Potential Theory and a Generalized Jensen-Nevanlinna Formula for Functions of Several Complex Variables

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1. Introduction. One of the striking differences between bounded domains D in complex n -space for $n = 1$ and for $n > 1$ is in the nature of the smallest closed subset B (*the Bergman-Šilov boundary*) of the closure $\bar{D}$ of D on which all complex functions analytic on $\bar{D}$ assume their maximum modulus. B is the whole topological boundary if $n = 1$, while if $n > 1$, B may be a proper subset of the topological boundary. Another difference lies in the fact that for $n > 1$ the real parts of analytic functions are not a dense subset of the space of continuous real functions on B . Because of these two differences, the methods of the theory of functions of several variables have rarely generalized the potential theoretic methods which have become classical in the theory of functions of one complex variable.

S. BERGMAN [1] has formulated the problem of finding classes $\mathcal{E}$ of functions including the real parts of analytic functions, the functions of each having their maxima on B , and, if possible, such that each class has boundary values dense in the space of continuous functions on B . In a previous paper [10] I examined this problem for a special class of bounded domains found by E. CARTAN: the symmetric homogeneous ones. Such a domain (*a Cartan domain*) has a transitive group of complex analytic transformations (*automorphisms*) including, for each point p , an automorphism with p alone as fixed point. It was shown that the additional requirement that the class $\mathcal{E}$ be carried into itself by automorphisms makes $\mathcal{E}$ unique. This class of functions is given by an integral formula of simple structure generalizing the Poisson integral. What is most important, the class $\mathcal{E}$ is the class of solutions of a certain second order elliptic partial differential equation $\Delta f = 0$, where Δ is the Laplace-Beltrami

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operator arising from the Bergman invariant metric. In the case of the unit disk, the equation is

$$(1 - |z|^2)^2 \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0;$$

the solutions of this equation in the unit disk are exactly the functions harmonic in the ordinary sense there. In several variables the situation is more complicated; the operator Δ degenerates on the boundary in such a way that the maximum principle is satisfied on the Bergman-Šilov boundary.

The object of the present paper is to apply this theory to give a generalization of the Jensen-Nevanlinna formula:

$$\log |f(z)| + J(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| P(z, \theta) d\theta.$$

Here the function J is given by

$$J(z) = - \sum \log \left| \frac{z - a_n}{1 - \bar{a}_n z} \right|$$

where the summation is over all zeros of f in the unit circle, each counted with its multiplicity, and P is the Poisson kernel for the unit circle. Each function

$$- \log \left| \frac{z - a}{1 - \bar{a}z} \right|$$

is the Green's function $G(z, a)$ for the Dirichlet problem in the unit circle. The essential properties of J are that J is harmonic except where $f = 0$, and there has a logarithmic singularity of a type opposite to $\log |f|$, and $J = 0$ on the boundary. The present paper examines only one of the four infinite classes of Cartan domains, since it relies on some explicit calculations and its methods are too cumbersome to apply to all domains. The two results are an existence and uniqueness theorem for the class $\mathcal{E}$ of solutions of the Dirichlet problem, and a generalization of the Jensen-Nevanlinna formula. The reasoning here actually proves the second theorem for all of the classical Cartan domains, but I have tried to indicate also how a general solution of the first problem leads to a general solution of the second.

The present paper differs from BERGMAN's work in the same direction in two aspects. First, the use of a single partial differential equation to define an extended class of functions makes it possible to use a certain number of ideas from classical potential theory. Second, the domains considered are in complex n -space, while BERGMAN's are all in 2-space; the domains considered here are homogeneous, while those considered by BERGMAN need not be. It is interesting to notice that the only domain common to the two theories is the bicylinder, by far the simplest case in either theory. Here the results of two quite different points of view are exactly the same, as was to be expected, since the class $\mathcal{E}$ is uniquely determined by the maximum and invariance conditions.

2. Functions of the extended class. The domains to be considered are defined by the inequalities

$$(1) \quad \left| \sum_1^n w_k^2 \right|^2 - 2 \sum_1^n |w_k|^2 + 1 > 0,$$

$$1 - \left| \sum_1^n w_k^2 \right|^2 > 0.$$

The generalized Cayley transform

$$(2) \quad \begin{aligned} z_k &= -2iw_k W^{-1}, \quad k = 1, \dots, n-1, \\ z_n + i &= 2(w_n - i)W^{-1}, \\ W &= \sum_1^{n-1} w_k^2 + (w_n - i)^2, \end{aligned}$$

maps the domain D given by (1) onto the domain D' given by

$$\begin{aligned} z_k &= x_k + iy_k, \quad x_k, y_k \text{ real}, \quad k = 1, \dots, n, \\ -\infty &< x_k < +\infty, \quad y_n > (y_1^2 + \dots + y_{n-1}^2)^{\frac{1}{2}}. \end{aligned}$$

The Bergman-Šilov boundary B of D is the set of points making both expressions in (1) equal to 0. They are given by

$$w_k = e^{i\varphi} u_k, \quad \sum_1^n u_k^2 = 1.$$

Here φ and the u_k are real. Except for a subset of lower dimension, B is mapped by (2) onto the set B' given by

$$y_1 = \dots = y_n = 0, \quad -\infty < x_k < +\infty.$$

Easily constructed one-to-one complex analytic transformations (called *automorphisms*) carry the point $(0, \dots, 0, i) = p$ into an arbitrary point of D' , which is therefore homogeneous. So, therefore, is D . Reflection in the origin of D gives the symmetry required by the definition of a symmetric homogeneous bounded complex domain [6]. The domain D' arises naturally when the Laplace transformation in several variables is used in connection with the wave equation [9]. For $n = 2$, D' is isomorphic to the product of two complex half planes. I shall calculate mainly with D' and B' in what follows, and often shall not explicitly state that similar statements hold for D and B . The complement of B' in the boundary of D' (called the residual boundary and denoted by R') consists of points like $(i, 0, \dots, 0, i) = q$. Any such point may easily be moved to q by an automorphism of D' .

On the residual boundary each point is contained in a certain submanifold isomorphic to a complex half plane, called the analytic slice through this point. The analytic slice through q above is given by

$$z_1 = z_n = \lambda, \quad \text{Im}(\lambda) \geq 0, \quad z_2 = \dots = z_{n-1} = 0.$$

In each of these domains there is a second order differential equation invariant under automorphisms. It is the Laplace-Beltrami operator arising from the Bergman metric [2]. For D it may be calculated from that of D' where it is

$$\Delta' = \sum_1^{n-1} (Y + 2y_k^2) \frac{\partial^2}{\partial z_k \partial \bar{z}_k} + (Y - 2y_n^2) \frac{\partial^2}{\partial z_n \partial \bar{z}_n} + 2 \sum_{i < k} y_i y_k \left(\frac{\partial^2}{\partial \bar{z}_i \partial z_k} + \frac{\partial^2}{\partial z_i \partial \bar{z}_k} \right),$$

where $z_k = x_k + iy_k$ and $Y = y_n^2 - y_1^2 - \cdots - y_{n-1}^2$ [10].

The differential operator Δ' is elliptic for D' , vanishes on B' , and on the residual boundary R' it degenerates partially, if it is defined on the closure of D' by continuity. Inspection of the point q shows that Δ' is the ordinary invariant Laplacian for the half plane in the analytic slice containing q .

If f is the real part of an analytic function it satisfies

$$\frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} = 0, \quad i, j = 1, \cdots, n.$$

Therefore $\Delta'f = 0$. In general, if $\sum_{i,j} g_{ij} dz_i d\bar{z}_j$ is a Kählerian metric [4], the Laplace-Beltrami operator is a pure second order differential operator

$$(3) \quad \Delta = \sum_{i,j} g^{ij} \frac{\partial^2}{\partial z_i \partial \bar{z}_j},$$

where $\sum_i g^{ii} g_{ik} = \delta_k^i$. Therefore $\Delta f = 0$. The Bergman metric for a domain is always Kählerian.

In the crucial definition below it is easier to consider the domain D rather than to impose boundary conditions at infinity for functions on D' .

Definition. The class of solutions of the invariant Laplacian Δ satisfying the regularity conditions below at the boundary of D will be denoted by $\mathcal{E}$, the extended class of functions; it is the class of functions f satisfying:

- (a) f is continuous on the closure of D .
- (b) In D , f is C^2 and satisfies $\Delta f = 0$.
- (c) On R , f is C^2 on every analytic slice and there satisfies $\Delta f = 0$.

Theorem 1. Let u be a continuous function on B . There is an unique function f of $\mathcal{E}$ with $f = u$ on B .

Proof: In a previous paper [10] I have examined the class $\mathcal{F}$ of real functions on $D \cup B$ which are continuous on B and satisfy the maximum principle

$$|f(z)| \leq \max_{b \in B} |f(b)|, \quad z \in D.$$

$\mathcal{F}$ contains the constants and is linear, and is invariant under the automorphisms g of D in the sense that $f \in \mathcal{F}$ implies $f_g \in \mathcal{F}$ where $f_g(z) = f(gz)$. It was shown that for $f \in \mathcal{F}$

$$(4) \quad f(z) = \int_B f(b) d\mu_z(b) = \int_B f(b)P(z, b) db.$$

An explicit formula for $P(z, b)$ has been given without proof by HUA [7]:

$$f(z) = \frac{1}{\pi\sigma_n} \int_B \frac{f(b)(1 + |z'^t z|^2 - 2z'^t \bar{z})^{\frac{1}{2}n} db}{|1 - 2z'^t b + z'^t z|^n}$$

where $'z$ and $\bar{z}$ are the transpose and conjugate respectively of the vector z considered as a 1 by n matrix, σ_n is the surface area of the unit sphere in real n -space, and $db = db_1 db_2 \cdots db_n$.

The corresponding formula for D' was calculated in [10] to be

$$f(u + iv) = \frac{2^n}{\pi\sigma_n} \int_{R^n} \frac{f(x)(v, v)^{\frac{1}{2}n} dx}{[(x - u, x - u) - (v, v)]^2 - 4(x - u, v)^2]^{\frac{1}{2}n}},$$

where $dx = dx_1 \cdots dx_n$ and $(s, t) = s_n t_n - s_1 t_1 - \cdots - s_{n-1} t_{n-1}$ is the Lorentz inner product of the two real vectors s and t .

Here $d\mu_z$ is the unique normalized Radon measure on B invariant under the *stability group* of z (the compact subgroup of automorphisms satisfying $gz = z$). $P(z, b)$ is a kernel giving the ratio of this measure to some fixed measure db , which, for the present paper, will be chosen to be invariant under the stability group of the origin. It was proved that $P(z, b)$ satisfies $\Delta P(z, b) = 0$, where the derivatives are with respect to z . The proof of Theorem 1 will consist of the fact that $\mathfrak{F}$ and $\mathfrak{E}$ are identical, essentially. Thus it is necessary to extend each function in $\mathfrak{F}$ to the closure of D , so as to satisfy the conditions of the definition. First, the uniqueness is shown. The functions of $\mathfrak{E}$ satisfy the invariance and linearity conditions for $\mathfrak{F}$, since Δ is linear and invariant; only the maximum condition remains: Let m be the maximum of $|f|$ on the closure of D . Since the operator Δ is strictly elliptic in D , $|f| < m$ there unless f is constant. It is possible that $|f| = m$ at the point q on R but then, if S is the analytic slice containing q , f must be constant on S since it is harmonic (in the ordinary sense) there. Hence $|f| = m$ on the boundary of S , which is contained in B . Uniqueness follows from linearity, and f is given by (4).

To show the existence define f by (4), $f = u$ on B ; it must be shown that f can be defined on R so as to satisfy the boundary requirements. Since B and R are homogeneous, it is necessary to consider boundary values only at one point in each, for instance the points transformed to the origin and q under the Cayley transformation (2). To find f at the origin consider the 1-parameter subgroup of scalar multiplication automorphisms $g_t z = e^{-t} z$. Then $\lim_{t \rightarrow \infty} g_t(p) = 0$. By the invariance properties of the class $\mathfrak{F}$ of functions

$$(5) \quad f(g_t z) = \int_{B'} f(g_t b) P(z, b) db.$$

Hence $\lim_{t \rightarrow \infty} f(g_t b)$ on B' must be evaluated; it is clearly $f(0)$ and the convergence of the integral to $f(0)$ is guaranteed by the uniform convergence of $\lim_{t \rightarrow \infty} f(g_t z)$ on bounded sets.

Next it will be shown that the limiting values of f on the residual boundary are to be found at a point q by constructing the function in the analytic slice S containing q which is harmonic on S (considering S as a complex half plane) and assumes the boundary values f on the boundary of S , $S \cap B$.

First, there is a 1-parameter group of automorphisms $\{g_t \mid t \text{ real}\}$ with $\lim_{t \rightarrow \infty} g_t p = q$. By a change of variables affecting only the first and last coordinates, the domain D' can be defined by

$$\begin{aligned} r_k &= u_k + iv_k, & u_k, v_k &\text{ real}, & 1 \leq k \leq n, \\ v_1 v_n - v_2^2 - \cdots - v_{n-1}^2 &> 0, & v_1 &> 0. \end{aligned}$$

Then $q = (0, \dots, 0, i)$. Define

$$\begin{aligned} (6) \quad g_t r_1 &= e^{-2t} r_1, \\ g_t r_k &= e^{-t} r_k, & 2 \leq k \leq n-1, \\ g_t r_n &= r_n. \end{aligned}$$

Using (5) and the same reasoning in the paragraphs after, the limit of $f(g_t p)$ is determined by $\lim_{t \rightarrow \infty} f(g_t b)$ for $b \in B'$. This limit depends only on the values of f on the line $u_1 = \cdots = u_n = v_1 = \cdots = v_{n-1} = 0$, which is the boundary of the analytic S' slice containing q . This limit is a continuous linear functional on the space of continuous functions on $B' \cap S'$; hence, by the Riesz theorem, it is given by an integral of the form

$$(7) \quad f(q) = \int_{B' \cap S'} f(b) d\mu_q(b).$$

Now, to return to D , the point q goes into a point on the residual boundary of D . The automorphisms of D leaving the origin fixed include

$$(8) \quad w \rightarrow Qw, \quad w \rightarrow e^{i\theta} w,$$

where Q is a real n by n orthogonal matrix. The analytic slice through q contains the points $(z, 0, \dots, 0, z)$ for z in the upper half plane, and these points go into points given by

$$w_1 = \frac{-2iz}{1 - 2iz}, \quad w_n - i = \frac{-2(z + i)}{1 - 2iz}, \quad w_2 = \cdots = w_{n-1} = 0.$$

I claim that there is a 1-parameter subgroup H of (8) which leaves the origin and the image of q fixed and which is transitive on $S \cap B$. Consider the subgroup of (8) which leaves the coordinates $w_2, \dots, w_{n-1}$ invariant; this is a two dimensional group which is the full group when $n = 2$. Thus the problem is reduced to the case $n = 2$; in this case, the domain D' is the product of two half planes, and D is a bicylinder, the product of two circles $|r_1| < 1, |r_2| < 1$, while the slice S is given by $|r_1| \leq 1, r_2 = 1$.

The group H is then given by multiplying r_1 by e^{it} . The subgroup H has the

same functional term in the general case as for the case $n = 2$. Thus in equation (7) H leaves q fixed and rotates $B \cap S$. From this equation, applied to f_n , for $h \in H$,

$$(9) \quad f(q) = f(hq) = \int_{B \cap S} f(hq) d\mu_q(b),$$

and, by integrating (9) over H ,

$$(10) \quad \begin{aligned} f(q) &= \int_H f(hq) dh = \int_H \int_{B \cap S} f(hb) d\mu_q(b) dh \\ &= \int_{B \cap S} \left\{ \int_H f(hb) dh \right\} d\mu_q(b). \end{aligned}$$

The expression in the braces in (10) is the average of f over the circle $B \cap S$ with respect to the rotations leaving q fixed; this is a constant, equal to the value at q of the harmonic function in S with boundary values f . The integral of this constant on $B \cap S$ with respect to μ_q is simply the constant itself. This shows that $f(q)$ is the value at q of the harmonic function in S with boundary values f on $B \cap S$. Thus the values of f on analytic slices satisfy $\Delta f = 0$.

There remains to prove only continuity on the closure of D . Again consideration of the two representative points q and the origin in D' suffices. Let g be a transformation of the form

$$(11) \quad g : z \rightarrow sQz,$$

where s is a scalar and Q is a real orthogonal matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad A \quad n-1 \quad \text{by} \quad n-1.$$

Every point near p in D is of the form $gg_t p$ where g_t is given by (6), s is near 1, Q is near the identity, and t is near ∞ . Give the point at infinity the usual neighborhoods, and put $g_\infty = \lim_{t \rightarrow \infty} g_t$.

Then f near p is given by

$$(12) \quad \int_{B'} f(gg_t b) db$$

using equation (5). The integral (12) is continuous in the topology of uniformly bounded and uniform convergence on compact sets, and the map mapping $gg_t p$ near q into $\int f(gg_t b) db$ is continuous also in this topology. Continuity of f at the origin follows in an analogous way, except that in (11) one considers s near 0 and it is necessary only to consider the transformations gz , g given by (11), as $s \rightarrow 0$.

It should be remarked that the case $n = 2$ is very special; in this case only is the domain isomorphic to a product domain, the bicylinder. In the bicylinder,

the functions of $\mathcal{E}$ are harmonic in each variable separately, but this is not true in general, and seems to shed no light on the special properties of the functions considered here. Of course, there is an obvious extension of the results of this paper to the product of any of the domains considered in it.

3. Plurisubharmonic functions.

Definition. A real function f on a connected domain E is called plurisubharmonic if f is upper semicontinuous and, for every complex vector $u = (u_1, \dots, u_n)$,

$$(13) \quad \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} u_i \bar{u}_j \geq 0.$$

Here the derivatives are the derivatives of the theory of distributions [8]. If g is analytic, $f = \log |g|$ is plurisubharmonic. The following fact from [8] will be useful: For every domain E_1 contained in E and of positive distance from its exterior there exists a sequence $f_1 \geq f_2 \geq \dots$ of C^∞ plurisubharmonic functions in E_1 with pointwise limit f . Recall also that if f is upper semicontinuous and h continuous, the set $\{x \mid f(x) \geq h(x)\}$ is closed in E .

Lemma 1. Let f be plurisubharmonic and let Δ be the Laplace-Beltrami operator corresponding to a Kählerian metric. For any bounded domain E_1 with $\bar{E}_1 \subset E$, and any C^2 function h which satisfies $\Delta h = 0$ on E_1 , $f \leq h$ on the boundary of E_1 implies $f \leq h$ on E_1 .

Proof. The property of the conclusion, along with upper semicontinuity, may be described by calling f subharmonic with respect to the metric. Let $\epsilon > 0$ and $f_1 \geq f_2 \geq \dots$ be a sequence of C^∞ plurisubharmonic functions converging to f on a neighborhood of $\bar{E}_1$. Then, on the boundary of E_1 , $k = f_n - (h + \epsilon) < 0$ for some integer n .

Let $\Delta = \sum g^{i\bar{j}} \partial^2 / \partial z_i \partial \bar{z}_j$ (3), and recall that the matrix $\partial^2 f_n / \partial z_i \partial \bar{z}_j$ is positive semidefinite (13). Then $\Delta k \geq 0$ on E . By the well known argument which establishes the maximum principle for Δ , $k \leq 0$ on E_1 . Since $f_n \geq f$, $f - (h + \epsilon) \leq 0$ on E_1 , hence $f \leq h$.

Lemma 2. Let f be subharmonic with respect to a metric, and suppose $f \leq h$ with $\Delta h = 0$. If $f = h$ at one point of E , then $f = h$ identically on E .

Proof. For p in E , let S_ϵ be the surface of the sphere of radius $\epsilon > 0$ about p . Then there is a Borel measure μ on S_ϵ with $g(p) = \int_{S_\epsilon} g(x) d\mu(x)$ for every solution of $\Delta g = 0$. Further, μ does not vanish on open sets. Since f is subharmonic, if $g \geq f$ on S_ϵ , $\Delta g = 0$, and g is continuous,

$$f(p) \leq g(p) \leq \int_{S_\epsilon} g(x) d\mu(x).$$

Since f is upper semicontinuous on S_ϵ , it is the limit of a decreasing sequence

of continuous functions, and these functions may be extended so as to be continuous solutions of $\Delta g = 0$ in the interior of S . Thus

$$f(p) \leq \int_{S_*} f(x) d\mu(x).$$

If $f(p) = h(p)$ and $f \leq h$,

$$f(p) \leq \int_{S_*} f(x) d\mu(x) \leq \int_{S_*} h(x) d\mu(x) = h(p),$$

so that equality must prevail, and, since μ does not vanish on open sets, $h = f$ on S_* . Since ϵ was an arbitrary small number, $h = f$ in a neighborhood of p . Therefore the points where $f = h$ form an open set; it is also closed, and therefore is E by connectedness. This completes the proof.

Now, to return to the original domain D , it is assumed that f is plurisubharmonic on a neighborhood of $\bar{D}$. The Kählerian metric will be the Bergman metric; since this is defined only on D , the previous lemmas hold only for domains with closure in D . They can be strengthened to cover D as follows.

Lemma 3. *Suppose h belongs to the class $\mathcal{E}$ of solutions of $\Delta h = 0$. Let f be plurisubharmonic on a neighborhood of $\bar{D}$. If $f \leq h$ on B , then $f \leq h$ on D .*

Proof. Let $\epsilon > 0$. The set $I = \{x \mid f(x) \geq h(x) + \epsilon\}$ is a closed subset of $\bar{D}$. If $I \subset D$, there is an open set E with $I \subset E \subset \bar{E} \subset D$. On the boundary of E , $f \leq h$; by Lemma 1, $f \leq h$ on E , hence I is empty. The other case is that I intersects the boundary of D . By hypothesis, I cannot intersect B , so it intersects the interior of some analytic slice S in the residual boundary. The restriction of a plurisubharmonic function to an analytic subvariety is plurisubharmonic, and if the subvariety is of complex dimension 1, the concepts “plurisubharmonic function” and “subharmonic function” coincide [8]. On S , $f \geq h + \epsilon$ at some interior points but on the boundary of S , $f \leq h$. Since f is subharmonic and h is harmonic, this is impossible. Thus on D , $f \leq h + \epsilon$ with ϵ an arbitrary positive number; hence $f \leq h$. This completes the proof.

Let G be the stability group of the origin. It is compact and consists of the linear transformations (11). Define

$$(Gf)(z) = \int_G f(gz) dg.$$

Here dg is normalized Haar measure on G . The integral exists or is $-\infty$ if f is upper semicontinuous on an open set E invariant under G .

Lemma 4. *If f is plurisubharmonic on E , so is Gf .*

Proof. It suffices to show that Gf is plurisubharmonic on an arbitrary invariant domain E_1 of positive distance from the boundary of E . Let $f_1 \geq f_2 \geq \dots$ be a sequence of C^∞ plurisubharmonic functions in E_1 converging to f . Then $Gf_1 \geq$

$Gf_2 \geq \dots$, and Gf_n converges to Gf . Since Gf_n is continuous, Gf is upper semi-continuous. Further,

$$\frac{\partial^2 f_n(gz)}{\partial z_i \partial \bar{z}_i}$$

is a positive semidefinite matrix for any $g \in G$. The integral with respect to g of this is then positive semidefinite. Thus Gf_n is plurisubharmonic; hence so is Gf .

4. The generalized Jensen-Nevanlinna formula. Recall that $P(w, b)$ is the Poisson kernel.

Theorem 2. *Let f be analytic on a neighborhood of $\bar{D}$. Then*

$$(14) \quad \int_B \log |f(b)| P(w, b) db = \log |f(w)| + J(w),$$

for w in D . $J(w) \geq 0$, and if J vanishes at some point w_0 in D , it must vanish identically. J vanishes identically if and only if f does not vanish in D . J has the same singularities as $-\log |f(w)|$ and, except for these singularities, satisfies $\Delta J = 0$. Also,

$$(15) \quad \int \log |f(rb)| db$$

is a convex function of $\log r$.

Proof. The convexity of (15) is proved first. By lemma 4, $a(w) = \int_G \log |f(gw)| dg$ is a plurisubharmonic function. The restriction of $a(w)$ to the complex plane $w_2 = \dots = w_n = 0$ is then plurisubharmonic, i.e. subharmonic in the ordinary sense, or $-\infty$. If $f(0) \neq 0$, $a(w)$ cannot be $-\infty$. The convergence of $\int_B \log |f(b)| P(w, b) db$ does not depend on w ; even if $f(0) = 0$, the origin could be moved to a point where f does not vanish, which guarantees the convergence of $a(w)$. But the value of a on the plane above depends only on $|w_1|$, since $a(w)$ is invariant under the group G . A subharmonic function of r only is a convex function of $\log r$, which proves the convexity of (15), since a on the circle of radius r in this plane is the integral (15).

Equation (14) may be taken as the definition of J ; it suffices to prove the remainder of Theorem 2 from this. Since $\log |f|$ is upper semicontinuous, there exists a decreasing sequence of continuous functions h_n on B converging to $\log |f|$. By Theorem 1 the h_n may be extended to satisfy $\Delta h_n = 0$ and to have the regularity properties on $\bar{D}$ of the class $\mathcal{E}$. By Lemma 3

$$\log |f(w)| \leq h_n(z) \leq \int_B h_n(b) P(w, b) db,$$

and by continuity of the integral under monotone convergence

$$\log |f(w)| \leq \int_B \log |f(b)| P(w, b) db.$$

This last inequality is equivalent to $J(w) \geq 0$. The special case $J(0) \geq 0$ follows from the convexity of $a(r)$, proved in the preceding paragraph. But by the homogeneity of D , $J(w) \geq 0$ also follows from this special case. If $J(w_0) = 0$ for w_0 in D , J must vanish identically in D by Lemma 2. Clearly if J vanishes identically, $\log |f|$ is non-singular, since $u = J + \log |f|$ satisfies $\Delta u = 0$ in D . Thus J has the same singularities as $\log |f|$, and $\Delta J = 0$, since $\Delta \log |f| = 0$, from (3).

If f does not vanish on the closure of D , $\log |f|$ is a function of the class $\mathcal{E}$, and is given by (4). Therefore J vanishes identically. If f vanishes on the boundary of D but not on D , $f_r(w) = f(rw)$ does not vanish inside D , for $r < 1$. Since f is defined for a neighborhood of $\bar{D}$, and $\int_B \log |f_r(b)| db$ is a convex function of r in a neighborhood of $r = 1$, hence is continuous there, $\log |f(0)| = \int_B \log |f(b)| db$, and $J(0) = 0$. Thus J vanishes at one point, and hence identically. This completes the proof.

Remarks. The proof as given here uses only the plurisubharmonic properties of $\log |f|$. Thus the theorem holds for other functions than $\log |f|$, but, for a pseudo-convex domain like D , the class of functions $\log |f|$ and the class of plurisubharmonic functions are not very far apart [5]. It would be possible to deduce Theorem 2 without using Theorem 1, since the inequality $J \geq 0$ follows from the convexity argument used in the course of the proof. Therefore Theorem 2 has really been proved for the classical Cartan domains, as treated in [10]. However, the argument used for the body of the proof of Theorem 2 is more general, provided only that an analog of Theorem 1 can be proved for more general domains, while the convexity argument in its full generality is peculiar to the Cartan domains. Other convexity theorems, generalizing the Hadamard three circle theorem, follow from a similar convexity argument for the classical Cartan domains, though. It would be easy to characterize J by $\log |f|$ and a condition amounting to vanishing on the boundary. The case where f is merely analytic in D and satisfies some regularity condition near B is much more complicated. It has been examined by BEURLING [3] in the unit disk in one dimension, and unpublished work by HENRY HELSON and the author has generalized BEURLING's work to the polycylinder. The method used here to prove Theorem 1 can be extended to all the classical Cartan domains, but the classification of their boundary points is more difficult. Here there are only two essentially different classes, and each point on the residual boundary is on an analytic slice of one complex dimension, but in the general case there are numerous different types of boundary points, and the ones not on the Bergman-Šilov boundary are analytic slices of varying dimensions.

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