be written simply as $\Omega = \omega_2$, and hence (6.2) gives

$$(6.3) \quad \frac{\partial \Omega}{\partial t} = \Omega^2$$

as the differential equation for the variation of the wave strength Ω .

By the result at the end of §5 the wave strength Ω can depend at most on the time t . Hence, integrating (6.3), we obtain

$$(6.4) \quad \Omega = \frac{\Omega_0}{1 - \Omega_0 t} ,$$

where Ω_0 is the wave strength at time $t = 0$. It follows from (6.4) that $\Omega \rightarrow \infty$ as $t \rightarrow T$ such that $T = 1/\Omega_0 > 0$. *Hence the plate will fracture along the Lüders band after the lapse of a time interval equal to $1/\Omega_0$ where Ω_0 is the strength of the wave at the beginning of the interval.*

¹² We have $\psi \geq 0$ in the plastic region except when *unloading* at the yield point for which $\psi < 0$. Cf. W. PRAGER & P. G. HODGE, JR. *Theory of Perfectly Plastic Solids*, loc. cit. p. 29; also R. HILL, *The Mathematical Theory of Plasticity*, loc. cit., p. 40. Hence $\bar{\psi} \geq 0$ when there is no unloading. But $\bar{\psi} = 0$ implies that $\omega_1 = 0$ and $\omega_2 = 0$ on account of (3.1); this contradicts the assumption at the beginning of §2. Hence $\bar{\psi} > 0$ as above stated.

Remark 1. By coordinate differentiation of the yield condition (1.1) and application of the compatibility conditions (2.4) we readily find that

$$(6.5) \quad \bar{\sigma}_{i,j}^* \xi_{i,j} = 0$$

on the wave surface Σ . Expanding (6.5) and taking account of (2.5) we have

$$(6.6) \quad \bar{\sigma}_{11}^* \xi_{11} + 2\bar{\sigma}_{12}^* \xi_{12} + \bar{\sigma}_{22}^* \xi_{22} = 0.$$

But the last term in (6.6) vanishes on account of (5.5), and it was shown in our previous work on Lüders bands that¹¹

$$(6.7) \quad \bar{\sigma}_{11}^* = 0; \quad \bar{\sigma}_{12}^* = \frac{\omega_1}{2\bar{\psi}},$$

relative to the y system. Since $\bar{\sigma}_{12}^* \neq 0$ it follows from (5.4) and (6.7) that (6.6) reduces to

$$\frac{\partial \bar{v}_1}{\partial t} = 0.$$

Now $\bar{v}_{1,1} = \bar{v}_{1,3} = 0$ from (2.1). Hence the time derivative of $\bar{v}_1$ and also its two derivatives with respect to the parametric coordinates y_1 and y_3 of Σ are equal to zero. In other words, *the velocity component v_1 is constant on Σ with respect to both position and time.* This velocity produces the *slipping effect* which is commonly associated with the Lüders band.

Remark 2. Since $\omega_2 > 0$, so that $\Omega = \omega_2$, it follows from (5.6) and (6.4) that

$$(6.8) \quad \frac{\partial \bar{v}_3}{\partial t} = \frac{\Omega_0 \bar{v}_3}{1 - \Omega_0 t}.$$

Hence, integrating (6.8), we have

$$(6.9) \quad (1 - \Omega_0 t) \bar{v}_3 = \phi(y_3).$$

As indicated, the constant of integration ϕ cannot involve the parametric coordinate y_1 since $\bar{v}_{3,1} = 0$ from (2.1). Differentiating (6.9) with respect to y_3 we obtain

$$(6.10) \quad -(1 - \Omega_0 t) \omega_2 = \phi'(y_3).$$

Hence, replacing the quantity ω_2 in (6.10) by the value given by the right member of (6.4), the equation (6.10) becomes

$$(6.11) \quad \phi'(y_3) = -\Omega_0.$$

Now when we integrate (6.11) we obtain

$$(6.12) \quad \phi = -\Omega_0 y_3 + C,$$

where C is an absolute constant independent of the time or position on the plane Σ . But from symmetry we must have $\bar{v}_3 = 0$ for $y_3 = 0$ at all times t .

Hence the constant C must be equal to zero so that (6.9) and (6.12) combine to give

$$(6.13) \quad \bar{v}_3 = -\frac{\Omega_0 y_3}{1 - \Omega_0 t}.$$

It is seen from (6.13) that particles in the plastic region contiguous to the plane Σ (and lying off the central plane $y_3 = 0$) must have a velocity which is directed at all times toward the center of the plate. A flow of this type obviously tends to produce the observed necking of the plastic material in the Lüders band.

Putting $\bar{v}_3 = dy_3/dt$ and integrating (6.13) we see that all particles on the plastic side of Σ are compressed into the y_3 axis at time $t = 1/\Omega_0$. The resulting *cut* along the plane Σ makes fracture inevitable.

Remark 3. Suppose that *unloading* takes place immediately after the formation of the Lüders band or, more specifically, before the lapse of the time T required for fracture according to the above result. But for unloading $\psi < 0$ in the plastic region.¹² Hence $\bar{\psi} < 0$ and hence $\omega_2 < 0$ from the second equation (3.1). It therefore follows from the definition (3.3) of the strength Ω of the wave that $\Omega = -\omega_2$. Hence (6.2) gives

$$(6.14) \quad \frac{\partial \Omega}{\partial t} = -\Omega^2.$$

Let $\Omega = \Omega_0$ at the time $t = 0$ when unloading begins. Then, integrating (6.14), we obtain

$$\Omega = \frac{\Omega_0}{1 + \Omega_0 t}.$$

Since $\Omega_0 > 0$ we can therefore state the following result. *If unloading is begun immediately after the formation of the Lüders band the wave strength Ω will approach zero asymptotically as the time t increases indefinitely and hence fracture will not occur.*

WEAK STATIONARY DISCONTINUITIES

In the Lüders band problem we were concerned with a strong discontinuity over the surface Σ since not all of the quantities $[v_i]$ were equal to zero (see §1). We shall now consider a related problem involving a *weak discontinuity*, i.e. such that $[\sigma_{ij}]$ and $[v_i]$ vanish over the surface of discontinuity separating the elastic and plastic regions in a thin flat plate under tension. More specifically, we shall make the following assumption concerning the order of the discontinuity over Σ . *Not all of the quantities $[\sigma_{ij,k}]$ vanish at points of Σ , i.e. the surface Σ has a discontinuity of order 1 relative to the stress.* Beyond the condition $[v_i] = 0$ no assumption will be made regarding the order of the discontinuity relative to the velocity.

The following discussion will be referred exclusively to the x coordinate system whose x_2 axis is in the direction of the tension τ and for which the x_1, x_2 plane is parallel to the flat sides of the plate. Since we assume the elastic region in a state of dynamical equilibrium the velocity components $v_i = 0$ in the elastic region. Hence the components $\bar{v}_i = 0$ in terms of our previous notation since the velocity, by assumption, is continuous over Σ . This use of a bar over a quantity to denote evaluation on the plastic side of the surface Σ will be continued throughout the remainder of this article.

7. Compatibility conditions of the first order. Wave strength. In view of the above assumptions, and the assumption that the normal velocity $G = 0$, the geometrical and kinematical conditions of compatibility of order 1 take the simple form

$$(7.1) \quad [v_{i,i}] = \bar{v}_{i,i} = \lambda_i \nu_i ; \quad \left[\frac{\partial v_i}{\partial t} \right] = \frac{\partial \bar{v}_i}{\partial t} = 0,$$

$$(7.2) \quad [\sigma_{ij,k}] = \bar{\sigma}_{ij,k} = \xi_{ij} \nu_k ; \quad \left[\frac{\partial \sigma_{ij}}{\partial t} \right] = \frac{\partial \bar{\sigma}_{ij}}{\partial t} = 0.$$

When the equations

$$(7.3) \quad v_{i,i} = 0,$$

$$(7.4) \quad \sigma_{ij,i} = \rho \frac{dv_i}{dt},$$

$$(7.5) \quad \frac{d\sigma_{ij}^*}{dt} = 2\mu(\epsilon_{ij} - \psi \sigma_{ij}^*)$$

are evaluated on the plastic side of the surface Σ and then combined with the compatibility conditions (7.1) and (7.2) the following relations are obtained, namely

$$(7.6) \quad \lambda_i \nu_i = 0,$$

$$(7.7) \quad \xi_{ij} \nu_i = 0,$$

$$(7.8) \quad \frac{1}{2} (\lambda_i \nu_i + \lambda_j \nu_j) = \frac{1}{K} (\sigma_{ab} \lambda_a \nu_b) \sigma_{ij}^*,$$

where use is made of the equation (1.3) defining the quantity ψ . In writing (7.8) we have omitted the bars over the σ 's since the quantities $\bar{\sigma}_{ij}$ are equal to the corresponding stress components σ_{ij} which result from the simple tension in the elastic region. Another relation which we shall need is obtained by differentiating the yield condition (1.1) with respect to x_k and then using the conditions (7.2). This gives

$$(7.9) \quad \sigma_{ij}^* \xi_{ij} = 0.$$

Multiplying (7.8) by ν_i and making use of (7.6) we obtain

$$(7.10) \quad \lambda_i = \frac{2}{K} (\sigma_{ab} \lambda_a \nu_b) \sigma_{ii}^* \nu_i .$$

When we multiply (7.10) by λ_i and again use (7.6) we find that

$$(7.11) \quad \lambda_i \lambda_i = \frac{2}{K} (\sigma_{ab} \lambda_a \nu_b)^2 .$$

Also we have

$$(7.12) \quad (\sigma_{ab} \lambda_a \nu_b) (\sigma_{ii}^* \nu_i \nu_i) = 0 ,$$

when we multiply (7.10) by ν_i and apply (7.6). It will be seen in the next section that the first of the two factors in the left member of (7.12) must vanish as a result of this equation.

Since the wave Σ is singular of order 1 relative to the stress not all of the above quantities ξ_{ij} can vanish. This fact is basic in the derivation of the following results. It is therefore appropriate to define the wave strength S by the equation

$$(7.13) \quad S = \sqrt{\xi_{ij} \xi_{ij}} .$$

8. Application of the condition of plane stress. Putting $j = 3$ in the first relation (7.2) it follows that $\xi_{i3} \nu_k = 0$ from the assumption of plane stress in the thin plate. Hence

$$(8.1) \quad \xi_{i3} = 0 .$$

Also when we take $k = 3$ in the first equation (7.2) we have $\xi_{ij} \nu_3 = 0$. But not all of the quantities ξ_{ij} can vanish, from the assumption that the surface Σ is singular of order 1 relative to the stress. Hence

$$(8.2) \quad \nu_3 = 0 .$$

The surface Σ must therefore be perpendicular to the flat sides of the plate, i.e. the x_1, x_2 coordinate plane.

Putting $i = 3$ in (7.10) we have

$$(8.3) \quad \lambda_3 = \frac{2}{K} (\sigma_{ab} \lambda_a \nu_b) \sigma_{3j}^* \nu_j .$$

But, in view of the plane stress condition, we see that

$$(8.4) \quad \sigma_{3j}^* = \sigma_{3j} - \frac{1}{3} \sigma_{aa} \delta_{3j} = 0 ,$$

for $j = 1, 2$. Hence from (8.2) and (8.4) it follows that the last factor in the right member of (8.3) must be equal to zero so that (8.3) becomes

$$(8.5) \quad \lambda_3 = 0 .$$

Expanding the equations (7.7) and (7.9) and taking account of the conditions (8.1) and (8.2) we now have

$$(8.6) \quad \begin{cases} \xi_{11}\nu_1 + \xi_{12}\nu_2 = 0, \\ \xi_{21}\nu_1 + \xi_{22}\nu_2 = 0, \\ \xi_{11}\sigma_{11}^* + 2\xi_{12}\sigma_{12}^* + \xi_{22}\sigma_{22}^* = 0. \end{cases}$$

The determinant of the coefficients of the ξ_{ij} in (8.6) must be equal to zero since not all of the ξ 's can vanish by assumption. Hence

$$(8.7) \quad \sigma_{22}^*\nu_1^2 - 2\sigma_{12}^*\nu_1\nu_2 + \sigma_{11}^*\nu_2^2 = 0.$$

Now if we put $i = j$ in (7.8) and sum on the repeated indices we see that the equations are satisfied identically on account of (7.6). Also (7.8) is seen to be satisfied identically if we take $i = 3$ and $j = 1, 2$ in view of (8.2), (8.4) and (8.5). Hence there are at most three independent equations in the set (7.8) and these are obtained by taking $i, j = (1, 2), (2, 2)$ and $(3, 3)$. When account is taken of (8.2) or (8.5) these equations become

$$(8.8) \quad \begin{cases} \lambda_1\nu_2 + \lambda_2\nu_1 = \frac{2}{K}(\sigma_{ab}\lambda_a\nu_b)\sigma_{12}^*, \\ \lambda_2\nu_2 = \frac{1}{K}(\sigma_{ab}\lambda_a\nu_b)\sigma_{22}^*, \\ (\sigma_{ab}\lambda_a\nu_b)\sigma_{33}^* = 0. \end{cases}$$

But for the elastic region under simple tension τ the components σ_{ij}^* are given by

$$(8.9) \quad \sigma_{11}^* = \sigma_{33}^* = -\frac{\tau}{3}; \quad \sigma_{22}^* = \frac{2\tau}{3}; \quad \sigma_{12}^* = \sigma_{13}^* = \sigma_{23}^* = 0.$$

Hence

$$(8.10) \quad \sigma_{ab}\lambda_a\nu_b = 0,$$

from the last equation (8.8). It now follows from (7.10) or (7.11) and (8.10) that

$$(8.11) \quad \lambda_i = 0.$$

Hence the first derivatives of the velocity components v_i are continuous across the surface Σ . All equations (8.8) and hence all equations in the set (7.8) are therefore satisfied as a consequence of the condition (8.11).

Making the substitutions (8.9) we find that (8.7) becomes

$$(8.12) \quad 2\nu_1^2 - \nu_2^2 = 0.$$

But since ν is a unit vector whose third component $\nu_3 = 0$ it follows from (8.12) that $\nu_1^2 = \frac{1}{3}$ and $\nu_2^2 = \frac{2}{3}$. Hence the surface Σ is a plane perpendicular to the flat sides of the plate and having an inclination θ given by $\tan^2 \theta = \frac{1}{2}$. These planes Σ therefore have inclinations of $\pm 35^\circ 16'$ and are geometrically identical with

the surfaces of separation of the elastic and plastic regions in the Lüders band problem.

9. Compatibility conditions of the second order. Since the first derivatives of the velocity components v_i are continuous across Σ the conditions of compatibility of order 2 become

$$(9.1) \quad \begin{cases} [v_{i,jk}] = \bar{v}_{i,jk} = \hat{\lambda}_{,j} \nu_k, \\ \left[\frac{\partial^2 v_i}{\partial x_j \partial t} \right] = \frac{\partial^2 \bar{v}_i}{\partial x_j \partial t} = 0, \\ \left[\frac{\partial^2 v_i}{\partial t^2} \right] = \frac{\partial^2 \bar{v}_i}{\partial t^2} = 0, \end{cases}$$

where the $\hat{\lambda}_i$ are functions defined on Σ . Recalling that the δ time derivative of a quantity is equal to its partial time derivative, since $G = 0$, and taking account of the fact that the coefficients $b_{\alpha\beta}$ of the second fundamental form are equal to zero for the plane Σ , the equations (61) and (65) of the paper on *extended compatibility conditions*⁸ give

$$(9.2) \quad \begin{cases} [\sigma_{ij,km}] = \bar{\sigma}_{ij,km} = \hat{\xi}_{ij} \nu_k \nu_m + \xi_{ij,\alpha} (\nu_k x_{m,\alpha} + \nu_m x_{k,\alpha}), \\ \left[\frac{\partial^2 \sigma_{ij}}{\partial x_k \partial t} \right] = \frac{\partial^2 \bar{\sigma}_{ij}}{\partial x_k \partial t} = \frac{\partial \xi_{ij}}{\partial t} \nu_k; \quad \left[\frac{\partial^2 \sigma_{ij}}{\partial t^2} \right] = \frac{\partial^2 \bar{\sigma}_{ij}}{\partial t^2} = 0. \end{cases}$$

These equations, in which the $\hat{\xi}_{ij}$ are suitable functions defined on Σ , are the geometrical and kinematical conditions of compatibility of order 2 for the stress.

Certain simple consequences of the above compatibility conditions can be observed immediately. Thus, putting $i = j$ in the first equation (9.1) and summing on the repeated indices, we have

$$(9.3) \quad \hat{\lambda}_{,j} \nu_j = 0,$$

in view of the condition of incompressibility of plastic material. Also from the plane stress condition and the first equation (9.2) we readily see that

$$(9.4) \quad \hat{\xi}_{i3} = 0; \quad \xi_{ij,3} = 0.$$

To obtain additional conditions let us differentiate the Prandtl-Reuss equations (5.7) partially with respect to x_k and evaluate at points of Σ . We thus have

$$(9.5) \quad \frac{\partial^2 \bar{\sigma}_{ij}^*}{\partial x_k \partial t} + \bar{\sigma}_{ij,km}^* \bar{v}_m + \bar{\sigma}_{ij,m}^* \bar{v}_{m,k} = 2\mu(\bar{\epsilon}_{ij,k} - \bar{\psi}_{,k} \sigma_{ij}^* - \bar{\psi} \bar{\sigma}_{ij,k}^*).$$

But $\bar{v}_m = 0$ and $\bar{v}_{m,k} = 0$. Hence we see from the equation (1.3) defining ψ that $\bar{\psi}$ also vanishes. The equations (9.5) therefore reduce to

$$(9.6) \quad \frac{\partial^2 \bar{\sigma}_{ij}^*}{\partial x_k \partial t} = 2\mu(\bar{\epsilon}_{ij,k} - \bar{\psi}_{,k} \sigma_{ij}^*).$$

Using the first equation (9.1) and the first equation (9.2) we now obtain

$$\begin{aligned}\bar{\epsilon}_{ij,k} &= \frac{1}{2}(\hat{\lambda}_i \nu_j + \hat{\lambda}_j \nu_i) \nu_k, \\ \frac{\partial^2 \bar{\sigma}_{ij}^*}{\partial x_k \partial t} &= \left(\frac{\partial \xi_{ij}}{\partial t} - \frac{1}{3} \frac{\partial \xi_{mm}}{\partial t} \delta_{ij} \right) \nu_k.\end{aligned}$$

Similarly after differentiation of (1.3) we find that

$$\bar{\Psi}_{,k} = \frac{\bar{\epsilon}_{ab,k} \sigma_{ab}}{K} = \frac{1}{K} (\hat{\lambda}_a \nu_b \sigma_{ab}) \nu_k.$$

Hence, making these substitutions, the equations (9.6) become

$$(9.7) \quad \frac{\partial \xi_{ij}}{\partial t} - \frac{1}{3} \frac{\partial \xi_{mm}}{\partial t} \delta_{ij} = \mu (\hat{\lambda}_i \nu_j + \hat{\lambda}_j \nu_i) - \frac{2\mu}{K} (\hat{\lambda}_a \nu_b \sigma_{ab}) \sigma_{ij}^*.$$

Now consider the relations

$$(9.8) \quad \frac{\partial^2 \bar{\sigma}_{ik}}{\partial x_k \partial t} = \frac{\partial \bar{p}}{\partial t} \left(\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_{i,k} \bar{v}_k \right) + \bar{p} \left(\frac{\partial^2 \bar{v}_i}{\partial t^2} + \frac{\partial^2 \bar{v}_i}{\partial x_k \partial t} \bar{v}_k + \bar{v}_{i,k} \frac{\partial \bar{v}_k}{\partial t} \right),$$

which result when we differentiate the equations of motion (7.4) with respect to the time t and evaluate on the plastic side of Σ . But

$$\bar{v}_k = \bar{v}_{i,k} = \frac{\partial \bar{v}_k}{\partial t} = \frac{\partial^2 \bar{v}_i}{\partial x_k \partial t} = \frac{\partial^2 \bar{v}_i}{\partial t^2} = 0.$$

Hence the right members of (9.8) vanish. But when we equate the left members of (9.8) to zero and make use of the second condition (9.2) we obtain

$$(9.9) \quad \frac{\partial \xi_{ik}}{\partial t} \nu_k = 0.$$

10. Stability of the wave. To deduce the differential equation for the variation in the strength of the wave we first put $i = j = 3$ in (9.7). Then, from (8.2) and the condition of plane stress, we obtain

$$(10.1) \quad \frac{1}{3} \frac{\partial \xi_{mm}}{\partial t} = \frac{2\mu}{K} (\hat{\lambda}_a \nu_b \sigma_{ab}) \sigma_{33}^*.$$

Next multiply (9.7) by $\nu_i \nu_j$ and sum on the repeated indices. Taking account of (9.3) and (9.9) we now find that

$$(10.2) \quad \frac{1}{3} \frac{\partial \xi_{mm}}{\partial t} = \frac{2\mu}{K} (\hat{\lambda}_a \nu_b \sigma_{ab}) \sigma_{ij}^* \nu_i \nu_j.$$

Hence combining (10.1) and (10.2) we have

$$(10.3) \quad (\hat{\lambda}_a \nu_b \sigma_{ab}) (\sigma_{ij}^* \nu_i \nu_j - \sigma_{33}^*) = 0.$$

But

$$\sigma_{ij}^* \nu_i \nu_j - \sigma_{33}^* = \sigma_{ij} \nu_i \nu_j = \tau \nu_2^2 \neq 0$$

for the values of the σ 's corresponding to the simple tension τ in the elastic region. Hence the first factor in (10.3) must vanish, *i.e.*

$$(10.4) \quad \hat{\lambda}_a \nu_b \sigma_{ab} = 0.$$

Now from (10.4) and (10.1) or (10.2) we have

$$(10.5) \quad \frac{\partial \xi_{mm}}{\partial t} = 0.$$

But from (10.4) and (10.5) the equations (9.7) reduce to

$$(10.6) \quad \frac{\partial \xi_{ij}}{\partial t} = \mu(\hat{\lambda}_{,i} \nu_j + \hat{\lambda}_{,j} \nu_i).$$

Multiplying (10.6) by ν_i and using (9.3) and (9.9) it now follows that

$$(10.7) \quad \hat{\lambda}_i = 0.$$

In other words, *the second derivatives of the velocity components v_i are continuous across the surface Σ .*

From (10.6) and (10.7) the partial derivatives of the ξ_{ij} with respect to the time t are equal to zero. Hence from (7.13) we have

$$(10.8) \quad \frac{\partial S}{\partial t} = 0.$$

A wave of this character will be said to be *stable*. Since the wave strength S does not vary with the time t fracture will not occur along the discontinuity surface Σ . As mentioned in §1 the *cloud effect*, produced on the highly polished flat sides of the plate when the yield point is reached, is due to the formation of weak discontinuities of this type.

WEAK DISCONTINUITIES WHOSE VELOCITY OF PROPAGATION IS DIFFERENT FROM ZERO

We shall now give a preliminary discussion of surfaces of discontinuity or wave surfaces Σ whose normal velocity of propagation G is different from zero. Specifically, we shall have $G > 0$ since the unit normal ν to the surface Σ is taken to be directed from the plastic side of Σ into the elastic region, *i.e.* the direction of ν coincides with the direction of propagation of the wave. Beyond this requirement on the velocity G the following conditions will be imposed relative to a fixed rectangular coordinate system x assumed to be dynamically admissible.

- (i) *The wave surface Σ is singular of order 1 relative to the set of functions consisting of the velocity components v_i and the stress components σ_{ij} .*
- (ii) *The velocity components $v_i = 0$ in the elastic region into which the wave is propagated.*

(iii) *The components of the stress tensor are constant in the elastic region and satisfy the condition of plane stress throughout the entire medium under consideration.*

Results obtained under these assumptions will be applied to the problem of the flat plate by taking the stress in the elastic region to be that due to simple tension and identifying the above x coordinate system with the x system used in the foregoing discussion.

11. Continuity of the density. Since $[v_i] = 0$ and $[\sigma_{ij}] = 0$ by assumption (i) the condition (1.6) is satisfied automatically. However, (1.5) reduces to $[\rho] = 0$ since G is different from zero, i.e. *the density ρ is continuous across the surface Σ .* Hence relations of the form

$$(11.1) \quad [\rho_{,i}] = \zeta v_i ; \quad \left[\frac{\partial \rho}{\partial t} \right] = -\zeta G,$$

hold along the surface Σ .

Now in the elastic region $v_{i,i} = 0$ from the above assumption (ii). But $v_{i,i}$ also vanishes in the plastic region from the condition of incompressibility (1.4). Moreover, $v_i = 0$ on each side of the surface Σ on account of (i) and (ii) and hence, on both the elastic and plastic sides of Σ , the equation of continuity reduces to

$$\frac{\partial \rho}{\partial t} = 0.$$

Since G does not vanish it therefore follows from the second relation (11.1) that $\zeta = 0$. *Hence the first derivatives of the density are also continuous across Σ .*

12. Results obtained from the conditions of compatibility of the first order. On account of (i), (ii) and (iii) we can write

$$(12.1) \quad [v_{i,i}] = \bar{v}_{i,i} = \lambda_i v_i ; \quad \left[\frac{\partial v_i}{\partial t} \right] = \frac{\partial \bar{v}_i}{\partial t} = -G \lambda_i ,$$

$$(12.2) \quad [\sigma_{ij,k}] = \bar{\sigma}_{ij,k} = \xi_{ij} v_k ; \quad \left[\frac{\partial \sigma_{ij}}{\partial t} \right] = \frac{\partial \bar{\sigma}_{ij}}{\partial t} = -G \xi_{ij} ,$$

where the λ_i and ξ_{ij} are functions defined on Σ . From (7.3) and the first condition (12.1) it follows immediately that

$$(12.3) \quad \lambda_i v_i = 0.$$

Evaluating (7.4) and (7.5) on the plastic side of Σ and applying (12.1) and (12.2) we have

$$(12.4) \quad \xi_{ij} v_i = -G \rho \lambda_i ,$$

$$(12.5) \quad G(\xi_{ij} - \frac{1}{3} \xi_{aa} \delta_{ij}) = -\mu(\lambda_i v_i + \lambda_j v_j) + \frac{2\mu}{K} (\sigma_{ab} \lambda_a v_b) \sigma_{ij}^* ,$$

where ρ is the density on the elastic side of Σ and σ_{ab} and σ_{ij}^* are the components of the stress and stress deviator in the elastic region. Similarly, by coordinate

differentiation of (1.1) and use of the condition (12.2) we obtain

$$(12.6) \quad \sigma_{ij}^* \xi_{ij} = 0.$$

From (i) not all of the quantities λ_i and ξ_{ij} are equal to zero at points of Σ . Now if all $\xi_{ij} = 0$ it follows from (12.4) that all $\lambda_i = 0$ and we have a contradiction. Conversely, if all $\lambda_i = 0$ we have from (12.4) and (12.5) that

$$(12.7) \quad \xi_{ij} \nu_j = 0; \quad \xi_{ij} - \frac{1}{3} \xi_{aa} \delta_{ij} = 0.$$

But if we multiply the second of these relations by ν_j and apply the first relation we obtain $\xi_{aa} = 0$. Hence $\xi_{ij} = 0$ from the second relation (12.7) and we again have a contradiction. *Hence not all of the ξ_{ij} can vanish and not all of the λ_i can vanish at points of Σ .*

13. Plane stress. The plane stress condition has been used in our previous work on Lüders bands¹¹ and also in the foregoing discussion. Recalling that this condition is given explicitly by

$$(13.1) \quad \sigma_{i3} = 0; \quad \sigma_{i3,k} = 0; \quad \sigma_{ij,3} = 0,$$

where the σ 's in these equations denote the components of stress both in the elastic and plastic portions of the body, let us now put $j = 3$ in (12.2). Then from (13.1) it follows that

$$(13.2) \quad \xi_{i3} = 0.$$

Putting $k = 3$ in (12.2) we have $\xi_{ij} \nu_3 = 0$. But not all ξ_{ij} vanish, from the italicized result at the end of §12. Hence

$$(13.3) \quad \nu_3 = 0.$$

Similarly, when we take $j = 3$ in (12.4) and use (13.2) we obtain

$$(13.4) \quad \lambda_3 = 0.$$

On account of (13.3) the condition (12.3) reduces to

$$(13.5) \quad \lambda_1 \nu_1 + \lambda_2 \nu_2 = 0.$$

Also from (13.2), (13.3) and (13.4) the set of conditions consisting of (12.4) and (12.6) becomes

$$(13.6) \quad \begin{cases} \xi_{11} \nu_1 + \xi_{12} \nu_2 = -G\rho \lambda_1, \\ \xi_{21} \nu_1 + \xi_{22} \nu_2 = -G\rho \lambda_2, \\ \xi_{11} \sigma_{11}^* + 2\xi_{12} \sigma_{12}^* + \xi_{22} \sigma_{22}^* = 0. \end{cases}$$

The determinant Δ of the coefficients of the ξ 's in the equations (13.6) is

$$(13.7) \quad \Delta = \begin{vmatrix} \nu_1 & \nu_2 & 0 \\ 0 & \nu_1 & \nu_2 \\ \sigma_{11}^* & 2\sigma_{12}^* & \sigma_{22}^* \end{vmatrix} = \sigma_{22}^* \nu_1^2 - 2\sigma_{12}^* \nu_1 \nu_2 + \sigma_{11}^* \nu_2^2.$$

Assuming $\Delta \neq 0$ we may solve the equations (13.6) to obtain

$$(13.8) \quad \begin{cases} \xi_{11} = \frac{2G\rho}{\Delta} (\sigma_{22}^* \lambda_2 \nu_2 + \sigma_{12}^* \lambda_1 \nu_2), \\ \xi_{12} = -\frac{G\rho}{\Delta} (\sigma_{11}^* \lambda_1 \nu_2 + \sigma_{22}^* \lambda_2 \nu_1), \\ \xi_{22} = \frac{2G\rho}{\Delta} (\sigma_{11}^* \lambda_1 \nu_1 + \sigma_{12}^* \lambda_2 \nu_1). \end{cases}$$

14. Simplification by canonical coordinates. A transformation to canonical coordinates z leaves invariant the plane stress condition (13.1), and hence the relations obtained in §13 remain unaltered by this transformation.¹³ In general, the selection of the system of canonical coordinates will depend on the point P of Σ under consideration. However, since the components σ_{ij} are constant in the x system by assumption (iii) we can choose a *fixed* system of canonical coordinates z which are related to the coordinates x by an orthogonal transformation $x \leftrightarrow z$ independent of the time t .

In the above system of canonical coordinates the relations (13.7) and (13.8) take the somewhat simpler forms

$$(14.1) \quad \Delta = \sigma_2^* \nu_1^2 + \sigma_1^* \nu_2^2$$

and

$$(14.2) \quad \begin{cases} \xi_{11} = \frac{2G\rho}{\Delta} (\sigma_2^* \lambda_2 \nu_2); & \xi_{22} = \frac{2G\rho}{\Delta} (\sigma_1^* \lambda_1 \nu_1), \\ \xi_{12} = \frac{-G\rho}{\Delta} (\sigma_1^* \lambda_1 \nu_2 + \sigma_2^* \lambda_2 \nu_1), \end{cases}$$

where the σ_i^* are the principal values of the stress deviator. To avoid the introduction of new symbols we use the same letters for the components of the quantities ξ , λ and ν in the canonical and x coordinate systems.

As observed in §8 there are only three independent equations (7.8). Similarly, the set of relations (12.5) contains but three independent equations which can be obtained by taking $i, j = (1, 2), (2, 2)$ and $(3, 3)$. In canonical coordinates these independent equations are readily seen to be given by

$$(14.3) \quad \begin{cases} G\xi_{12} + \mu(\lambda_1 \nu_2 + \lambda_2 \nu_1) = 0, \\ G(\xi_{22} - \frac{1}{3}\xi_{aa}) + 2\mu \left\{ 1 + \frac{1}{K} (\sigma_1^* - \sigma_2^*) \sigma_2^* \right\} \lambda_2 \nu_2 = 0, \\ \frac{G}{3} \xi_{aa} - \frac{2\mu}{K} (\sigma_1^* - \sigma_2^*) \sigma_3^* \lambda_2 \nu_2 = 0. \end{cases}$$

¹³ See T. Y. THOMAS, *On the propagation of weak discontinuities in perfectly plastic solids*, Jour. of Math. and Mech. **6** (1957) pp. 67–86.

Now when we eliminate the quantities ξ_{ij} from (14.3) by the substitution (14.2) and make use of (13.5) and the identity

$$(14.4) \quad \sigma_1^* + \sigma_2^* + \sigma_3^* = 0,$$

we find that

$$\begin{aligned} \frac{G^2 \rho}{\Delta} (\sigma_1^* \lambda_1 \nu_2 + \sigma_2^* \lambda_2 \nu_1) - \mu (\lambda_1 \nu_2 + \lambda_2 \nu_1) &= 0, \\ \left\{ \frac{G^2 \rho}{\Delta} (\sigma_1^* - \sigma_3^*) - 3\mu - \frac{3\mu}{K} (\sigma_1^* - \sigma_2^*) \sigma_2^* \right\} \lambda_1 \nu_1 &= 0, \\ \left(\frac{G^2 \rho}{\Delta} + \frac{3\mu}{K} \sigma_3^* \right) (\sigma_1^* - \sigma_2^*) \lambda_1 \nu_1 &= 0. \end{aligned}$$

But, using the proportionality

$$\lambda_1, \lambda_2 \propto \nu_2, -\nu_1,$$

which is equivalent to (13.5), the quantities λ_1 and λ_2 can be eliminated from the above equations. We thus obtain

$$(14.5) \quad \frac{G^2 \rho}{\mu \Delta} (\sigma_1^* \nu_2^2 - \sigma_2^* \nu_1^2) + \nu_1^2 - \nu_2^2 = 0,$$

$$(14.6) \quad \left\{ \frac{G^2 \rho}{\mu \Delta} (\sigma_1^* - \sigma_3^*) - \frac{3}{K} (\sigma_1^* - \sigma_2^*) \sigma_2^* - 3 \right\} \nu_1 \nu_2 = 0,$$

$$(14.7) \quad \left(\frac{G^2 \rho}{\mu \Delta} + \frac{3\sigma_3^*}{K} \right) (\sigma_1^* - \sigma_2^*) \nu_1 \nu_2 = 0.$$

The equations (14.5), (14.6) and (14.7) give conditions on the components of the unit normal ν and lead to a determination of the possible wave surfaces Σ . The various cases which can arise are discussed in the following sections.

15. Case I ($\nu_1 = \pm 1, \nu_2 = 0$). Equations (14.6) and (14.7) are satisfied automatically. The determinant $\Delta = \sigma_2^*$ from (14.1). Hence the equation (14.5) reduces to

$$(15.1) \quad G^2 = \frac{\mu}{\rho}.$$

Hence the wave surface Σ is a plane perpendicular to the z_1 axis of a system of canonical coordinates and is propagated with a constant velocity G given by (15.1).

From the equations (13.5) we have $\lambda_1 = 0$. Hence $\lambda_2 \neq 0$ from (13.4) and the fact that not all of the λ 's can vanish by the result at the end of §12. Consequently the equations (14.2) become

$$(15.2) \quad \xi_{11} = 0; \quad \xi_{12} = \mp G \rho \lambda_2; \quad \xi_{22} = 0,$$

with the minus and plus signs in the second of these equations corresponding to $\nu_1 = +1$ and $\nu_1 = -1$ respectively. Hence in a system of canonical coordinates all of the ξ 's vanish except ξ_{12} which is given by the second equation (15.2).

16. Case II ($\nu_1 = 0, \nu_2 = \pm 1$). Equations (14.6) and (14.7) are again satisfied automatically and the equation (14.5) again reduces to (15.1) since now $\Delta = \sigma_1^*$. From (13.5) it follows that $\lambda_2 = 0$. Hence $\lambda_1 \neq 0$ and the equations (14.2) become

$$(16.1) \quad \xi_{11} = 0; \quad \xi_{12} = \mp G\rho\lambda_1; \quad \xi_{22} = 0,$$

in which the $-$ and $+$ signs correspond to $\nu_2 = +1$ and $\nu_2 = -1$ respectively.

For this case the wave surface Σ is a plane perpendicular to the z_2 axis of the system of canonical coordinates and is propagated with constant velocity G given by (15.1). All components ξ vanish in the canonical system except the component ξ_{12} which is given by the second equation (16.1).

17. Case III ($\nu_1\nu_2 \neq 0, \sigma_1^* = \sigma_2^*$). Equation (14.7) is satisfied. Now $\Delta = \sigma_1^* = \sigma_2^*$. Hence equations (14.5) and (14.6) become

$$(17.1) \quad (G^2\rho - \mu)(\nu_1^2 - \nu_2^2) = 0,$$

$$(17.2) \quad G^2\rho(\sigma_1^* - \sigma_3^*) - 3\mu\sigma_1^* = 0.$$

But $\sigma_3^* = -2\sigma_1^*$ for this case. Making this substitution in (17.2) and taking account of the fact that σ_1^* must be different from zero¹⁴ we see that the equation reduces to (15.1), in consequence of which equation (17.1) is automatically satisfied.

Equations (14.2) giving the ξ 's become

$$\begin{cases} \xi_{11} = 2G\rho\lambda_2\nu_2; & \xi_{22} = 2G\rho\lambda_1\nu_1, \\ \xi_{12} = -G\rho(\lambda_1\nu_2 + \lambda_2\nu_1). \end{cases}$$

From the first two of these relations and (13.2) and (13.5) we see that the combination $\xi_{aa} = 0$. Since the components ν_1 and ν_2 are not determined by these conditions, Σ can be any surface perpendicular to the x_1, x_2 coordinate plane.¹⁵

18. Case IV ($\nu_1\nu_2 \neq 0, \sigma_1^* \neq \sigma_2^*$). On account of (13.5) and the fact that λ_1 and λ_2 cannot both vanish we see immediately that neither λ_1 nor λ_2 can vanish. Equation (14.7) yields

$$(18.1) \quad \frac{G^2\rho}{\mu\Delta} + \frac{3\sigma_3^*}{K} = 0.$$

¹⁴ If $\sigma_1^* = 0$ we have $\sigma_1^* = \sigma_2^* = \sigma_3^* = 0$ from the conditions of Case III and the identity (14.4). But this violates the yield condition (1.1) in which the material constant K is positive.

¹⁵ The x_1, x_2 coordinate plane is parallel to the z_1, z_2 plane as is immediately observed from the form of the transformation relating the coordinates of the x and z systems. See T. Y. THOMAS, *On the propagation of weak discontinuities in perfectly plastic solids, loc. cit.*, p. 70.

Eliminating the term $G^2/\mu\Delta$ by means of (18.1) we find that equation (14.6) can be written

$$(18.2) \quad (\sigma_1^* - \sigma_3^*)\sigma_3^* + (\sigma_1^* - \sigma_2^*)\sigma_2^* + K = 0.$$

But

$$(18.3) \quad K = (\sigma_1^*)^2 + (\sigma_2^*)^2 + (\sigma_3^*)^2,$$

from the condition (1.1). Now, replacing K in (18.2) by its value as given by (18.3), it is easily seen that the resulting equation is satisfied identically.

Using (18.1) and replacing ν_2^2 by its value $1 - \nu_1^2$ we obtain

$$(18.4) \quad \nu_1^2 = \frac{3\sigma_1^*\sigma_3^* + K}{3\sigma_1^*\sigma_3^* + 3\sigma_2^*\sigma_3^* + 2K}$$

from (14.5). But, using (18.3) and the identity (14.4), it can readily be shown that

$$3\sigma_1^*\sigma_3^* + K = (\sigma_2^* - \sigma_1^*)(\sigma_2^* - \sigma_3^*),$$

$$3\sigma_1^*\sigma_3^* + 3\sigma_2^*\sigma_3^* + 2K = (\sigma_1^* - \sigma_2^*)^2.$$

Making these substitutions in (18.4) we obtain

$$(18.5) \quad \nu_1^2 = \frac{\sigma_2^* - \sigma_3^*}{\sigma_2^* - \sigma_1^*}.$$

Hence

$$(18.6) \quad \nu_2^2 = \frac{\sigma_3^* - \sigma_1^*}{\sigma_2^* - \sigma_1^*}.$$

The surface Σ is therefore a plane, perpendicular to the x_1, x_2 coordinate plane,¹⁵ such that the components ν_1 and ν_2 of its unit normal ν , relative to the canonical system, are determined from the stress in the elastic region in accordance with (18.5) and (18.6).

From (18.5), (18.6) and the equation (14.1) for Δ we have

$$(18.7) \quad \Delta = \frac{\sigma_1^*(\sigma_3^* - \sigma_1^*)}{\sigma_2^* - \sigma_1^*} + \frac{\sigma_2^*(\sigma_2^* - \sigma_3^*)}{\sigma_2^* - \sigma_1^*}.$$

But by an easily manipulation, involving use of the identity (14.4), it can be shown that the above relation (18.7) reduces to $\Delta = -2\sigma_3^*$. Hence from (18.1) we obtain

$$G^2 = \frac{6\mu}{\rho K} (\sigma_3^*)^2$$

as the equation for the determination of the velocity G .

TRANSITION FROM ELASTIC TO PLASTIC STATE WITHOUT FRACTURE

Let us now return to the case of the thin flat plate under simple tension τ in the x_2 direction with the x_1, x_2 plane parallel to the flat sides of the plate. For this case the stress components σ_{ij} in the elastic region are given by $\sigma_{22} = \tau$ and $\sigma_{ij} = 0$ otherwise, and it is easily seen that the canonical coordinate system, introduced in §14, can be identified with the x coordinate system. Thus we have

$$\sigma_1^* = \sigma_{11}^* = \frac{-\tau}{3}; \quad \sigma_2^* = \sigma_{22}^* = \frac{2\tau}{3}; \quad \sigma_3^* = \sigma_{33}^* = \frac{-\tau}{3}.$$

Since $\sigma_1^* \neq \sigma_2^*$ it follows that Case III in §17 cannot be realized for the problem of the flat plate under simple tension. Also, if we use the above values of the σ_i^* , we find that $\nu_1^2 = 1$ from (18.5). Hence Case IV reduces to Case I for the flat plate problem.

We shall now treat Case I in detail. It will be found that points of infinite wave strength will not develop on the wave surface Σ for this case and consequently the elastic region in the flat plate will become plastic without fracture (see §1).

19. Preliminary remarks. Wave strength. Let us suppose for definiteness that $\nu_1 = 1$. Then Σ is a plane, perpendicular to the x_1 axis, which is propagated across the plate with a velocity given by (15.1) in the direction of the positive x_1 axis. As already observed λ_2 cannot vanish. We can therefore define the wave strength Λ of the wave Σ as the absolute value $|\lambda_2|$, *i.e.*

$$(19.1) \quad \Lambda = |\lambda_2|.$$

Explicit equations for the wave surface Σ are given by

$$(19.2) \quad x_1 = Gt + \text{const.}; \quad x_2 = x_2; \quad x_3 = x_3,$$

in which the x_2 and x_3 coordinates are used as the parametric coordinates of Σ . By δ time differentiation of these relations we have

$$\frac{\delta x_1}{\delta t} = G = \sqrt{\frac{\mu}{\rho}}; \quad \frac{\delta x_2}{\delta t} = 0; \quad \frac{\delta x_3}{\delta t} = 0.$$

20. Compatibility conditions of the second order. Since $G = \text{const.}$ and the coefficients $b_{\alpha\beta} = 0$ for the plane Σ it follows that the geometrical compatibility conditions (29) of order 2 and the kinematical compatibility conditions (42) and (43) of order 2 in the paper on *extended compatibility conditions*⁸ give us the following relations:

$$(20.1) \quad [v_{i,jk}] = \bar{v}_{i,jk} = \hat{\lambda}_i \nu_j \nu_k + \lambda_{i,\alpha} (\nu_j x_{k,\alpha} + \nu_k x_{j,\alpha}),$$

$$(20.2) \quad \left[\frac{\partial^2 v_i}{\partial x_j \partial t} \right] = \frac{\partial^2 \bar{v}_i}{\partial x_j \partial t} = \left(-G \hat{\lambda}_i + \frac{\delta \lambda_i}{\delta t} \right) \nu_j - G \lambda_{i,\alpha} x_{j,\alpha},$$

$$(20.3) \quad \left[\frac{\partial^2 v_i}{\partial t^2} \right] = \frac{\partial^2 \bar{v}_i}{\partial t^2} = G^2 \hat{\lambda}_i - 2G \frac{\delta \lambda_i}{\delta t},$$

where the $\hat{\lambda}_i$ are new functions defined on the plane Σ and the comma denotes partial differentiation with respect to the parametric coordinates x_2, x_3 of Σ . Correspondingly we have

$$(20.4) \quad [\sigma_{ij,km}] = \bar{\sigma}_{ij,km} = \hat{\xi}_{ij} \nu_k \nu_m + \xi_{ij,\alpha} (\nu_k x_{m,\alpha} + \nu_m x_{k,\alpha}),$$

$$(20.5) \quad \left[\frac{\partial^2 \sigma_{ij}}{\partial x_k \partial t} \right] = \frac{\partial^2 \bar{\sigma}_{ij}}{\partial x_k \partial t} = \left(-G \hat{\xi}_{ij} + \frac{\delta \xi_{ij}}{\delta t} \right) \nu_k - G \xi_{ij,\alpha} x_{k,\alpha},$$

$$(20.6) \quad \left[\frac{\partial^2 \sigma_{ij}}{\partial t^2} \right] = \frac{\partial^2 \bar{\sigma}_{ij}}{\partial t^2} = G^2 \hat{\xi}_{ij} - 2G \frac{\delta \xi_{ij}}{\delta t}.$$

If we put $j = 3$ in (20.4) and take account of the plane stress condition (13.1) we immediately obtain

$$(20.7) \quad \hat{\xi}_{i3} \nu_k \nu_m + \xi_{i3,\alpha} (\nu_k x_{m,\alpha} + \nu_m x_{k,\alpha}) = 0.$$

But $\xi_{i3,\alpha}$ vanishes from (13.2). Hence (20.7) gives

$$(20.8) \quad \hat{\xi}_{i3} = 0.$$

Next, taking $m = 3$ in (20.4) and using the fact that $\nu_3 = 0$, we obtain

$$(20.9) \quad \xi_{ij,\alpha} \nu_k x_{3,\alpha} = 0.$$

Now from (19.2) we see that $x_{3,2} = 0$ and $x_{3,3} = 1$ in the above relations (20.9). Hence (20.9) reduces to $\xi_{i2,3} = 0$. But, since all of the ξ 's have been shown to vanish with the exception of ξ_{12} , this provides us with a single new relation, namely $\xi_{12,3} = 0$. Hence $\Lambda_{,3} = 0$ from (15.2) and the definition (19.1) of the strength of the wave Σ . In other words, *the quantities ξ_{12} and Λ do not vary on Σ in the x_3 direction at any time.*

Putting $i = j$ in (20.1), summing on the repeated indices, and taking account of the incompressibility condition (1.4), we have

$$(20.10) \quad \hat{\lambda}_i \nu_i \nu_k + \lambda_{i,\alpha} \nu_i x_{k,\alpha} + \lambda_{i,\alpha} x_{i,\alpha} \nu_k = 0.$$

Since $\lambda_1 = 0$ these relations are satisfied identically when $k = 2, 3$. However, if $k = 1$ equation (20.10) yields the relation

$$(20.11) \quad \hat{\lambda}_1 + \lambda_{2,2} = 0.$$

21. Further consequences of the compatibility conditions. Stability of the wave. Additional conditions of a dynamical nature can be obtained from the yield condition (1.1), the equations of motion (7.4), and the Prandtl-Reuss equations (7.5). Thus, differentiating (1.1) twice partially with respect to the time t and evaluating on the plastic side of Σ , we have

$$(21.1) \quad \sigma_{ij}^* \frac{\partial^2 \bar{\sigma}_{ij}}{\partial t^2} + \frac{\partial \bar{\sigma}_{ij}^*}{\partial t} \frac{\partial \bar{\sigma}_{ij}}{\partial t} = 0,$$

where we have omitted the star on the second derivatives and on one of the first derivatives as is evidently permissible in view of the identity $\sigma_{ii}^* = 0$. Now from (12.2) we obtain

$$\frac{\partial \bar{\sigma}_{ii}^*}{\partial t} = -G\xi_{ii} + \frac{1}{3}G\xi_{aa}\delta_{ii} = -G\xi_{ii} = \frac{\partial \bar{\sigma}_{ii}}{\partial t},$$

since $\xi_{aa} = 0$. Making this substitution for the first derivatives and eliminating the second derivatives in (21.1) by means of (20.6) we find that

$$(21.2) \quad \sigma_{ii}^* \left(G^2 \hat{\xi}_{ii} - 2G \frac{\partial \xi_{ii}}{\partial t} \right) + G^2 \xi_{ii} \xi_{ii} = 0.$$

But when we make the substitutions (15.2) and (19.1), use the values of σ_{ii}^* appropriate to the simple tension τ , and take account of the condition (20.8), the equation (21.2) is seen to reduce to

$$(21.3) \quad 2\hat{\xi}_{22} = \hat{\xi}_{11} - 6 \left(\frac{G^2 \rho^2}{\tau} \right) \lambda_2^2.$$

Now differentiate the equations (7.4) with respect to x_m , multiply by ν_m , and evaluate on the plastic side of Σ . We thus obtain

$$(21.4) \quad \bar{\sigma}_{ii,im}\nu_m = \rho \frac{\partial^2 \bar{v}_i}{\partial x_m \partial t} \nu_m + \rho \bar{v}_{i,i} \bar{v}_{j,m} \nu_m,$$

when use is made of the conditions $\rho = \bar{\rho}$ and $\partial \bar{\rho} / \partial x_m = 0$ which follow from the facts (1) that ρ is constant in the elastic region and (2) that both ρ and its derivatives are continuous across Σ by the result in §11. But, using (12.1), (20.2) and (20.4), we find

$$\begin{cases} \bar{v}_{i,i} \bar{v}_{j,m} \nu_m = \lambda_i \lambda_1 = 0; & \frac{\partial^2 \bar{v}_i}{\partial x_m \partial t} \nu_m = -G\hat{\lambda}_i + \frac{\partial \lambda_i}{\partial t}, \\ \bar{\sigma}_{ii,im}\nu_m = \hat{\xi}_{ii}\nu_i + \xi_{ii,\alpha} x_{i,\alpha} = \hat{\xi}_{i1} + \xi_{i2,2}. \end{cases}$$

Making these substitutions in (21.4) the resulting equations become

$$(21.5) \quad \hat{\xi}_{i1} + \xi_{i2,2} = \rho \left(-G\hat{\lambda}_i + \frac{\partial \lambda_i}{\partial t} \right).$$

If we take $i = 1, 2$ the equations (21.5) give

$$(21.6) \quad \hat{\xi}_{11} + \xi_{12,2} = -\rho G\hat{\lambda}_1,$$

$$(21.7) \quad \hat{\xi}_{12} = -\rho G\hat{\lambda}_2 + \rho \frac{\partial \lambda_2}{\partial t},$$

since $\lambda_1 = 0$ and $\xi_{22} = 0$. Similarly, taking $i = 3$ it follows from (13.2), (13.4) and (20.8) that

$$(21.8) \quad \hat{\lambda}_3 = 0.$$

Let us now apply this procedure to the Prandtl-Reuss equations, *i.e.* we differentiate the equations (7.5) with respect to the time t and evaluate on the plastic side of Σ to obtain

$$(21.9) \quad \frac{\partial^2 \bar{\sigma}_{ij}^*}{\partial t^2} + \bar{\sigma}_{ij,k}^* \frac{\partial \bar{v}_k}{\partial t} = 2\mu \left(\frac{\partial \bar{\epsilon}_{ij}}{\partial t} - \frac{\partial \bar{\Psi}}{\partial t} \sigma_{ij}^* - \bar{\Psi} \frac{\partial \bar{\sigma}_{ij}^*}{\partial t} \right),$$

where the bar has been omitted from the σ_{ij}^* , due to the continuity of the stress components along Σ . Now

$$(21.10) \quad \bar{\Psi} = \frac{\bar{\epsilon}_{ab} \sigma_{ab}}{K} = \frac{\bar{\epsilon}_{22} \tau}{K} = \frac{\lambda_2 \nu_2 \tau}{K} = 0,$$

since $\nu_2 = 0$. Also it readily follows by application of the appropriate compatibility conditions and auxiliary relations from the foregoing discussion that

$$(21.11) \quad \frac{\partial^2 \bar{\sigma}_{ij}^*}{\partial t^2} = G^2 (\hat{\xi}_{ij} - \frac{1}{3} \hat{\xi}_{aa} \delta_{ij}) - 2G \frac{\delta \xi_{ij}}{\delta t},$$

$$(21.12) \quad \bar{\sigma}_{ij,k}^* \frac{\partial \bar{v}_k}{\partial t} = 0; \quad \frac{\partial \bar{\Psi}}{\partial t} = -\frac{G}{K} (\tau \lambda_{2,2} - \rho G \lambda_2^2),$$

$$(21.13) \quad \frac{\partial \bar{\epsilon}_{ij}}{\partial t} = \frac{1}{2} \left\{ -G (\hat{\lambda}_i \nu_j + \hat{\lambda}_j \nu_i) + \frac{\delta \lambda_i}{\delta t} \nu_j + \frac{\delta \lambda_j}{\delta t} \nu_i - G (\lambda_{i,\alpha} x_{j,\alpha} + \lambda_{j,\alpha} x_{i,\alpha}) \right\},$$

when the elastic region is in a state of simple tension τ . Making the substitutions (21.10)—(21.13) the equations (21.9) become

$$(21.14) \quad \begin{cases} G^2 (\hat{\xi}_{ij} - \frac{1}{3} \hat{\xi}_{aa} \delta_{ij}) - 2G \frac{\delta \xi_{ij}}{\delta t} - \frac{2\mu G}{K} (\tau \lambda_{2,2} - \rho G \lambda_2^2) \sigma_{ij}^* \\ = \mu \left\{ -G (\hat{\lambda}_i \nu_j + \hat{\lambda}_j \nu_i) + \frac{\delta \lambda_i}{\delta t} \nu_j + \frac{\delta \lambda_j}{\delta t} \nu_i - G (\lambda_{i,\alpha} x_{j,\alpha} + \lambda_{j,\alpha} x_{i,\alpha}) \right\}. \end{cases}$$

To realize the consequences of the conditions (21.14) we must give the indices i, j their special values 1, 2, 3 and then reduce the resulting equations by means of the various relations which we have established. Thus taking $i = 1, j = 2$ we find that equation (21.14) can be reduced to give

$$(21.15) \quad \frac{\delta \lambda_2}{\delta t} = 0,$$

when use is made of (15.1), (15.2), (19.2) and (21.7); the reduction, of course, also requires the use of the values 1, 0, 0 for the components ν_1, ν_2, ν_3 and the fact that $\sigma_{12}^* = 0$, corresponding to the condition of simple tension in the elastic region. In view of (19.1) the condition (21.15) permits us to state the following result. *The strength Λ of the plane wave Σ does not vary along the normal directions during its propagation.*

Next take $i = 1, j = 3$. But when this is done and use is made of (13.4), (20.8) and (21.8) we find that the equation (21.14) is satisfied identically.

However, when we take $i = 2, j = 3$ equation (21.14) yields the condition

$$(21.16) \quad \lambda_{2,3} = 0.$$

In other words, *the strength Λ of the wave Σ does not vary at any time along the direction perpendicular to the flat sides of the plate.*

Let us now put $i = j = 1$ in (21.14). Then, in the first instance, equation (21.14) gives

$$(21.17) \quad G^2(\hat{\xi}_{11} - \frac{1}{3}\hat{\xi}_{aa}) + \frac{2\mu\tau G}{3K}(\tau\lambda_{2,2} - \rho G\lambda_2^2) + 2\mu G\hat{\lambda}_1 = 0.$$

But when account is taken of the equations (15.2), (20.11), (21.3) and (21.6) we find that $\lambda_{2,2} = 0$, *i.e. the strength of the wave does not vary at any time in the direction of the tension.* Taking $i = j = 2$ and $i = j = 3$ no conditions arise from (21.14) beyond those already at our disposal.

It may be concluded from the above results that points of infinite wave strength are not necessarily associated with the wave Σ at any given time $t = t_0$, and it follows also from (21.15) that such points cannot arise as a consequence of the wave propagation. Moreover, the wave will be *stable* in the sense that its strength does not vary during the propagation (see §10), and we must therefore expect that, for this case, the elastic region of the flat plate will pass into the plastic state without the occurrence of fracture.

FRACTURE IN THE DIRECTION OF TENSION

We shall now investigate Case II of §16. Due to the general similarity of this case with Case I most of the preceding formulæ can be applied to the present discussion without modification. However, there is an essential difference between these two cases which arises from the difference in the orientation of the wave surface Σ relative to the direction of the applied load. As a consequence of this difference we shall find that there is a tendency for the occurrence of *fracture points* (see §23) on the wave surface Σ of Case II which will cause a crack in the direction of the tension in the plate.

22. Basic formulæ of the problem. We recall that for Case II (§16) the wave surface Σ is a plane perpendicular to the x_2 direction, *i.e.* the direction of the tension, and to the flat sides of the plate and that it is propagated with a velocity G given by (15.1). For definiteness we assume that the components of the unit normal ν have the values 0, 1, 0 so that the propagation is in the direction of the *positive* x_2 axis. The equations of Σ are therefore of the form

$$(22.1) \quad x_1 = x_1; \quad x_2 = Gt + \text{const.}; \quad x_3 = x_3,$$

in which the coordinates x_1, x_3 are also taken as parametric coordinates on Σ .

Since $\nu_2 = +1$ the minus sign applies in the second equation (16.1). Now since all ξ 's and all λ 's vanish except ξ_{12} and λ_1 , respectively, let us put $\xi_{12} = \xi$

and $\lambda_1 = \lambda$ for brevity. Let us also define the strength Λ for the wave Σ as the absolute value of the quantity λ . Hence we have

$$(22.2) \quad \xi = -\rho G\lambda; \quad \Lambda = |\lambda|.$$

Equations (20.1)—(20.6) also hold for this case. Putting $j = 3$ in (20.4) we recover the condition (20.8). Also taking $m = 3$ in (20.4) we are again led to the relation $\Lambda_{,3} = 0$ so that the italicized statement at the end of §20 is still valid. Equations (20.10) likewise hold and are found to be satisfied identically for $k = 1, 3$. Putting $k = 2$ in (20.10) and using the above symbol λ we now obtain, in place of (20.11), the equation

$$(22.3) \quad \hat{\lambda}_2 + \lambda_{,1} = 0.$$

Equation (21.2) holds but is now seen to reduce to

$$(22.4) \quad \hat{\xi}_{11} = 2\hat{\xi}_{22} + 6\left(\frac{G^2\rho^2}{\tau}\right)\lambda^2,$$

instead of (21.3). We now have the equations (21.4) in which

$$\begin{cases} \bar{v}_{i,i}\bar{v}_{i,m}\nu_m = 0; & \frac{\partial^2 \bar{v}_i}{\partial x_m \partial t} \nu_m = -G\hat{\lambda}_i + \frac{\delta \lambda_i}{\delta t}, \\ \bar{\sigma}_{ij,im}\nu_m = \hat{\xi}_{ij}\nu_j + \xi_{ij,\alpha}x_{j,\alpha} = \hat{\xi}_{i2} + \xi_{i1,1}, \end{cases}$$

as can readily be observed when use is made of (22.1) and the compatibility conditions. Making these substitutions in (21.4) we obtain

$$(22.5) \quad \hat{\xi}_{i2} + \xi_{i1,1} = \rho\left(-G\hat{\lambda}_i + \frac{\delta \lambda_i}{\delta t}\right).$$

Taking $i = 3$ in (22.5) we again find the condition (21.8). However, when we put $i = 1, 2$ in (22.5) we see that

$$(22.6) \quad \hat{\xi}_{12} = \rho\left(-G\hat{\lambda}_1 + \frac{\delta \lambda}{\delta t}\right),$$

$$(22.7) \quad \hat{\xi}_{22} = \rho G(\lambda_{,1} - \hat{\lambda}_2).$$

These two relations replace (21.6) and (21.7). Combining (22.4) and (22.7) and eliminating the quantity $\hat{\lambda}_2$ by means of (22.3) we have

$$(22.8) \quad \hat{\xi}_{11} = 4\rho G\lambda_{,1} + 6\left(\frac{G^2\rho^2}{\tau}\right)\lambda^2,$$

$$(22.9) \quad \hat{\xi}_{22} = 2\rho G\lambda_{,1}.$$

Equations (21.9) continue to hold as well as (21.10) since now $\lambda_2 = 0$. Also for this case we have (21.11), the first equation (21.12) and the equations (21.13). The second equation (21.12) must, however, be replaced by

$$\frac{\partial \bar{\Psi}}{\partial t} = \frac{G}{K} (\tau \lambda_{,1} + \rho G \lambda^2).$$

Hence we find that equations (21.9) lead to the conditions

$$(22.10) \quad \left\{ \begin{aligned} & G^2({}_i\hat{\xi}_i - \frac{1}{3}\hat{\xi}_{aa}\delta_{ii}) - 2G \frac{\delta \xi_{ij}}{\delta t} + \frac{2\mu G}{K} (\tau \lambda_{,1} + \rho G \lambda^2) \sigma_{ij}^* \\ & = \mu \left\{ -G(\hat{\lambda}_i \nu_i + \hat{\lambda}_j \nu_j) + \frac{\delta \lambda_i}{\delta t} \nu_i + \frac{\delta \lambda_j}{\delta t} \nu_j - G(\lambda_{i,\alpha} x_{i,\alpha} + \lambda_{j,\alpha} x_{j,\alpha}) \right\}. \end{aligned} \right.$$

Reduction of (22.10) for $i, j = 1, 2$ gives the condition

$$(22.11) \quad \frac{\delta \lambda}{\delta t} = 0,$$

i.e. *the strength Λ of the wave does not vary along the normal directions during its propagation*. Taking $i, j = 1, 3$ and $i, j = 2, 3$ in (22.10) we obtain the two relations

$$(22.12) \quad \lambda_{,3} = 0; \quad \hat{\lambda}_3 = 0,$$

the first of which expresses the fact that *the wave strength is constant at any time along the direction perpendicular to the flat sides of the plate*.

Taking $i = j = 1$ in (22.10) the resulting equation is seen immediately to reduce to

$$(22.13) \quad G(2\hat{\xi}_{11} - \hat{\xi}_{22}) - \frac{2\mu\tau}{K} \rho G^2 \lambda^2 + 3\mu \lambda_{,1} = 0.$$

Then, making the substitutions (22.8) and (22.9) and replacing K by its value $\frac{2}{3}\tau^2$ in terms of the tension τ , we find that (22.13) gives the condition

$$(22.14) \quad \lambda_{,1} + \frac{\rho G}{\tau} \lambda^2 = 0.$$

This same condition is also obtained when we take $i = j = 3$ in (22.10). Finally, taking $i = j = 2$ the equation (22.10) is seen to be satisfied identically.

23. Fracture points. Propagation of cracks. The above relation (22.14) supplements the two italicized statements at the end of §22 in that it gives information regarding the variation of the wave strength in the x_1 direction, i.e. across the width of the plate. Now from (22.11) and the first equation (22.12) we see that λ , or the wave strength Λ , will be independent of the time and will depend at most on the single coordinate x_1 . Hence if λ has the *finite* value λ_0 when $x_1 = 0$ we find, by integration of (22.14), that

$$(23.1) \quad \lambda = \frac{\lambda_0}{1 - \frac{x_1}{A}},$$

where

$$(23.2) \quad A = \frac{-\tau}{\rho G \lambda_0} = \frac{-\tau}{\lambda_0 \sqrt{\rho \mu}}.$$

The algebraic sign of the quantity A will depend on the algebraic sign of the initial discontinuity λ_0 as well as the algebraic sign of τ , with $\tau > 0$ for tension and $\tau < 0$ for compression⁹ which are equally permissible from the mathematical standpoint in this discussion.

From the form of the relation (23.1) we see that $\lambda \rightarrow \infty$ as $x_1 \rightarrow A$. Points on the wave surface Σ for which $x_1 = A$ will be referred to as *fracture points* and at such points the wave strength Λ will be infinite. It follows from (23.1) and (23.2) that these fracture points will always occur on the wave Σ provided the initial discontinuity λ_0 is sufficiently large.

Assuming fracture at points of infinite wave strength (see §1) the existence of fracture points on the wave at the initial time $t = 0$ will result in a *crack* which will be propagated in the direction of the tension or compression with the velocity G , given by (15.1), as a consequence of the propagation of the wave. Such a crack will produce fracture of the plate in a plane perpendicular to the flat sides and parallel to the direction of the tension or compression. As indicated at the end of §1 the thin flat plate can be regarded as a limiting case of a hollow cylinder of indefinitely large radius and hence the longitudinal cracks produced in hollow cylinders when subjected to compression¹⁰ can be considered a confirmation of this result.

When the above fracture points do not arise the elastic region of the plate will, of course, pass into the plastic state as a result of the wave propagation without the mishap of fracture.

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