On the Orthogonal Equivalence of Sets of Real Symmetric Matrices¹

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1. Introduction. In a letter written in May, 1957, CARL B. ALLENDORFER stated the following conjecture. *Let A_1, A_2, A_1^*, A_2^* be n -rowed real symmetric matrices, x_1 and x_2 be independent indeterminates over the field $\Re$ of all real numbers. Then there exists a real orthogonal matrix P such that $PA_1P' = A_1^*$ and $PA_2P' = A_2^*$ if and only if the matrices $X = x_1A_1 + x_2A_2$ and $X^* = x_1A_1^* + x_2A_2^*$ have the same characteristic determinant.* ALLENDORFER also stated that he had verified the conjecture for $n = 2$.

The conjecture is actually false for $n \geq 3$, and we shall construct classes of counterexamples. However, we shall derive a correct consequence of the determinant condition.

We shall consider the following general formulation of the *determinant condition*. Let $A_1, \dots, A_m$ be n -rowed real symmetric matrices and $\mathfrak{J}$ be the vector space of all matrices

$$(1) \quad A = a_1A_1 + \dots + a_mA_m,$$

for real numbers $a_1, \dots, a_m$. We let $x_1, \dots, x_m$ be algebraically independent indeterminates over $\Re$ and

$$(2) \quad X = x_1A_1 + \dots + x_mA_m.$$

Then we call X the *general element* of $\mathfrak{J}$. We also consider a second set $A_1^*, \dots, A_m^*$ of n -rowed real symmetric matrices and the vector space $\mathfrak{J}^*$ of all matrices

$$(3) \quad A^* = a_1A_1^* + \dots + a_mA_m^*,$$

for real a_i . If A and A^* are given by (1) and (2), for the same a_i , the mapping S defined by

$$(4) \quad A \rightarrow A^S = A^*$$

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is a linear mapping over $\mathfrak{H}$ of $\mathfrak{J}$ onto $\mathfrak{J}^*$. Then S induces the mapping $X \rightarrow X^*$ of the general element X of $\mathfrak{J}$ onto an *image general element*

$$(5) \quad X^* = x_1 A_1^* + \cdots + x_m A_m^*$$

of $\mathfrak{J}^*$. We shall then say that *the determinant condition holds for $\mathfrak{J}$ with $\mathfrak{J}^*$ relative to S* if the relation

$$(6) \quad \Delta(\mathfrak{J}, \mathfrak{J}^*, S) = |xI - X| - |xI - X^*| = 0$$

holds *identically* in $x, x_1, \dots, x_n$.

We have already indicated that the determinant condition does not, in general, imply the existence of a real orthogonal matrix P such that $PA_i P' = A_i^*$. However, let $\mathfrak{J}$ be *Jordan-closed*, that is, let $2A \cdot B = AB + BA$ be in $\mathfrak{J}$ for every A and B of $\mathfrak{J}$. Then we shall see that the condition $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ holds if and only if $\mathfrak{J}^*$ is also Jordan-closed and S is a Jordan algebra isomorphism of $\mathfrak{J}$ onto $\mathfrak{J}^*$. Moreover, the determinant condition then does imply the existence of a real orthogonal matrix P such that $PA_i P' = A_i^*$ except when the Jordan algebra $\mathfrak{J}$ has a simple component $\mathfrak{J}_i$ of degree two and dimension $m_i = 4q_i + 2$. In that case the enveloping associative *Clifford* algebra $\mathfrak{B}_i$ of $\mathfrak{J}_i$ contains a real symmetric matrix H_i which generates its center, $H_i = A_{i1} \cdots A_{im_i}$ for A_{ij} in $\mathfrak{J}_i$, and there is then a corresponding element $H_i^* = A_{i1}^* \cdots A_{im_i}^*$ which generates the center of the enveloping Clifford algebra of $\mathfrak{J}_i^*$. We shall show, finally, that in the Jordan-closed case there exists a real orthogonal matrix P such that $PA_i P' = A_i^*$ for $i = 1, \dots, m$ if and only if $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ and the trace condition $\tau(H_i) = \tau(H_i^*)$ holds for every component $\mathfrak{J}_i$ of degree two and dimension $4q_i + 2$ of $\mathfrak{J}$.

2. A class of counterexamples. Define

$$(7) \quad A_1 = A_1^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}, \quad A_2^* = \begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix},$$

and compute the characteristic determinant of

$$(8) \quad X = yA_1 + zA_2 = \begin{bmatrix} 0 & za & zb \\ za & y & zc \\ zb & zc & -y \end{bmatrix}.$$

This determinant is the polynomial

$$(9) \quad \phi(x, X) = |xI - X| = x^3 - \alpha(X)x^2 + \beta(x)x - \gamma(x),$$

where $\alpha(X)$ is the trace of X , $\beta(x)$ is the sum of the two-rowed principal minors of X , and $\gamma(X) = |X|$. Then $\alpha(X) = 0$,

$$(10) \quad -\beta(X) = (a^2 + b^2 + c^2)z^2 + y^2,$$

and

$$(11) \quad \gamma(X) = (a^2 - b^2)yz^2 + 2(abc)z^3.$$

It follows that if $X^* = yA_1^* + zA_2^*$ then $\alpha(X^*) = 0$,

$$(12) \quad -\beta(X^*) = (d^2 + e^2 + f^2)z^2 + y^2,$$

and

$$(13) \quad \gamma(X^*) = (d^2 - e^2)yz^2 + 2(def)z^3.$$

Hence $\Delta(\mathfrak{Y}, \mathfrak{Y}^*, S) = 0$ if and only if

$$(14) \quad a^2 + b^2 + c^2 = d^2 + e^2 + f^2, \quad a^2 - b^2 = d^2 - e^2, \quad abc = def.$$

These conditions are clearly satisfied by any real numbers a, b, c, d, e, f satisfying the equations

$$(15) \quad a = e, \quad b = f, \quad c = d, \quad b^2 + c^2 = 2a^2.$$

For the first and last equations of (14) are satisfied identically. The equation $a^2 - b^2 = d^2 - e^2$ is then equivalent to $a^2 - b^2 = c^2 - a^2$ and hence to $b^2 + c^2 = 2a^2$. We are now able to derive the following result.

Theorem 1. *Let A_1, A_2, A_1^*, A_2^* be given by (7) for real numbers a, b, c, d, e, f satisfying (15). Then $\Delta(\mathfrak{Y}, \mathfrak{Y}^*, S) = 0$, and if $b^2 \neq c^2$ there exists no real orthogonal matrix P such that $PA_1P' = A_1, PA_2P' = A_2^*$.*

For A_1 is a diagonal matrix with distinct diagonal elements, and so $PA_1 = A_1P$ if and only if P is a diagonal matrix. If P is a real orthogonal matrix and $P = \text{diag}\{\lambda, \mu, \nu\}$, then $\lambda^2 = \mu^2 = \nu^2 = 1$. It follows that

$$(16) \quad PA_2P' = \begin{bmatrix} 0 & \lambda\mu a & \lambda\nu b \\ \lambda\mu a & 0 & \mu\nu c \\ \lambda\nu b & \mu\nu c & 0 \end{bmatrix} = A_2^*$$

only if $a^2 = d^2, b^2 = e^2, c^2 = f^2$. However, (15) implies that when this occurs $c^2 = f^2 = b^2$, contrary to hypothesis. This completes our proof.

Note that, in particular, the values $b = f = 2, a = e = \sqrt{2}, c = d = 0$ satisfy (15) and $b^2 \neq c^2$. Thus we have the counterexample with

$$(17) \quad A_2 = \begin{bmatrix} 0 & \sqrt{2} & 2 \\ \sqrt{2} & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad A_2^* = \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 2 \\ \sqrt{2} & 2 & 0 \end{bmatrix}.$$

It is easy to extend this result to the case $n > 3$. Indeed, we define three n -rowed real symmetric matrices A_{1n}, A_{2n}, A_{2n}^* by

$$(18) \quad A_{1n} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{2n} = \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{2n}^* = \begin{bmatrix} A_2^* & 0 \\ 0 & 0 \end{bmatrix},$$

where A_1, A_2, A_2^* are given by (7) and (15) with $b^2 \neq c^2$. Take $A_{1n}^* = A_{1n}$ and see that if $X_n = yA_{1n} + zA_{2n}$ and $X = yA_1 + zA_2$ then

$$(19) \quad |xI - X_n| = \begin{vmatrix} xI_3 - (yA_1 + zA_2) & 0 \\ 0 & xI_{n-3} \end{vmatrix} = x^{n-3} |xI_3 - X|.$$

Thus $|xI - X_n^*| = x^{n-3} |xI_3 - X^*| = |xI - X_n|$, and the determinant condition holds. If P is a real n -rowed orthogonal matrix such that $PA_{1n}P' = A_{1n}$ and $PA_{2n}P' = A_{2n}^*$ then we have $PX_nP' = X_n^*$. Write

$$(20) \quad P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix},$$

where P_1 has three rows and columns, and obtain

$$(21) \quad \begin{aligned} PX_nP' &= \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} P' = \begin{pmatrix} P_1X & 0 \\ P_3X & 0 \end{pmatrix} \begin{pmatrix} P'_1 & P'_3 \\ P'_2 & P'_4 \end{pmatrix} \\ &= \begin{pmatrix} P_1XP'_1 & P_1XP'_3 \\ P_3XP'_1 & P_3XP'_3 \end{pmatrix} = \begin{pmatrix} X^* & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

It follows from the fact that $|X| = |X^*| = \gamma(X) \neq 0$ that P_1 is nonsingular $P_3 = 0, P_1XP'_1 = X^*$. But

$$(22) \quad P'P = \begin{pmatrix} P'_1 & 0 \\ P'_2 & P'_4 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ 0 & P_4 \end{pmatrix} = I,$$

and so $P'_1P_1 = I_3$; P_1 is orthogonal, contrary to Theorem 1. Note that (22) implies that $P_2 = 0$, a result used later. Thus, while the determinant condition holds for the n -rowed matrices $A_{1n}, A_{2n}, A_{1n}^* = A_{1n}, A_{2n}^*$, no real orthogonal matrix P exists with the property $PA_{1n}P' = A_{1n}^*, PA_{2n}P' = A_{2n}^*$.

3. Preliminary results. We define the vector spaces $\mathfrak{J}$ and $\mathfrak{J}^*$ over $\mathfrak{R}$, their elements A and A^* , and their general elements X and X^* as in the Introduction. Let us say that the set $A_1, \dots, A_m$ is *Jordan-closed* (and that the space $\mathfrak{J}$ is *Jordan-closed*) if

$$(23) \quad A \cdot B = \frac{1}{2}(AB + BA)$$

is in $\mathfrak{J}$ for every A and B of $\mathfrak{J}$. Then $\mathfrak{J}$ is what is called a *special Jordan algebra*. It should be clear that $\mathfrak{J}$ is Jordan-closed if and only if there exist m^3 real numbers g_{ijk} such that $A_iA_j + A_jA_i = \sum_{k=1}^m g_{ijk}A_k$ for $i, j = 1, \dots, m$. Let us now observe some simple consequences of the determinant condition which are independent of Jordan-closure.

Lemma 1. *Let A be any element of $\mathfrak{J}$ and A^* the image element of $\mathfrak{J}^*$. Then the determinant condition implies that there exists a real orthogonal matrix P such that $PAP' = A^*$.*

For the condition $|xI - X| = |xI - X^*|$ specializes to the condition $|xI - A| = |xI - A^*|$ and the *principal axis theorem* implies the existence of P .

If A is in $\mathfrak{S}$ and $A^* = 0$ Lemma 1 implies that $PAP' = 0$ and hence $A = 0$. We state this result as follows:

Lemma 2. *The determinant condition implies that the mapping $A \rightarrow A^*$ is a vector space isomorphism of $\mathfrak{S}$ onto $\mathfrak{S}^*$.*

We shall also use the following property.

Lemma 3. *Let $\Delta(\mathfrak{S}, \mathfrak{S}^*, S) = 0$, $B_1, \dots, B_r$ be any elements of $\mathfrak{S}$, $B_1^*, \dots, B_r^*$ be their images under S in $\mathfrak{S}^*$, and $y_1, \dots, y_r$ be scalars which are either real numbers or (algebraically independent) indeterminates over $\mathfrak{R}$. Then $Y = y_1B_1 + \dots + y_rB_r$ and $Y^* = y_1B_1^* + \dots + y_rB_r^*$ are similar in $\mathfrak{R} = \mathfrak{R}(y_1, \dots, y_r)$ and so have the same rank.*

There is no loss of generality if we assume that $y_1, \dots, y_s$ are algebraically independent indeterminates over $\mathfrak{R}$ and that $y_{s+1}, \dots, y_r$ are real. We can then replace $B_{s+1}, \dots, B_r$ by $B = y_{s+1}B_{s+1} + \dots + y_rB_r$ and $B_{s+1}^*, \dots, B_r^*$ by $B^* = y_{s+1}B_{s+1}^* + \dots + y_rB_r^*$ and so reduce to the case $r = s + 1$. The determinant condition can then be specialized to imply that $|xI - Y| = |xI - Y^*|$, where $Y = y_1B_1 + \dots + y_rB_r$ and $Y^* = y_1B_1^* + \dots + y_rB_r^*$. We can also use the homogeneity of the determinant condition to reduce to the case where $y_1, \dots, y_r$ are independent indeterminates over $\mathfrak{R}$. Thus it actually suffices to show that the original determinant condition $|xI - X| = |xI - X^*|$ implies that X and X^* are similar in $\mathfrak{R} = \mathfrak{R}(x_1, \dots, x_m)$.

The field $\mathfrak{R}$ is the quotient field of the polynomial ring $\mathfrak{L} = \mathfrak{R}[x_1, \dots, x_m]$. The ring $\mathfrak{L}$ contains all of the elements of X and X^* , and it is known² that X and X^* are similar in $\mathfrak{R}$ if and only if $xI - X$ and $xI - X^*$ have the same $\mathfrak{R}$ -elementary divisors. Each such elementary divisor is a power $[\phi(x)]^e$ of an irreducible polynomial of the ring $\mathfrak{R}[x]$. Since the product of these elementary divisors for X is $|xI - X|$, the determinant condition $|xI - X| = |xI - X^*|$ implies that the $\mathfrak{R}$ -elementary divisors of X are the same as those of its image X^* if we can show that every elementary divisor is simple, that is, $e = 1$ for every irreducible factor $\phi(x)$ of $|xI - X|$.

It is well known that $xI - X$ has a multiple elementary divisor if and only if the matrix ring $\mathfrak{R}[X]$ contains a nilpotent element. When this occurs it should be clear that $\mathfrak{L}[X]$ contains a nilpotent element, that is, there exists a nonzero matrix $N = N(x_1, \dots, x_m)$ in $\mathfrak{L}[X]$ such that $N^t = 0$ identically in $x_1, \dots, x_m$. There will then exist real numbers $a_1, \dots, a_m$ such that $N_0 = N(a_1, \dots, a_m) \neq 0$ and $N_0^t = 0$. But N_0 is a real nonzero symmetric matrix and has a nonzero real characteristic root λ ; $\lambda^t \neq 0$ is a characteristic root of $N_0^t = 0$, which is impossible. Hence X and X^* are similar and our proof is complete.

² This aspect of the similarity theory is treated in Chapter IV of the author's *Modern Higher Algebra*.

The following result³ reduces our orthogonal equivalence problem for sets of real symmetric matrices to one of similarity.

Lemma 4. *Let $\mathfrak{S}$ and $\mathfrak{S}^*$ be vector spaces of real n -rowed symmetric matrices and P be a real nonsingular matrix such that the mapping $A \rightarrow A^* = PAP^{-1}$ of $\mathfrak{S}$ into $\mathfrak{S}^*$ is an isomorphism of $\mathfrak{S}$ onto $\mathfrak{S}^*$. Then there exists a real orthogonal matrix U such that $A^* = UAU'$ for every A of $\mathfrak{S}$.*

For let $PAP^{-1} = A^*$ for every A of $\mathfrak{S}$. Then we transpose to obtain $(P')^{-1}AP' = A^*$ for every A of $\mathfrak{S}$, and so $A = P^{-1}A^*P = P'A^*(P')^{-1}$,

$$(24) \quad (PP')A^* = A^*(PP'),$$

for every A of $\mathfrak{S}$. There exists a real orthogonal matrix Q such that

$$(25) \quad QPP'Q' = D = \text{diag} \{ \delta_1 I^{(1)}, \delta_2 I^{(2)}, \dots, \delta_t I^{(t)} \},$$

where the δ_i are the *distinct* characteristic roots of PP' and the number of rows in the identity matrix $I^{(i)}$ is the multiplicity μ_i of the root δ_i . Since PP' is positive definite every $\delta_i > 0$ and $D = \Delta^2 = \Delta\Delta'$, where $\Delta = \text{diag} \{ \sqrt{\delta_1} I^{(1)}, \dots, \sqrt{\delta_t} I^{(t)} \}$ is real. Define

$$(26) \quad A_0 = QAQ', \quad A_0^* = QA^*Q', \quad P_0 = QPQ'$$

for every A of $\mathfrak{S}$, and see that the condition $PAP^{-1} = A^*$ is equivalent to

$$(27) \quad P_0 A_0 P_0^{-1} = A_0^*.$$

However,

$$(28) \quad P_0 P_0' = QPP'Q' = D = \Delta^2,$$

and so the matrix

$$(29) \quad P_1 = \Delta^{-1}P_0$$

is orthogonal. The condition $PP'A^* = A^*PP'$ is equivalent to $DA_0^* = A_0^*D$ for every A of $\mathfrak{S}$, and it follows that $A_0^* = \text{diag} \{ G_1, \dots, G_t \}$, where G_i has μ_i rows and columns. Then $\Delta A_0^* \Delta^{-1} = A_0^*$, and (27) becomes

$$(30) \quad P_1 A_0 P_1' = A_0^*$$

for every A of $\mathfrak{S}$. We have thus proved that $(P_1 Q)A(Q'P_1') = QA^*Q'$ and so $UAU' = A^*$ for every A of $\mathfrak{S}$, where $U = Q'P_1Q$ is the desired real orthogonal matrix.

4. Reduction to the nonsingular central simple case. The determinant condition is preserved if we replace $\mathfrak{S}$ by $P\mathfrak{S}P'$, $\mathfrak{S}^*$ by $Q\mathfrak{S}^*Q'$, and S by the mapping $PAP' \rightarrow QA^*Q'$, where P and Q are any real orthogonal matrices. We shall use this property repeatedly in our arguments. Let us now derive the following simple result.

³ The basic idea of the proof of this result was given on page 117 of *Modern Higher Algebra*.

Lemma 5. *Let E be an idempotent matrix in $\mathfrak{S}$ and A be an element of $\mathfrak{S}$ such that $EA = AE = A$. Then the determinant condition implies that $E^*A^* = A^*E^* = A^*$.*

For every idempotent matrix of rank r is orthogonally equivalent to the matrix

$$(31) \quad E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

By Lemma 1 we may take $E = E^*$ to have the form (31). By direct computation we see that $EA = AE = A$ if and only if

$$(32) \quad A = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix},$$

where B is an r -rowed real symmetric matrix. The matrix

$$(33) \quad yE + zA = \begin{bmatrix} yI_r + zB & 0 \\ 0 & 0 \end{bmatrix}$$

has the r -rowed nonzero principal minor $|yI_r + zB|$ and clearly has rank r . Lemma 3 implies that $yE + zA$ and $yE^* + zA^*$ are similar in $\mathfrak{N}(y, z)$, and so $yE + zA^*$ must have rank r . Write

$$(34) \quad A^* = \begin{bmatrix} C & D \\ D' & H \end{bmatrix}, \quad yE + zA^* = \begin{bmatrix} yI_r + zC & zD \\ zD' & zH \end{bmatrix},$$

for a partitioning of A^* with C an r -rowed symmetric matrix. The matrix $yI_r + zC$ is nonsingular for any C and so $G = G(y, z) = (yI_r + zC)^{-1}$ exists. The rank of $yE + zA^*$ is the same as the rank of

$$(35) \quad \begin{bmatrix} I_r & 0 \\ -zD'G & I_{n-r} \end{bmatrix} (yE + zA^*) = \begin{bmatrix} G^{-1} & zD \\ 0 & zH - z^2D'GD \end{bmatrix}.$$

But G^{-1} is nonsingular and has r rows. Hence the rank condition implies that $zH - z^2D'GD = 0$, and so $H - zD'GD = B(y, z) = 0$ identically in y and z . The matrix $G(y, 0) = (yI_r)^{-1}$ and so $B(y, 0) = H = 0$, $D'GD = 0$, $D'D = D'[G(1, 0)]D = 0$. This is possible for a real matrix D only for $D = 0$, and so

$$(36) \quad A^* = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix};$$

$A^*E = EA^* = A^*$ as desired.

We can now reduce our study to the case where the n -rowed identity matrix is the unity element E of $\mathfrak{S}$. For otherwise E has rank $r < n$ and we can use Lemma 1 to take

$$(37) \quad E = E^* = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

By Lemma 5 we have

$$(38) \quad A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^* = \begin{bmatrix} A_0^* & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad X^* = \begin{bmatrix} X_0^* & 0 \\ 0 & 0 \end{bmatrix}.$$

It should be clear that the mapping $A \rightarrow A_0$ is an isomorphic mapping of $\mathfrak{J}$ onto the algebra $\mathfrak{J}_0$ of the real symmetric matrices A_0 with general element X_0 . The mapping $A^* \rightarrow A_0^*$ maps the vector space $\mathfrak{J}^*$ onto $\mathfrak{J}_0^*$ and it should be clear that the mapping $A \rightarrow A^*$ is an isomorphism of $\mathfrak{J}$ onto $\mathfrak{J}^*$ if and only if the mapping $A_0 \rightarrow A_0^*$ is an isomorphism of $\mathfrak{J}_0$ onto $\mathfrak{J}_0^*$. If P is an orthogonal matrix such that $PAP' = A^*$ then the computations of (21) and (22) imply that P_1 is orthogonal and $P_1 A_0 P_1' = A_0^*$. We see finally that $|xI - X| = x^{n-r} |xI_r - X_0|$, and so $|xI - X| = |xI - X^*|$ if and only if $|xI_r - X_0| = |xI_r - X_0^*|$. Thus our study is reduced to the *nonsingular case* where $E = I$. We also have the following result.

Lemma 6. *Let I be the unity element of J and $I^* = I$. Then the determinant condition holds if and only if*

$$(39) \quad |X| = |X^*|.$$

For if the determinant condition holds Lemma 3 implies that X and X^* are similar and so have the same determinant. Conversely, let (39) hold. By hypothesis $I = I^*$. Then we can take $I = A_1$ and have $X = x_1 A_1 + \cdots + x_m A_m$ and $X^* = x_1 I + x_2 A_2^* + \cdots + x_m A_m^*$. Put $y_1 = x - x_1$, $y_i = -x_i$ for $i = 2, \dots, m$ and see that (39) implies that if $Y = y_1 A_1 + \cdots + y_m A_m$ and $Y^* = y_1 A_1^* + \cdots + y_m A_m^*$ then $|xI - Y| = |xI - Y^*|$ as desired.

Suppose now that $\mathfrak{J}$ is a Jordan algebra having I as a unity element and that $\mathfrak{J}$ is not simple. Then $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$, where $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are both *semisimple* Jordan algebras. Let E_i be the unity element of $\mathfrak{J}_i$ so that $E_1 + E_2 = I$. We have then seen that we can take

$$(40) \quad E_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

and so

$$(41) \quad E_2 = I - E_1 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

The algebra $\mathfrak{J}_1$ consists of all elements B_1 of $\mathfrak{J}$ such that $E_1 B_1 = B_1 E_1 = B_1$ and $\mathfrak{J}_2$ consists of all elements B_2 of $\mathfrak{J}$ such that $E_2 B_2 = B_2 E_2 = B_2$. It follows that every element A of $\mathfrak{J}$ has the form

$$(42) \quad A = B_1 + B_2, \quad B_1 = \begin{bmatrix} A_{10} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & A_{20} \end{bmatrix},$$

with B_i in $\mathfrak{J}_i$.

We assume that $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ so that Lemma 1 implies that $I^* = I$. We have already seen that we can take $E_1^* = E_1$, from which $E_2^* = I^* - E_1^* = I - E_1 = E_2$. By Lemma 5 every element A^* of $\mathfrak{J}^*$ has the form

$$(43) \quad A^* = B_1^* + B_2^*, \quad B_1^* = \begin{bmatrix} B_{10}^* & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2^* = \begin{bmatrix} 0 & 0 \\ 0 & B_{20}^* \end{bmatrix},$$

where B_i^* is in $\mathfrak{J}_i^*$. The general elements of $\mathfrak{J}$ and $\mathfrak{J}^*$ are then

$$(44) \quad X = \begin{bmatrix} X_{10} & 0 \\ 0 & X_{20} \end{bmatrix}, \quad X^* = \begin{bmatrix} X_{10}^* & 0 \\ 0 & X_{20}^* \end{bmatrix},$$

where the meaning of X_{10} , X_{10}^* , X_{20} , X_{20}^* should be obvious. The relation $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ is now seen to be equivalent to $|xI_r - X_{10}| \cdot |xI_{n-r} - X_{20}| = |xI_r - X_{10}^*| \cdot |xI_{n-r} - X_{20}^*|$. We can specialize this result to the case $X_{20} = X_{20}^* = 0$, and we obtain $|xI_r - X_{10}| = |xI_r - X_{10}^*|$. It should now be clear that S induces a mapping S_1 of $\mathfrak{J}_1$ onto $\mathfrak{J}_1^*$ defined by $B_1 \rightarrow B_1^* = B_1 S_1$ such that $\Delta(\mathfrak{J}_1, \mathfrak{J}_1^*, S_1) = 0$. Also S induces a mapping S_2 of $\mathfrak{J}_2$ onto $\mathfrak{J}_2^*$ such that $\Delta(\mathfrak{J}_2, \mathfrak{J}_2^*, S_2) = 0$. Conversely, it should be clear that if $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$, so that we can assume (40), (41), (42), then the determinant conditions $\Delta(\mathfrak{J}_1, \mathfrak{J}_1^*, S_1) = \Delta(\mathfrak{J}_2, \mathfrak{J}_2^*, S_2) = 0$ imply that $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$.

It should also be clear that if the determinant condition $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ implies that S is an isomorphism of any *simple* Jordan algebra $\mathfrak{J}$ onto $\mathfrak{J}^*$, then the determinant condition also implies the same result for arbitrary (necessarily semisimple) Jordan algebras $\mathfrak{J}$. It remains to discuss the orthogonal equivalence property.

If P_1 is an r -rowed real orthogonal matrix such that $P_1 B_{10} P_1' = B_{10}^*$ and P_2 is an $(n - r)$ -rowed orthogonal matrix such that $P_2 B_{20} P_2' = B_{20}^*$, then we put

$$(45) \quad P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

and see that P is orthogonal, $P B_1 P' = B_1^*$, $P B_2 P' = B_2^*$ and so $P A P' = A^*$ for every A of $\mathfrak{J}$. Conversely, let $P A P' = A^*$ for every A of $\mathfrak{J}$. As we have seen, we can take $E_1 = E_1^*$ to have the form (40). Then the argument of (21) and (22) implies that P has the form (45) where clearly P_1 and P_2 are orthogonal. Also $P B_1 P' = B_1^*$, $P B_2 P' = B_2^*$ and so the existence of P for $\mathfrak{J}, \mathfrak{J}^*, S$ is equivalent to the existence of P_i for the simple components $\mathfrak{J}_i$ of $\mathfrak{J}$ and their images $\mathfrak{J}_i^*$ under S . This completes the reduction of our study to the case where $\mathfrak{J}$ is a simple Jordan algebra.

The *center* of a *simple* Jordan algebra $\mathfrak{J}$ is a field. When $\mathfrak{J}$ is an algebra over the field of all real numbers and the elements of $\mathfrak{J}$ are real symmetric matrices,

the center of $\mathfrak{J}$ can only be $\mathfrak{K}$ itself. Hence $\mathfrak{J}$ must be a *central* simple Jordan algebra.

Every simple Jordan algebra $\mathfrak{J}$ has a unity element E , and we have shown that it is sufficient to consider the case where $E = I$ is the n -rowed identity matrix. Then

$$(46) \quad I = E_1 + \cdots + E_t,$$

where the E_i are pairwise orthogonal *primitive* idempotents of $\mathfrak{J}$, and t is called the *degree* of $\mathfrak{J}$. The primitivity of the E_i is equivalent to the hypothesis that the set $\mathfrak{J}_{ii}$ of all A_{ii} of $\mathfrak{J}$ such that $E_i A_{ii} = A_{ii} E_i = A_{ii}$ is a field. But $\mathfrak{J}_{ii}$ is a field of real symmetric matrices, and so necessarily

$$\mathfrak{J}_{ii} = E_i \mathfrak{K}$$

is one-dimensional. Thus $\mathfrak{J}$ is a so-called *reduced* Jordan algebra. We now proceed to discuss the case $t = 2$.

5. Simple algebras of degree two. Every simple Jordan algebra $\mathfrak{J}$ of degree two over $\mathfrak{K}$ as center has a basis $A_1, \cdots, A_\mu, A_{\mu+1} = I$ where $m = \mu + 1$ is the dimension of $\mathfrak{J}$ over $\mathfrak{K}$, $A_i A_j + A_j A_i = 0$ for $i \neq j$ and $i, j = 1, \cdots, \mu$, and $A_i^2 = \alpha_i I$. The α_i are real nonzero numbers and must be positive in the case where every A_i is symmetric. Hence we can assume that

$$(47) \quad A_i^2 = I, \quad A_i A_j = -A_j A_i \quad (i \neq j; \quad i, j = 1, \cdots, \mu).$$

It is easy to show that when (47) holds the algebra $\mathfrak{J}$ is simple if and only if $m > 2$ and thus $\mu \geq 2$.

Lemma 7. *Let $\mathfrak{J}$ be a simple Jordan algebra of degree two. Then S is a Jordan algebra isomorphism of $\mathfrak{J}$ onto $\mathfrak{J}^*$ if and only if $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$.*

For if $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ we use Lemma 1 to see that A_i^* is orthogonally equivalent to A_i , and so $(A_i^*)^2$ is orthogonally equivalent to $A_i^2 = I$, $A_i^{*2} = I$. But, if A and B are in $\mathfrak{J}$ and $A^2 = B^2 = I$, $AB + BA = 0$, then $(A + B)^2 = A^2 + B^2 + AB + BA = 2I$. By Lemma 1 we have $(A^* + B^*)^2 = 2I = (A^*)^2 + (B^*)^2 + A^* B^* + B^* A^* = 2I + A^* B^* + B^* A^*$. Hence we can conclude that $A_i^* A_j^* + A_j^* A_i^* = 0$ for $i \neq j$, and $\mathfrak{J}$ and $\mathfrak{J}^*$ have the same multiplication table. It follows that $(A \cdot B)^* = A^* \cdot B^*$ for every A and B of $\mathfrak{J}$. By Lemma 2 we conclude that S is a Jordan algebra isomorphism of $\mathfrak{J}$ onto $\mathfrak{J}^*$. Conversely, let S be an algebra isomorphism of $\mathfrak{J}$ onto $\mathfrak{J}^*$. If $X = x_0 I + x_1 A_1 + \cdots + x_\mu A_\mu$ then $(X - x_0 I)^2 = (x_1^2 + \cdots + x_\mu^2) I$, that is, X is a root of

$$\phi(x, X) = x^2 - 2x_0 x + (x_0^2 + \cdots + x_\mu^2) = 0.$$

The polynomial $\phi = \phi(x, X)$ has coefficients in $\mathfrak{K} = \mathfrak{K}[x_0, x_1, \cdots, x_\mu]$ and is quadratic. Thus ϕ is reducible in $\mathfrak{K} = \mathfrak{K}(x_0, x_1, \cdots, x_\mu)$ if and only if $\phi = 0$ has a root α in $\mathfrak{K}$, $(\alpha - x_0)^2 + x_1^2 + \cdots + x_\mu^2 = 0$ identically in $x_0, \cdots, x_\mu$. This is clearly impossible in $\mathfrak{K}$. But then the characteristic determinant of X is

$[\phi(x, x)]^n$ where $n = 2\nu$. The isomorphism S implies that the characteristic determinant of X coincides with that of $X^* = x_0I + x_1A_1^* + \cdots + x_\mu A_\mu^*$, since $\phi(x, X^*) \equiv \phi(x, X)$. Thus $\Delta(\mathfrak{F}, \mathfrak{F}^*, S) = 0$ and the proof of our lemma is complete.

We shall now consider the question of orthogonal equivalence in the case where we assume that $\Delta(\mathfrak{F}, \mathfrak{F}^*, S) = 0$. Write $\mu = 2\rho + \sigma$, where $\sigma = 0$ or 1 , and define

$$\begin{aligned} U_1 &= A_1, & V_1 &= A_2, \\ (48) \quad U_j &= (A_1 \cdots A_{2j-2})A_{2j-1}, \\ V_j &= (A_1 \cdots A_{2j-2})A_{2j} \quad (j = 2, \cdots, \rho). \end{aligned}$$

It is easy to show that *when j is odd* we have

$$(49) \quad U_j = U'_j, \quad V_j = V'_j, \quad U_j^2 = V_j^2 = I, \quad V_j U_j + U_j V_j = 0,$$

and *when j is even* we have

$$(50) \quad U_j = -U'_j, \quad V_j = -V'_j, \quad U_j^2 = V_j^2 = -I, \quad V_j U_j + U_j V_j = 0.$$

Define

$$(51) \quad \mathfrak{B}_j = I\mathfrak{R} + U_j\mathfrak{R} + V_j\mathfrak{R} + U_j V_j\mathfrak{R} \quad (j = 1, \cdots, \rho).$$

Then $\mathfrak{B}_j$ is a total matrix algebra of degree two when j is odd, but is isomorphic to the algebra of real quaternions when j is even. It is also easy to see that

$$(52) \quad U_i U_k - U_k U_i = U_i V_k - V_k U_i = V_i V_k - V_k V_i = 0 \\ (j \neq k; \quad j, k = 1, \cdots, \rho).$$

It follows that the enveloping associative algebra $\mathfrak{B}$, consisting of all polynomials in $A_1, \cdots, A_\mu$, has the form

$$(53) \quad \mathfrak{B} = \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_\rho \times C,$$

where $C = I\mathfrak{R}$ when $\mu = 2\rho$ and otherwise

$$(54) \quad C = H\mathfrak{R},$$

where H is the product

$$(55) \quad H = A_1 A_2 \cdots A_\mu = U_{\rho+1}$$

of all of the basal elements A_i of $\mathfrak{F}$. It should also be clear that if

$$(56) \quad U_j^* = (A_1^* \cdots A_{2j-2}^*)A_{2j-1}^*, \\ V_j^* = (A_1^* \cdots A_{2j-2}^*)A_{2j}^* \quad (j = 1, \cdots, \rho),$$

and

$$(57) \quad \mathfrak{B}_j^* = I\mathfrak{R} + U_j^*\mathfrak{R} + V_j^*\mathfrak{R} + U_j^* V_j^*\mathfrak{R} \quad (j = 1, \cdots, \rho),$$

then the enveloping associative algebra $\mathfrak{B}^*$ of $\mathfrak{J}^*$ is

$$(58) \quad \mathfrak{B}^* = \mathfrak{B}_1^* \times \cdots \times \mathfrak{B}_\rho^* \times C^*,$$

where $C^* = I\mathfrak{N}$ when $\mu = 2\rho$ and

$$(59) \quad C^* = H^*\mathfrak{N}, \quad H^* = A_1 \cdots A_\mu = U_{\rho+1}^*,$$

when $\mu = 2\rho + 1$. When $\mu = 2\rho$ and $m = 2\rho + 1$ is odd the mapping S induces an isomorphism T , defined by $U_i T = U_i^*$, $V_i T = V_i^*$, of the simple algebra $\mathfrak{B}$ onto $\mathfrak{B}^*$. When $m = 4q$ then $\mu = 4q - 1 = 2\rho + 1$ and $\rho + 1 = 2q$ is even, $H = U_{\rho+1}$ is skew, C and C^* are isomorphic to the field of all complex numbers, and S again induces an isomorphism T of the simple algebra $\mathfrak{B}$ onto $\mathfrak{B}^*$. But then the standard theory of associative algebras implies that there exists a nonsingular matrix P such that $PYP^{-1} = YT$ for every Y of $\mathfrak{B}$. In particular, $PAP^{-1} = A^*$ for every A of J . By Lemma 4 we have the following result.

Lemma 8. *Let $\mathfrak{J}$ be a simple Jordan algebra of degree two and dimension $m \equiv 0, 1, 3 \pmod{4}$ over $\mathfrak{R}$. Then there exists a real orthogonal matrix P such that $PAP' = A^*$ for every A of $\mathfrak{J}$ if and only if $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$.*

There remains the case where $\mathfrak{J}$ has degree two and dimension $m = 4q + 2$. We shall call such simple algebras *irregular* algebras. Then $H = U_{\rho+1}$ is symmetric and $H^2 = I$. It should be clear that S induces an isomorphism T of $\mathfrak{B}_1 \times \cdots \times \mathfrak{B}_\rho$ onto the algebra $\mathfrak{B}_1^* \times \cdots \times \mathfrak{B}_\rho^*$ defined by $U_i T = U_i^*$, $V_i T = V_i^*$ for $j = 1, \dots, \rho$. However, when the mapping T is extended to include $HT = H^*$ it need not be an isomorphism. Indeed, $I^* = I$ and it is actually possible that $H = I$, $H^* = -I$, $(H - I)^* \neq H^* - I^*$. The matrix H is always orthogonally equivalent to

$$(60) \quad \begin{bmatrix} I_\lambda & 0 \\ 0 & -I_{n-\lambda} \end{bmatrix},$$

and so has trace $\tau(H) = 2\lambda - n$. Thus H and H^* have the same index λ if and only if $\tau(H) = \tau(H^*)$. If $PAP' = A^*$ for every A of $\mathfrak{J}$, it should be clear that the mapping $Y \rightarrow PYP'$ is an isomorphism T of $\mathfrak{B}$ onto $\mathfrak{B}^*$ such that $\tau(H) = \tau(H^*)$. Conversely, let $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ and $\tau(H) = \tau(H^*)$. Then H and H^* are orthogonally equivalent and both can be taken to have the form (60). The property $YH = HY$ for every Y of $\mathfrak{B}_0 = \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_\rho$ and the property $Y^*H = HY^*$ for every Y^* of $\mathfrak{B}_0^* = \mathfrak{B}_1^* \times \cdots \times \mathfrak{B}_\rho^*$ imply that every Y and Y^* have the form

$$(61) \quad Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}, \quad Y^* = \begin{bmatrix} Y_1^* & 0 \\ 0 & Y_2^* \end{bmatrix},$$

where Y_1 and Y_1^* have λ rows and Y_2 and Y_2^* have $n - \lambda$ rows. It should then be clear that the set $\mathfrak{B}_{01}$ of all of the real matrices Y_1 forms an associative simple

algebra homomorphic to the algebra $\mathfrak{B}_0$, and that the mapping $Y_1 \rightarrow Y_1^*$ is an isomorphism of $\mathfrak{B}_{10}$ onto the algebra $\mathfrak{B}_{01}^*$ of matrices Y_1^* . It follows that there exists a nonsingular matrix P_1 such that $P_1 Y_1 P_1^{-1} = Y_1^*$. Similarly there exists a nonsingular matrix P_2 such that $P_2 Y_2 P_2^{-1} = Y_2^*$. Thus we can write

$$(62) \quad P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad PYP^{-1} = Y^*, \quad PHP^{-1} = H = H^*.$$

But then $PAP^{-1} = A^*$ for every A of $\mathfrak{J}$. By Lemma 4 we have the following result.

Lemma 9. *Let $\mathfrak{J}$ be an irregular simple Jordan algebra (of degree two and dimension $4q + 2$), and let $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ so that the mapping $A \rightarrow A^*$ is an isomorphism S of $\mathfrak{J}$ onto $\mathfrak{J}^*$. Let the generating central element H of the enveloping Clifford algebra $\mathfrak{B}$ of $\mathfrak{J}$ be mapped, by the extension of S to $\mathfrak{B}$ and to the enveloping algebra $\mathfrak{B}^*$ of $\mathfrak{J}^*$, onto a generating central element H^* of $\mathfrak{B}^*$. Then there exists a real orthogonal matrix P such that $PAP' = A^*$ for every A of $\mathfrak{J}$ if and only if H and H^* have the same trace.*

Our results thus provide a second class of counterexamples to the conjecture that the determinant condition implies orthogonal equivalence, even when $\mathfrak{J}$ is Jordan-closed. Indeed, the first element of this new class is that where $m = 6$. Take $A_i = A_i^*$ for $i = 1, 2, 3, 4$, where $A_i^2 = I$, $A_i A_j + A_j A_i = 0$ for $i \neq j$, and $i, j = 1, 2, 3, 4$. Define $A_5 = A_1 A_2 A_3 A_4$ and we have $A_5^2 = I$, $A_5 A_i = -A_i A_5$ for $i = 1, 2, 3, 4$. Define $A_5^* = -A_5$ so that we again have $(A_5^*)^2 = I$, $A_5^* A_i^* = -A_i^* A_5^*$ for $i = 1, 2, 3, 4$. Then $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ as desired. If P were a real orthogonal matrix such that $PA_i P' = A_i^*$ for $i = 1, 2, 3, 4$ then the relation $PA_5 P' = A_5^*$ would not hold since actually $PA_5 P' = A_5 = -A_5^*$.

6. The general case. Let $\mathfrak{J}$ be a simple Jordan algebra of degree $t > 2$, so that we can assume that I is the unity element of J and that (46) holds. Then we can write

$$(63) \quad \mathfrak{J} = E_1 \mathfrak{R} + \cdots + E_t \mathfrak{R} + \sum_{i \leq j} \mathfrak{J}_{ij},$$

where $\mathfrak{J}_{ij} = \mathfrak{J}_{ji}$ consists of all elements A of $\mathfrak{J}$ such that

$$(64) \quad E_i \cdot A = E_j \cdot A = \frac{1}{2} A.$$

We can always use an orthogonal transformation to take $E_i = \text{diag} \{0, \dots, 0, I_s, 0, \dots, 0\}$, where $n = ts$ and the s -rowed identity matrix I_s occurs in the i^{th} place. Then every element of $\mathfrak{J}_{ij}$ is a matrix g_{ij} with an s -rowed matrix g in its i^{th} row and j^{th} column, g' in its j^{th} row and i^{th} column and zeros elsewhere. The s -rowed matrices g all lie in a quadratic algebra $\mathfrak{G}$ of dimension 1, 2 or 4 over $\mathfrak{R}$ such that the mapping

$$(65) \quad g \rightarrow g'$$

is the conjugate operation of $\mathfrak{G}$. For

$$(66) \quad gg' = N(g)I_s ,$$

where $N(g)$ is real. Indeed, it should be obvious that

$$(67) \quad N(g) > 0$$

for every nonzero g of $\mathfrak{G}$, and so $\mathfrak{G}$ must be a division algebra. Note that

$$(68) \quad g_{ii}^2 = [N(g)](E_i + E_i),$$

for every g_{ij} of J_{ij} .

The subspace $\mathfrak{G}_{ij} = \mathfrak{I}_{ij} + E_i\mathfrak{N} + E_j\mathfrak{N}$ is a simple Jordan algebra mapped by S onto a subspace $\mathfrak{G}_{ij}^* = \mathfrak{I}_{ij}^* + E_i^*\mathfrak{N} + E_j^*\mathfrak{N}$ of $\mathfrak{I}^*$. By Lemma 3 the condition $\Delta(\mathfrak{G}_{ij}, \mathfrak{G}_{ij}^*, S) = 0$ holds and we use Lemma 7 to see that S induces an isomorphism S_{ij} of $\mathfrak{G}_{ij}$ onto $\mathfrak{G}_{ij}^*$. But it should then be clear that S will be an isomorphism of $\mathfrak{I}$ onto $\mathfrak{I}^*$ if and only if

$$(69) \quad (g_{ij} \cdot h_{ik})^* = g_{ij}^* \cdot h_{ik}^*$$

for every nonzero g and h of $\mathfrak{G}$ and all $i < j < k$. Since the required relation is one for the subalgebras $\mathfrak{G}_{ijk} = E_i\mathfrak{N} + E_j\mathfrak{N} + E_k\mathfrak{N} + \mathfrak{I}_{ij} + \mathfrak{I}_{ik} + \mathfrak{I}_{jk}$ and $\mathfrak{G}_{ijk}^* = E_i^*\mathfrak{N} + E_j^*\mathfrak{N} + E_k^*\mathfrak{N} + \mathfrak{I}_{ij}^* + \mathfrak{I}_{ik}^* + \mathfrak{I}_{jk}^*$ of degree three it is sufficient to consider the case $t = 3$. By symmetry it is sufficient to show that $(g_{12} \cdot h_{23})^* = g_{12}^* \cdot h_{23}^*$.

We already know that we can always take

$$(70) \quad E_1 = E_1^* = \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = E_2^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_3 = E_3^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e \end{bmatrix},$$

where the s -rowed identity matrix e is the unity element of $\mathfrak{G}$. Then

$$(71) \quad g_{12} = \begin{bmatrix} 0 & g & 0 \\ g' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

for $g \neq 0$ in $\mathfrak{G}$. But $gg' = \alpha e$ where $\alpha > 0$. We can then normalize so that $\alpha = 1$ and so assume that the matrix g is orthogonal. Then

$$(72) \quad P = \begin{bmatrix} g' & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix}$$

is an orthogonal matrix such that $PE_iP' = PE_i^*P' = E_i = E_i^*$ and

$$(73) \quad Pg_{12}P' = e_{12} = \begin{pmatrix} 0 & e & 0 \\ e & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can assume also that h_{23} has been normalized so that

$$(74) \quad h_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & h' & 0 \end{pmatrix},$$

where h is orthogonal. Then

$$(75) \quad Q = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & h \end{pmatrix}$$

is orthogonal. Moreover it should be clear that $QE_iQ' = QE_i^*Q' = E_i = E_i^*$, $Qe_{12}Q' = e_{12}$, and

$$(76) \quad Qh_{23}Q' = e_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{pmatrix}.$$

By symmetry we can take

$$(77) \quad e_{12}^* = e_{12}, \quad e_{23}^* = e_{23},$$

where we only use properties of the Jordan algebras $\mathfrak{S}_{12}^*$, $\mathfrak{S}_{23}^*$. By direct multiplication we have

$$(78) \quad 2e_{12} \cdot e_{23} = e_{13} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ e & 0 & 0 \end{pmatrix}.$$

However, we have *not* assumed that $\mathfrak{S}^*$ is Jordan-closed and so it remains only to show that $e_{13}^* = e_{13}$. We do know that $e_{13}^2 = E_1 + E_3$ from which $e_{13}^{*2} = (E_1 + E_3)^* = E_1 + E_3$. It follows that

$$(79) \quad e_{13}^* = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a' & 0 & 0 \end{pmatrix}$$

where a is an orthogonal matrix.

We can now apply Lemma 3 to see that the determinants of the matrices

$$(80) \quad M = \begin{bmatrix} e & e & e \\ e & 0 & xe \\ e & xe & 0 \end{bmatrix} = E_1 + e_{12} + e_{13} + xe_{23},$$

$$M^* = \begin{bmatrix} e & e & a \\ e & 0 & xe \\ a' & xe & 0 \end{bmatrix} = E_1^* + e_{12}^* + e_{13}^* + xe_{23}^*$$

are equal identically in the indeterminate x . But $|M^*| = |M_1|$ where

$$(81) \quad M_1 = \begin{bmatrix} e & 0 & 0 \\ -e & e & 0 \\ -a' & 0 & e \end{bmatrix} M^* = \begin{bmatrix} e & e & a \\ 0 & -e & xe - a \\ 0 & xe - a' & -e \end{bmatrix}.$$

Also $|M_1| = |M_2|$ where

$$(82) \quad M_2 = \begin{bmatrix} e & 0 \\ xe - a' & e \end{bmatrix} \begin{bmatrix} -e & xe - a \\ xe - a' & -e \end{bmatrix} = \begin{bmatrix} -e & xe - a \\ 0 & (xe - a')(xe - a) - e \end{bmatrix}.$$

We deduce that $|M^*| = \epsilon |(xe - a')(xe - a) - e| = \epsilon |x^2e - (a + a')xe|$ where $\epsilon = |-e| = 1$ or -1 . It now follows that $|M| = \epsilon |x^2e - 2xe| = |M^*|$ if and only if $|xe - (a + a')| = |xe - 2e|$. But then the matrices $a + a'$ and $2e$ have the same characteristic determinant. Since both are real symmetric matrices they must be similar. Hence $a + a' = a + a'^{-1} = 2e$, $(a - e)^2 = 0$, $a = e + b$ where $b^2 = 0$. Since a is orthogonal we have $aa' = (e + b)(e + b') = e + b + b' + bb' = e$, $b + b' + bb' = 0$, $b(b + b' + bb') = bb' = 0$. This is possible for a real matrix b only if $b = 0$, $a = e$, and $e_{13}^* = e_{13}$ as desired. We have completed a proof of the following result.

Theorem 2. *Let $\mathfrak{J}$ and $\mathfrak{J}^*$ be vector spaces of real symmetric matrices, S be a linear mapping $A \rightarrow A^*$ of $\mathfrak{J}$ onto $\mathfrak{J}^*$, and $\mathfrak{J}$ be Jordan closed. Then the determinant condition $\Delta(\mathfrak{J}, \mathfrak{J}^*, S) = 0$ holds if and only if S is a Jordan algebra isomorphism of $\mathfrak{J}$ onto $\mathfrak{J}^*$.*

The enveloping associative algebra $\mathfrak{A}$ of a simple special Jordan algebra $\mathfrak{J}$ of degree $t > 2$ with unity element I is easily seen to be the simple associative algebra $\mathfrak{G}_t$ of all t -rowed square matrices with elements in the associative coefficient algebra $\mathfrak{G}$. It should then be clear that the mapping $A \rightarrow A^*$ of $\mathfrak{J}$ onto $\mathfrak{J}^*$ induces an isomorphism T of $\mathfrak{A}$ onto the enveloping associative algebra

$\mathfrak{B}^*$ of $\mathfrak{S}^*$ and that $A^* = A^T$ for every A of $\mathfrak{S}$. We can combine this result with Lemma 4 and Lemma 9⁴ to yield the following result.

Theorem 3. *Let the condition of Theorem 2 hold⁵ so that the mapping $A \rightarrow A^*$ is a Jordan algebra isomorphism of $\mathfrak{S}$ onto $\mathfrak{S}^*$. Then there exists a real orthogonal matrix P such that $PAP' = A^*$ for every A of $\mathfrak{S}$ if and only if the trace condition $\tau(H_i) = \tau(H_i^*)$ holds for every irregular simple component $\mathfrak{S}_i$ of $\mathfrak{S}$.*

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⁴ Let M_n be the algebra of all real n -rowed square matrices and let the identity matrix I be the unity element of two simple associative subalgebras A and A^* of M_n . Then it is well-known that if $A \rightarrow A^*$ is an isomorphism of A onto A^* there exists a nonsingular matrix P such that $A^* = PAP^{-1}$.

⁵ It should be clear that Theorems 2 and 3 hold if the elements of $\mathfrak{S}$ and $\mathfrak{S}^*$ are complex Hermitian matrices and P is unitary.