New Inequalities for Entire Functions

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1. Introduction. Let $f(z)$ be regular for $x \geq 0$ and of exponential type $\tau < \pi$. If $\lim_{n \rightarrow \infty} f(n)$ exists, it was shown by CARTWRIGHT that $\lim_{x \rightarrow \infty} f(x)$ exists, and by PLANCHEREL & PÓLYA, in analogy to S. BERNSTEIN's theorem on derivatives, that $\lim_{x \rightarrow \infty} f'(x) = 0$. The converse of the first statement is trivially true, while the converse of the second is false (consider, for example, $f(z) = z \int_0^\pi t^{-\frac{1}{2}} \sin zt \, dt$). One of our objects is to obtain generalizations of the second statement which have valid converses. We shall show, in particular, that $\lim_{n \rightarrow \infty} f(n)$ exists if and only if $\sum_1^\infty f'(n)$ converges. This is plausible since $\lim f(n)$ exists if and only if $\lim f(x)$ exists, *i.e.* if and only if $\int_0^\infty f'(t) \, dt$ converges. We deduce from this, answering a question of GAIER, a (C, 1) analogue of CARTWRIGHT's theorem: $f(n)$ approaches a limit in the (C, 1) sense if and only if $f(x)$ approaches a limit in the (C, 1) sense. That is, $m^{-1} \sum_1^m f(k)$ has a limit as $m \rightarrow \infty$ if and only if $x^{-1} \int_0^x f(t) \, dt$ has a limit as $x \rightarrow \infty$. More generally, if one of $m^{-1} \sum_1^m f(k + \lambda)$ and $x^{-1} \int_0^x f(t + \lambda) \, dt$ has a limit for some λ , the other has a limit for every positive λ . From this, in turn, we deduce that $f(n)$ approaches a limit (C, 1) if and only if $\sum f'(n)$ is summable (C, 1); that is, $\lim_{m \rightarrow \infty} m^{-1} \sum_0^m f(k)$ exists if and only if $\lim_{m \rightarrow \infty} m^{-1} \sum_0^m (m - k) f'(k)$ exists. In particular, if $f(x)$ approaches a limit (C, 1), then $f'(x)$ approaches zero (C, 1).

There are corresponding results with boundedness replacing convergence, and more precise results for entire functions. We now state our results in more detail.

Let $f(z)$ be an entire function of exponential type τ (that is, of order 1 and type τ or less, or of order less than 1). If $|f(x)| \leq M$ for all real x then, according to a famous theorem of S. BERNSTEIN, $|f'(x)| \leq M\tau$. The converse is, of course, not true; we cannot even infer the boundedness of $f(x)$ from that of $f'(x)$. Our first theorem is a generalization of BERNSTEIN's theorem that has a valid converse in the following sense: If $\tau < \pi$, $f(x)$ is bounded if and only if all sums $\sum_p^q f'(x + k)$ have a common bound (independent of p and q). More precisely, we prove

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Theorem 1. Let $f(z)$ be an entire function of exponential type τ .

(a) If $\tau \leq \pi$ and $|f(x)| \leq M$ then

$$(1.1) \quad \left| \sum_{k=0}^n f'(k+x) \right| \leq M\pi, \quad n = 0, \pm 1, \pm 2, \dots$$

(b) If $\tau < \pi$ and (1.1) holds for a single x then

$$(1.2) \quad |f(x_1) - f(x_2)| \leq A(\tau)M$$

for all real x_1, x_2 , where $A(\tau)$ depends only on τ . If $\tau = \pi$ and (1.1) is valid uniformly for $0 \leq x \leq 1$ then (1.2) is true.

Part (a) contains BERNSTEIN's theorem for $\tau = \pi$ (and the general case follows from this). Relation (1.2) of course implies that $f(x)$ is bounded. The question of whether the constant π in (1.1) is sharp is discussed at the end of this section.

In our next result boundedness is replaced by convergence.

Theorem 2. Let $f(z)$ be an entire function of exponential type π . Then $f(x)$ approaches a limit as $x \rightarrow +\infty$ if and only if

$$(1.3) \quad \sum_{k=1}^{\infty} f'(k+x)$$

converges uniformly on $0 \leq x \leq 1$. If f is of type less than π , the convergence of (1.3) for one real x implies its uniform convergence in every interval $x \geq x_0 > -\infty$.

The corresponding theorem for a half-plane is as follows.

Theorem 3. Let $f(z)$ be regular and of exponential type $\tau < \pi$ in the half-plane $x > 0$. Then $f(x)$ is bounded for real $x > 1$ (or approaches a limit as $x \rightarrow +\infty$) if and only if $\sum_1^{\infty} f'(k)$ has bounded partial sums (or converges).

A special case of Theorem 3, that $f'(x) \rightarrow 0$ if $f(x)$ approaches a limit, was given by PLANCHEREL & PÓLYA ([7], p. 243); it was rediscovered by GAIER [5], who applied it to the theory of Borel summability.

An equivalent statement of the convergence part of Theorem 3 is that if $F(z)$ is regular and of exponential type less than π in the half-plane $x > 0$, then $\int_0^{\infty} F(t)dt$ converges if and only if $\sum F(n)$ converges; in other words, the integral test for convergence applies to $\sum F(n)$ when $F(z)$ is regular and of exponential type less than π in a right-hand half-plane.

Since, under the hypotheses of Theorem 3, $\sum f'(k)$ converges when $f(k)$ approaches a limit as $k \rightarrow \infty$, it is reasonable to expect that the stronger hypothesis that $\sum f(k)$ converges should imply even more. This is, in fact, the case: By applying Theorem 3 to a sum $\sum_m^n f(x+k)$ we see that if $\sum f(k)$ converges then

$$\sum_{j=p}^q \sum_{k=m}^n f'(k+j) \rightarrow 0$$

as p and $m \rightarrow \infty$ ($q \geq p; n \geq m$). For example,

$$\sum_{r=1}^n rf'(r) + \sum_{r=n}^{2n} (2n-r)f'(r)$$

approaches a limit as $n \rightarrow \infty$. Similarly, under the hypotheses of Theorem 1 we can say more about f'' than about f' : We have

$$\left| \sum_{j=p}^q \sum_{k=m}^n f''(k+j) \right| \leq \pi^2 M.$$

We shall use Theorem 3 to answer, at least in part, a question asked by GAIER [6].

Theorem 4. *Let $f(z)$ be regular and of exponential type $\tau < \pi$ in the half-plane $x > 0$. If either*

$$(1.4) \quad \frac{1}{n} \sum_{k=0}^n f(k+\lambda)$$

or

$$(1.5) \quad \frac{1}{x} \int_0^x f(t+\lambda) dt$$

is bounded (or approaches a limit as $n \rightarrow \infty$ or $x \rightarrow \infty$) for a single positive λ , then the other is bounded (or approaches a limit) for every positive λ .

In other words, the sequences $\{f(k+\lambda)\}$ are (C, 1)-bounded or (C, 1)-convergent if and only if the same is true for the function $f(x)$. The corresponding result for (C, 0) is trivial in one direction and true by CARTWRIGHT's theorem in the other.

The form of Theorem 3 suggests that there should be similar theorems in which summability replaces convergence. We shall verify this for (C, 1) summability.

Theorem 5. *Let $f(z)$ be regular and of exponential type $\tau < \pi$ in the half-plane $x > 0$. Then $n^{-1} \sum_{k=1}^n f(k)$ is bounded or approaches a limit as $n \rightarrow \infty$ if and only if $m^{-1} \sum_{j=1}^m (m-j)f'(j)$ is bounded or approaches a limit as $m \rightarrow \infty$ (i.e., $\sum f'(k)$ has (C, 1)-bounded partial sums or is summable (C, 1)).*

It is reasonable to conjecture that Theorems 4 and 5 can be extended to Cesàro summability of higher order, but the proofs may be expected to become more complicated as the order increases.²

In proving several of our theorems we require a generalization of an important lemma of A. J. MACINTYRE. We formulate this as follows.

² We are informed by D. GAIER that W. D. LANG has proved, by quite different methods, the following stronger form of Theorem 4: If $f(z)$ is of exponential type less than π in $x > 0$, then $f(x)$ converges (C, k) ($k > 0$) if $f(n)$ converges (C, k); the converse is true if $f(z)$ is of type less than 2π .

Theorem 6. Let $f(z)$ be regular and of exponential type τ in the half-plane $x > 0$, with $\limsup_{x \rightarrow \infty} x^{-1} \log |f(x)| \leq 0$. If $\alpha > \tau$ and σ is any positive number, we can write $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is regular and of exponential type α in $x > 0$ and $f_1(x) = O(x^{-\sigma})$ as $x \rightarrow +\infty$, while $f_2(z)$ is an entire function of exponential type α and $f_2(x) = O(|x|^{-\sigma})$ as $x \rightarrow -\infty$.

It is worth remarking that each derivative of $f_1(x)$ is $O(x^{-\sigma})$ as $x \rightarrow +\infty$, while each derivative of $f_2(x)$ is $O(|x|^{-\sigma})$ as $x \rightarrow -\infty$.

MACINTYRE's lemma ([3], p. 80; [6]) is the case $\sigma = 1$, and the general case is deducible from this.

A consequence of Theorem 6 is the following result which answers a question raised by J. KOREVAAR. The corresponding result for entire functions is known to be true [2].

Theorem 7. Let $f(z)$ be regular and of exponential type $\tau < \pi$ in the half-plane $x > 0$. If $p > 0$ and $\sum_1^\infty |f(\lambda_n)|^p$ converges, where the sequence $\{\lambda_n\}$ satisfies $|\lambda_n - n| \leq L$ and $|\lambda_n - \lambda_m| \geq \delta > 0$ for $n \neq m$, then $\int_1^\infty |f(x)|^p dx$ converges.

We can use Theorem 7 to establish an "absolute" analogue of Theorem 3. This is the case $p = 1$ of the following theorem.

Theorem 8. If $f(z)$ is regular and of exponential type $\tau < \pi$ in the half-plane $x > 0$ then

$$\sum_{n=1}^{\infty} |f(n+1) - f(n)|^p$$

converges ($p > 0$) if and only if $\sum |f'(n)|^p$ converges.

The convergence of $\sum |f'(n)|^p$ is equivalent to the convergence of $\int^\infty |f'(x)|^p dx$.

We conclude this summary with some remarks on the constant in (1.1). The example $f(z) = \sin \pi z$ shows that the constant π is precise when n is even; but it is not precise when n is odd. Dropping the condition $\tau \leq \pi$, we may ask generally for the smallest $K(\tau, n)$ such that

$$\left| \sum_{k=0}^n f'(k+x) \right| \leq K(\tau, n)M$$

when $|f(x)| \leq M$. By BERNSTEIN's theorem, we certainly have $K(\tau, n) \leq (n+1)\tau$. In another paper [4] we have attacked this problem by a variational method and shown that the extremal function must have a special form, in general involving hyperelliptic integrals. For $n = 1$ (that is, for $f'(x) + f'(x+1)$) the extremal function is expressed in terms of elliptic integrals for certain values of τ . We have, in fact, proved the following result in [4]. Let χ_n, γ_n be the numbers defined by

$$\begin{aligned} \frac{1}{2}\chi_n \tan \frac{1}{2}\chi_n &= 1, & 2n\pi < \chi_n < (2n+1)\pi, \\ \gamma_0 &= \pi, & \frac{1}{2}\gamma_n \tan \frac{1}{2}\gamma_n &= -\frac{\gamma_n^2 + \pi^2}{\gamma_n^2 - \pi^2}, & (2n+1)\pi < \gamma_n < (2n+2)\pi, & n \geq 1. \end{aligned}$$

Then if $0 < \tau \leq \chi_0 = 1.72$, or if $\gamma_n \leq \tau \leq \chi_{n+1}$, $K(\tau, 1) = 2 \mid \tau \cos \frac{1}{2}\tau \mid$ and the maximum is attained for a function of the form $\pm \sin \tau z$; in $\chi_n < \tau < \gamma_n$, $K(\tau, 1)$ is the largest value (for $0 < \theta < \frac{1}{2}\pi$) of $2 \mid \psi'(\frac{1}{2}) \cos \psi(\frac{1}{2}) \mid$, where

$$\psi(z) = \int_0^{\tau z} \frac{w^2 + \mathbf{K}(\mathbf{K} - 2\mathbf{E})}{\{w^4 - 2w^2\mathbf{K}^2 \cos 2\theta + \mathbf{K}^4\}^{\frac{1}{2}}} dw,$$

and $\mathbf{K}$ and $\mathbf{E}$ denote the complete elliptic integrals of argument $\cos \theta$; an extremal function is $f(z) = \pm \sin \psi(z)$.

2. Proof of Theorem 6. We begin by proving Theorem 6, since it will be used several times in what follows. Consider first the case $\sigma = 2$. If $f(z)$ satisfies the hypotheses of Theorem 6 the function $zf(z)$ does so also, and by MACINTYRE'S lemma we can write $zf(z) = g_1(z) + g_2(z)$, where $g_1(z)$ is regular and of exponential type α in $x > 0$ and $g_1(x) = O(1/x)$ as $x \rightarrow +\infty$, while $g_2(z)$ is an entire function of exponential type α such that $g_2(x) = O(1/\mid x \mid)$ as $x \rightarrow -\infty$. Then if we write

$$f_1(z) = z^{-1}g_1(z) + cz^{-2} \sin \delta z,$$

$$f_2(z) = z^{-1}g_2(z) - cz^{-2} \sin \delta z,$$

we have the desired decomposition provided that we choose $0 < \delta < \alpha$ and c so that the pole (if any) of $z^{-1}g_2(z)$ is cancelled.

Now consider the general case, where without loss of generality we suppose that σ is a positive integer greater than 2. We write $z^{\sigma-1}f(z) = g_1(z) + g_2(z)$, where $g_1(z)$ and $g_2(z)$ have the same properties as before, and set

$$f_1(z) = z^{-\sigma+1}g_1(z) + \sum_{\nu=1}^{\sigma-1} c_\nu \frac{\sin^{\sigma-1} \delta z}{z^{\sigma-1+\nu}},$$

$$f_2(z) = z^{-\sigma+1}g_2(z) - \sum_{\nu=1}^{\sigma-1} c_\nu \frac{\sin^{\sigma-1} \delta z}{z^{\sigma-1+\nu}}.$$

Here $0 < (\sigma - 1)\delta < \alpha$. Now the function $z^{-\sigma+1}g_2(z)$ has a pole of order at most $\sigma - 1$ at $z = 0$, and $z^{-\sigma+1+\nu} \sin^{\sigma-1} \delta z$ has a pole of precise order ν at $z = 0$. We choose $c_{\sigma-1}$, then $c_{\sigma-2}$, and so on in such a way that the possible pole of $z^{-\sigma+1}g_2(z)$ at the origin is cancelled. Then $f(z) = f_1(z) + f_2(z)$ is the desired decomposition.

It remains to justify the remark that the derivatives of $f_1(z)$ and $f_2(z)$ satisfy the same inequalities, namely $f_1^{(\nu)}(x) = O(x^{-\sigma})$ as $x \rightarrow +\infty$ and $f_2^{(\nu)}(x) = O(\mid x \mid^{-\sigma})$ as $x \rightarrow -\infty$, for $\nu = 1, 2, \dots$. The first of these statements is representative. By considering $z^\sigma f_1(z)$ we reduce it to the proposition that a regular function $g(z)$ of exponential type α in $x > 0$ which is bounded on the positive real axis has its derivative bounded for large positive x . Although we have not found an explicit statement of this fact in the literature, the corresponding o -result is known ([7], p. 243; [5]), and the argument follows familiar lines. From the Phragmén-Lindelöf principle we deduce ([3], p. 82) that $\mid g(x + iy) \mid \leq$

$Ke^{\beta|v|}$ ($\beta > \alpha$) for $x \geq 1$, and then Cauchy's integral representation of $g'(z)$ shows that $g'(x) = O(1)$ as $x \rightarrow +\infty$.

3. Proof of the direct part of Theorems 1 and 2. Let $f(z)$ be an entire function of exponential type π that is bounded by M on the real axis. Then any translation $f(z + c)$, where c is real, satisfies the same conditions. We use a known interpolation formula ([3], p. 211):

$$\begin{aligned} f'(\nu) &= \frac{4}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^2} f(\nu + k + \tfrac{1}{2}) \\ (3.1) \qquad &= \frac{4}{\pi} \sum_{j=-\infty}^{\infty} \frac{(-1)^{j-\nu}}{(2j-2\nu+1)^2} f(j + \tfrac{1}{2}). \end{aligned}$$

If $n \geq 0$, we have, accordingly,

$$(3.2) \qquad \sum_{\nu=0}^n f'(\nu) = \frac{4}{\pi} \sum_{j=-\infty}^{\infty} f(j + \tfrac{1}{2}) \{h(j-n) - h(j+1)\},$$

where we have set

$$(3.3) \qquad h(k) = \sum_{p=k}^{\infty} \frac{(-1)^p}{(2p+1)^2}.$$

Now

$$(3.4) \qquad (-1)^k h(k) > 0,$$

$k \geq 0$. But it is readily shown that $h(-k) = h(k)$, so (3.4) is true for all k . Thus we have

$$(3.5) \qquad \sum_{\nu=0}^n f'(\nu) = \frac{4}{\pi} \sum_{j=-\infty}^{\infty} (-1)^j f(j + \tfrac{1}{2}) \{(-1)^n |h(j-n)| + |h(j+1)|\}.$$

In particular, $f(z) = \sin \pi z$ satisfies our hypotheses. For this function and for an even n , (3.5) yields

$$\sum_{j=-\infty}^{\infty} |h(j)| = \frac{\pi^2}{8}.$$

Returning to the general case, we have from (3.5)

$$\left| \sum_{\nu=0}^n f'(\nu) \right| \leq \frac{4M}{\pi} \sum_{j=-\infty}^{\infty} \{|h(j-n)| + |h(j+1)|\} = \pi M, \quad n = 0, 1, 2, \dots$$

If we apply this to $f(\pm z + c)$ we obtain (1.1).

Now suppose that $f(x) \rightarrow L$ as $x \rightarrow +\infty$. Write

$$H = \sum_{j=-\infty}^{\infty} h(j).$$

Then we have

$$\begin{aligned} \sum_{j=-\infty}^{\infty} (-1)^j f(x + j + \tfrac{1}{2}) (-1)^n |h(j - n)| \\ = \sum_{j=-\infty}^{\infty} \{f(x + n + j + \tfrac{1}{2}) - L\} h(j) + LH \\ = LH + \epsilon(n, x), \end{aligned}$$

where $\epsilon(n, x) \rightarrow 0$ as $n \rightarrow \infty$, uniformly in x on $[0, 1]$. Thus, from (3.5), we have

$$\sum_{\nu=0}^n f'(\nu + x) = \frac{4}{\pi} \sum_{j=-\infty}^{\infty} f(x + j + \tfrac{1}{2}) (-1)^j |h(j + 1)| + LH + \epsilon(n, x).$$

and so $\sum f'(\gamma + x)$ converges uniformly for $0 \leq x \leq 1$.

If n is odd, (3.5) leads to a smaller bound than πM for the sum in (1.1). To illustrate this for $n = 1$ we write

$$\begin{aligned} f'(1) + f'(0) &= \frac{4}{\pi} \sum_{j=-\infty}^{\infty} f(j + \tfrac{1}{2}) \{h(j - 1) - h(j + 1)\} \\ &= \frac{4}{\pi} \sum_{j=-\infty}^{\infty} f(j + \tfrac{1}{2}) \left\{ \frac{(-1)^{j-1}}{[2(j-1) + 1]^2} + \frac{(-1)^j}{(2j + 1)^2} \right\}. \end{aligned}$$

For $j < 0$ the sum in $\{\dots\}$ has the sign of $(-1)^j$; for $j = 0$ it is 0; for $j > 0$ it has the sign of $(-1)^{j-1}$. Hence

$$\begin{aligned} |f'(1) + f'(0)| &\leq \frac{4M}{\pi} \sum_{-\infty}^{-1} \left\{ \frac{-1}{[2(j-1) + 1]^2} + \frac{1}{(2j + 1)^2} \right\} \\ &\quad + \frac{4M}{\pi} \sum_1^{\infty} \left\{ \frac{1}{[2(j-1) + 1]^2} - \frac{1}{(2j + 1)^2} \right\} \\ &= \frac{8M}{\pi} < \pi M. \end{aligned}$$

The bound $8M/\pi$ is not precise since for equality we should have to have $f(j + \frac{1}{2}) = \delta M (-1)^j$ for $j < 0$ and $f(j + \frac{1}{2}) = \delta M (-1)^{j+1}$ for $j \geq 0$, where δ is a complex number of modulus 1. There is no entire function of exponential type π which is bounded by M on the real axis and assumes these values.

To see this, it is enough to consider the case when $\delta = M = 1$, and instead of $f(z)$ we consider $g(z) = f(z + \frac{1}{2})$, which would have $g(j) = (-1)^j$, $j > 0$; $g(j) = (-1)^{j+1}$, $j \leq 0$. By writing $g(z) = g_1(z) + ig_2(z)$, where g_1 and g_2 are entire functions of exponential type π which are real on the real axis, we reduce the problem to the consideration of the function g_1 , which is real on the real axis. (Since $g_2(j) = 0$ for every integer j , $g_2(z)$ is a constant multiple of $\sin \pi z$; see [3], p. 156.) Now we show, more generally, that if $g_1(z)$ is an entire function of exponential type π , real on the real axis, with $g_1(j) = \pm 1$ for all integers j , then $g_1(z)$ is either ± 1 or $\pm \cos \pi z$, neither of which is consistent with our data. To prove this, consider $g'_1(z)$, which is again an entire function of exponential

type π , bounded on the real axis (by BERNSTEIN's theorem), and vanishing at the integers since $g_1(z)$ takes its maximum and minimum values at the integers. By a Phragmén-Lindelöf theorem ([3], p. 82) we have $|g_1'(z)| \leq \pi e^{\pi|v|}$, and consequently ([3], p. 156) $g_1'(z)$ is a constant multiple of $\sin \pi z$, say $g_1'(z) = \pi a \sin \pi z$. Then $g_1(z) = -a \cos \pi z + b$, and $g_1(j) = -a(-1)^j + b = \pm 1$. Taking $j = 0, 1$, we have $-a + b = \pm 1$, $a + b = \pm 1$, and hence either $b = \pm 1$ and $a = 0$ or $a = \pm 1$ and $b = 0$. The first case gives us $g_1(z) = \pm 1$; the second, $g_1(z) = \pm \cos \pi z$.

Likewise a bound for $\sum_0^n f'(\nu)$ which is less than πM can be obtained for every positive odd integer n . We first note that if $\phi(k) = (-1)^k h(k)$ then $\phi(k)$ is a decreasing function of k for $k \geq 0$. For

$$\phi(k) - \phi(k+1) = \frac{1}{(2k+1)^2} - \frac{2}{(2k+3)^2} + \frac{2}{(2k+5)^2} - \dots$$

If the terms are regrouped, the first group being

$$\frac{1}{(2k+1)^2} - \frac{2}{(2k+3)^2} + \frac{1}{(2k+5)^2},$$

it is clear from the convexity of $1/x^2$ that each group is positive. Since $h(k)$ is an even function it follows that $|h(k)| > |h(\nu)|$ for $|\nu| < k$.

If n is a positive odd integer, relation (3.2) shows that $|\sum_0^n f'(\nu)| \leq 4MB/\pi$, where

$$B = \sum_{j=-\infty}^{\infty} |h(j-n) - h(j+1)|.$$

Now $h(j-n) - h(j+1)$ has the sign of $-h(j+1)$ if $j < \frac{1}{2}(n-1)$, it is zero for $j = \frac{1}{2}(n-1)$, and it has the sign of $h(j-n)$ for $j > \frac{1}{2}(n-1)$. Thus

$$\begin{aligned} B &= \sum_{j=-\infty}^{\frac{1}{2}(n-3)} \{h(j-n) - h(j+1)\}(-1)^j - \sum_{j=\frac{1}{2}(n+1)}^{\infty} \{h(j-n) - h(j+1)\}(-1)^j \\ &= 2h(0) + 4 \sum_{\nu=1}^{\frac{1}{2}(n-1)} (-1)^\nu h(\nu) + 2(-1)^{\frac{1}{2}(n+1)} h(\tfrac{1}{2}n + \tfrac{1}{2}) \\ &= 2\{1 + 3^{-2} + 5^{-2} + \dots + n^{-2}\}. \end{aligned}$$

Hence we have

$$\left| \sum_{\nu=0}^n f'(\nu) \right| \leq \frac{8M}{\pi} \{1 + 3^{-2} + \dots + n^{-2}\}$$

if n is a positive odd integer. This bound is less than πM but it is not precise, by the same argument as for $n = 1$.

4. Proof of the converse part of Theorems 1 and 2. To prove part (b) of Theorem 1 we suppose without loss of generality that (1.1) holds for $x = 0$, since otherwise we consider $f(z+c)$ instead of $f(z)$. Let $f(z)$ be an entire function of exponential type $\tau < \pi$. For each positive integer m , the function

$$(4.1) \quad \phi_m(z) = \sum_{k=0}^m f'(z+k)$$

is an entire function of exponential type τ , and by (1.1)

$$|\phi_m(n)| \leq 2M\pi, \quad n = 0, \pm 1, \pm 2, \dots$$

By CARTWRIGHT'S theorem ([3], p. 180) we then have

$$|\phi_m(x)| \leq A(\tau)M, \quad -\infty < x < \infty,$$

where $A(\tau)$ depends only on τ . Thus

$$(4.2) \quad \left| \sum_{k=0}^n f'(x+k) \right| \leq A(\tau)M, \quad -\infty < x < \infty.$$

In other words, (1.1) holds uniformly in x with a larger constant. If $x_2 > x_1$, we then have

$$(4.3) \quad f(x_2) - f(x_1) = \int_0^1 \left\{ \sum_{\nu=0}^{\lfloor x_2-x_1 \rfloor - 1} f'(x_1 + \nu + t) \right\} dt + \int_{\lfloor x_2-x_1 \rfloor}^{x_2-x_1} f'(x_1 + t) dt,$$

and conclusion (b) of Theorem 1 follows. The last statement of (b) follows by the same argument since we assume (4.2) to begin with when $\tau = \pi$.

Similarly, if (1.4) converges uniformly and we take $x_2 = x$ and $x_1 = 0$ in (4.3) we have

$$f(x) - f(0) = \int_0^1 \sum_{\nu=0}^{\lfloor x \rfloor - 1} f'(\nu + t) dt + \int_{\lfloor x \rfloor}^x f'(t) dt,$$

and so $f(x)$ approaches a limit.

It remains to establish the last sentence of Theorem 2. Now $f(z)$ is an entire function, so if the series (1.3) converges uniformly in $x \geq x_0$ where x_0 is some real number, it must converge uniformly in $x \geq x_1$, where x_1 is any real number. By translating $f(z)$, if necessary, we may suppose that $f(z)$ is an entire function of exponential type $\tau < \pi$ and that the series (1.3) converges for $x = 0$. We shall show that (1.3) converges uniformly for $x \geq 1$, and the theorem will follow.

The convergence of $\sum f'(k)$ implies that $f'(k) = o(1)$, and since $f'(z)$ is again an entire function of exponential type τ another theorem of CARTWRIGHT'S ([3], p. 180) shows that $f'(x) = o(1)$ as $x \rightarrow \infty$.

We apply Theorem 6 to $f(z)$ with $\sigma = 2$. Thus $f(z) = f_1(z) + f_2(z)$, where f_1 and f_2 are entire functions of exponential type α , $\tau < \alpha < \pi$; $f_1(x) = O(1/x^2)$ as $x \rightarrow +\infty$, and $f_2(x) = O(1/x^2)$ as $x \rightarrow -\infty$. Then

$$\sum_{k=1}^{\infty} f'(x+k) = \sum_{k=1}^{\infty} f'_1(x+k) + \sum_{k=1}^{\infty} f'_2(x+k).$$

We have remarked that $f'_1(x) = O(1/x^2)$ as $x \rightarrow +\infty$, so $\sum f'_1(x+k)$ converges uniformly for $x \geq 1$. Since $f'_2(k) = f'(k) - f'_1(k)$, the convergence of $\sum f'(k)$ implies the convergence of $\sum f'_2(k)$.

Define

$$\phi_m(z) = \sum_{k=1}^m f'_2(k+z).$$

Then $\phi_m(z)$ is bounded at the positive integers, with a bound independent of m . It is also bounded at the negative integers since $f'_2(x) = O(1/x^2)$ as $x \rightarrow -\infty$. By CARTWRIGHT'S theorem ([3], p. 180) it is bounded on the real axis, with a bound λ which is independent of m . Then by a Phragmén-Lindelöf argument ([3], p. 82) we have

$$(4.4) \quad |\phi_m(x+iy)| \leq \lambda e^{\alpha|y|}$$

over the entire plane.

Now $\phi_m(z) - \phi_n(z)$ converges to zero at the integers $\nu = 0, 1, 2, \dots$ as $m, n \rightarrow \infty$; indeed the convergence is uniform in ν . Then the method of proof used in [3], pp. 182–183, shows that $\phi_m(x) - \phi_n(x) \rightarrow 0$ uniformly in $x \geq 1$ as $m, n \rightarrow \infty$. This also follows by a normal family argument. For if $\psi_{m,n}(z) = \phi_m(z) - \phi_n(z)$ has modulus greater than some positive ρ at some point of the part $x \geq 1$ of the real axis, let μ be an integer such that the translate $\psi_{m,n}(z + \mu)$ has modulus greater than ρ somewhere in the interval $1 \leq x \leq 2$. Here μ depends on m, n . Now (4.4) shows that the functions $\psi_{m,n}(z + \mu)$ are dominated by $2\lambda e^{\alpha|y|}$ in the half-plane $x > 0$, and they tend to zero at the integers $z = 0, 1, 2, \dots$. Thus a subsequence tends to a limit $\psi(z)$ which vanishes at the integers, is regular in $x > 0$, and satisfies $|\psi(x+iy)| \leq 2\lambda e^{\alpha|y|}$. But CARLSON'S theorem ([3], p. 153) then shows that $\psi(z)$ vanishes identically. Thus $\phi_m(x) - \phi_n(x)$ converges uniformly to zero in $x \geq 1$ as $m, n \rightarrow \infty$.

5. Proof of Theorem 3. Let $f(z)$ be of exponential type $\tau < \pi$ in the half-plane $x > 0$. If $f(x)$ is bounded for $x > 1$ or if $\sum f'(k)$ has bounded partial sums the conditions of Theorem 6 are satisfied. For in the latter case $f'(k)$ is bounded, so by CARTWRIGHT'S theorem $f'(x) = O(1)$ as $x \rightarrow \infty$, and then $f(x) = O(x)$ as $x \rightarrow \infty$. Thus we can apply Theorem 6 with $\sigma = 2$, and write $f(z) = f_1(z) + f_2(z)$, where f_1 is of exponential type $\alpha < \pi$ in $x > 0$, f_2 is an entire function of exponential type α , and $f_1(x) = O(x^{-2})$ as $x \rightarrow \infty$, $f_2(x) = O(x^{-2})$ as $x \rightarrow -\infty$. As we remarked before, we have $f'_1(x) = O(x^{-2})$ as $x \rightarrow \infty$, so $\sum f'_1(k)$ converges.

We shall show that $f_2(x)$ is bounded for $x > 1$ (or approaches a limit as $x \rightarrow \infty$) if and only if $\sum f'_2(k)$ has bounded partial sums (or converges). The theorem will follow from this statement.

If $f_2(x)$ is bounded, Theorem 1 shows that $\sum_1^\infty f'_2(k)$ has bounded partial sums. If $f_2(x)$ has a limit as $x \rightarrow \infty$, Theorem 2 shows that $\sum_1^\infty f'_2(k)$ converges.

Conversely, if $\sum_1^\infty f'_2(k)$ converges, Theorem 2 states that (1.3) converges uniformly for $x \geq 0$, and that $f_2(x)$ approaches a limit as $x \rightarrow \infty$. If the partial sums of $\sum_1^\infty f'_2(k)$ are bounded then inequality (1.1) holds for some M since $f'_2(x) = O(1/x^2)$ as $x \rightarrow -\infty$. Then part (b) of Theorem 1 shows that $f_2(x)$ is bounded.

6. Proof of Theorem 4. Let $f(z)$ be regular and of exponential type $\tau < \pi$ in the half-plane $x > 0$. If (1.4) is bounded as $n \rightarrow \infty$ then $f(n + \lambda) = O(n)$, so the function $f(z + \lambda)/z$ is bounded at the positive integers. Since it is of exponential type $\tau < \pi$ in the half-plane $x > 0$, CARTWRIGHT'S theorem shows that it is bounded as $x \rightarrow \infty$. Thus if either (1.4) or (1.5) is bounded as n or $x \rightarrow \infty$, the function $f(z)$ satisfies the hypotheses of Theorem 6. We use that theorem with $\sigma = 2$ to write $f(z) = f_1(z) + f_2(z)$ with the usual conditions on f_1 and f_2 . Then both

$$\frac{1}{n} \sum_{k=0}^n f_1(k + \lambda), \quad \frac{1}{x} \int_0^x f_1(t + \lambda) dt$$

approach zero as $n, x \rightarrow \infty$, respectively, since $f_1(x) = O(x^{-2})$ as $x \rightarrow \infty$.

It will then be sufficient to prove the theorem for the entire function $f_2(z)$. Consider the function

$$F(z) = \frac{1}{z} \int_0^z f_2(t) dt.$$

Then $F(x) \rightarrow 0$ as $x \rightarrow -\infty$. We suppose first that either

$$(6.1) \quad |F(x)| \leq M$$

or

$$(6.2) \quad \lim_{x \rightarrow \infty} F(x) = S,$$

and we are to show, respectively, that for each positive λ

$$\frac{1}{n} \sum_{k=0}^n f_2(k + \lambda)$$

is bounded or approaches a limit as $n \rightarrow \infty$.

We have, respectively, as $n \rightarrow \infty$,

$$(6.3) \quad \begin{aligned} \frac{1}{n} \sum_{k=1}^n f_2(k + \lambda) &= \frac{1}{n} \sum_{k=1}^n \{(k + \lambda)F'(k + \lambda) + F(k + \lambda)\} \\ &= \frac{1}{n} \sum_{k=1}^n (k + \lambda)F'(k + \lambda) + \begin{cases} O(1) \\ S + o(1). \end{cases} \end{aligned}$$

If either (6.1) or (6.2) is true then $F(x)$ is bounded for $x > 1$, so Theorem 3 shows that $\sum_1^\infty F'(k)$ has bounded partial sums. Thus the function

$$\phi_m(z) = \sum_{k=1}^m F'(k + z)$$

is bounded at the positive integers with a bound which is independent of m . Now $F'(x) = O(1/x^2)$ as $x \rightarrow -\infty$ so $\phi_m(z)$ is likewise bounded at the negative integers. Since $\phi_m(z)$ is an entire function of exponential type $\alpha < \pi$, CARTWRIGHT'S

theorem shows that $\phi_m(z)$ is bounded on the real axis with a bound independent of m .

Now by partial summation

$$(6.4) \quad \begin{aligned} \sum_{k=1}^n (k + \lambda) F'(k + \lambda) &= \sum_{k=1}^n (k + \lambda) (\phi_k(\lambda) - \phi_{k-1}(\lambda)) \\ &= - \sum_{k=1}^{n-1} \phi_k(\lambda) + (n + \lambda) \phi_n(\lambda). \end{aligned}$$

Thus the left-hand side of (6.3) is bounded as $n \rightarrow \infty$.

In case (6.2) is true, Theorem 3 shows that $\sum F'(k)$ converges, and Theorem 2 then shows that $\sum F'(k + \lambda)$ converges. Thus $\phi_k(\lambda)$ has a limit as $k \rightarrow \infty$, so (6.3) and (6.4) show that

$$\frac{1}{n} \sum_{k=1}^n f(k + \lambda)$$

has a limit as $n \rightarrow \infty$. This establishes one part of Theorem 4.

Now assume that

$$(6.5) \quad \left| \frac{1}{n} \sum_{k=1}^n f_2(k) \right| \leq M$$

or

$$(6.6) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f_2(k) = L.$$

We are to show respectively that, for each positive λ ,

$$\frac{1}{x} \int_0^x f_2(t + \lambda) dt$$

is bounded or approaches a limit as $x \rightarrow \infty$. If we define

$$\psi_n(z) = \frac{1}{n} \sum_{k=0}^n f_2(z + k)$$

then

$$\frac{1}{x} \int_0^x f_2(t + \lambda) dt = \frac{n}{x} \int_0^{x-[x]} \psi_n(t + \lambda) dt + \frac{n-1}{x} \int_{x-[x]}^1 \psi_{n-1}(t + \lambda) dt,$$

where $n = [x]$. Thus it will be sufficient to show in the respective cases that $\psi_n(x)$ is bounded for $0 \leq x \leq 1$ by a number independent of n , or tends boundedly to a limit in $0 \leq x \leq 1$ as $n \rightarrow \infty$.

If either (6.5) or (6.6) is true then the function $\psi_n(z)$ is bounded at the non-negative integers by a number which is independent of n . Since $f_2(x) = O(1/x^2)$ as $x \rightarrow -\infty$, the function $\psi_n(z)$ is also bounded at the negative integers. Since $\psi_n(z)$ is of exponential type less than π , CARTWRIGHT'S theorem shows that

$|\psi_n(x)| \leq N$ over the whole real axis, where N is independent of n . Then ([3], p. 82) we have

$$(6.7) \quad |\psi_n(x + iy)| \leq Ne^{\alpha|y|}.$$

Now suppose (6.6) is true. Then $\psi_n(z) \rightarrow L$ at each positive or negative integer as $n \rightarrow \infty$. By (6.7) the functions $\psi_n(z)$ form a normal family. We assert that $\lim_{n \rightarrow \infty} \psi_n(z) \equiv L$. For, if this is not true, there is a sequence of n 's for which $\psi_n(z) \rightarrow \psi(z)$ where $\psi(z) \not\equiv L$. But $\psi(z) - L$ is an entire function of exponential type less than π , vanishing at the integers. Then by CARLSON'S theorem $\psi(z) - L \equiv 0$.

7. Proof of Theorem 5. If either

$$\frac{1}{n} \sum_{k=1}^n f(k)$$

or

$$\frac{1}{n} \sum_{j=1}^n (n - j)f'(j)$$

is bounded in n then $f(z)$ satisfies the conditions of Theorem 6. For in the first case $f(n)/n = O(1)$ so by CARTWRIGHT'S theorem $f(x)/x = O(1)$. In the second case $f'(n)/n = O(1)$ so $f'(x)/x = O(1)$; and thus $f(x) = O(x^2)$. We use Theorem 6 again with $\sigma = 2$ to write $f(z) = f_1(z) + f_2(z)$ as before. Then both

$$\frac{1}{n} \sum_{k=1}^n f_1(k)$$

and

$$\frac{1}{n} \sum_{j=1}^n (n - j)f_1'(j)$$

approach limits as $n \rightarrow \infty$. We may therefore confine our attention to the entire function $f_2(z)$.

Now $f_2(z)$ is an entire function of exponential type $\alpha < \pi$ such that $z^2 f_2(z)$ is bounded on the negative real axis. If $\alpha < \beta < \pi$, an application of the Phragmén-Lindelöf principle shows that $|z^2 f_2(z)| \leq Me^{\beta|y|}$ for $x \leq 0$. Also, if either $n^{-1} \sum_1^n f_2(k)$ or $n^{-1} \sum_1^n (n - j)f_2'(j)$ is bounded as $n \rightarrow \infty$ then $f_2(x) = O(x^2)$ as $x \rightarrow \infty$. Hence $|f_2(z)| \leq (1 + |z|^2)Me^{\beta|y|}$ for $x \geq 0$. Thus the function

$$s(z) = \sum_{j=-\infty}^0 f_2(z + j) - \sum_{j=-\infty}^0 f_2(j)$$

is an entire function of exponential type β and $s(0) = 0$.

Clearly

$$(7.1) \quad s(n) = \sum_{\nu=1}^n f_2(\nu)$$

and

$$(7.2) \quad \sum_{\nu=0}^{n-1} s'(\nu) = \sum_{j=1}^{n-1} (n-j)f'_2(j) + n \sum_{j=-\infty}^0 f'_2(j).$$

If

$$\frac{1}{n} \sum_{j=1}^{n-1} (n-j)F'_2(j)$$

is bounded or approaches a limit as $n \rightarrow \infty$ the same is true of

$$\frac{1}{n} \sum_{\nu=0}^{n-1} s'(\nu),$$

and according to Theorem 4 (in which the restriction $\lambda > 0$ is unnecessary for entire functions)

$$\frac{1}{x} \int_0^x s'(t) dt = \frac{s(x) - s(0)}{x}$$

is bounded or approaches a limit as $x \rightarrow \infty$. Then one part of the theorem follows by (7.1).

Now write $\sigma(z) = s(z)/z$ and suppose that $\sigma(n) = n^{-1} \sum_1^n f_2(\nu)$ is bounded or approaches a limit as $n \rightarrow \infty$. Then $\sum_1^n \sigma'(k)$ is bounded or approaches a limit as $n \rightarrow \infty$, by Theorem 3. But $s'(\nu) = \nu\sigma'(\nu) + \sigma(\nu)$ so that

$$\begin{aligned} \frac{1}{n} \sum_1^n s'(\nu) &= \frac{1}{n} \sum_1^n \nu\sigma'(\nu) + \frac{1}{n} \sum_1^n \sigma(\nu) \\ &= \frac{1}{n} \sum_1^n (n-\nu)\sigma'(\nu) + \frac{1}{n} \sum_1^n \sigma(\nu) - \sum_1^n \sigma'(\nu) \\ &= \frac{1}{n} \sum_{\nu=0}^n \sum_{j=0}^{\nu} \sigma'(j) + \frac{1}{n} \sum_1^n \sigma(\nu) - \sum_1^n \sigma'(\nu). \end{aligned}$$

Each sum on the right-hand side is bounded or approaches a limit as $n \rightarrow \infty$, so the remainder of Theorem 5 follows from (7.2).

8. Proof of Theorem 8. Let $f(z)$ be regular and of exponential type $\tau < \pi$ in the half-plane $x > 0$. Suppose first that $\sum_1^\infty |f(n)|^p$ converges, where $p > 0$. By Theorem 7, $\int_1^\infty |f'(x)|^p dx$ is finite. Then if $p > 1$,

$$|f(n+1) - f(n)|^p \leq \left\{ \int_n^{n+1} |f'(x)| dx \right\}^p \leq \int_n^{n+1} |f'(x)|^p dx$$

by Hölder's inequality; therefore $\sum |f(n+1) - f(n)|^p$ converges. If $p < 1$ we must argue differently (and the same argument works for all positive p). We have $|f(n+1) - f(n)|^p = |f'(\lambda_n)|^p$, where $n < \lambda_n < n+1$. We certainly have $\lambda_{n+2} - \lambda_n \geq 1$. Since $\int_1^\infty |f'(x)|^p dx$ converges, $\sum |f'(\lambda_{2n})|^p$ and $\sum |f'(\lambda_{2n+1})|^p$ converge ([3], p. 101), so $\sum |f'(\lambda_n)|^p = \sum |f(n+1) - f(n)|^p$ converges.

Conversely, let $\sum |f(n+1) - f(n)|^p$ converge. Then by the law of the mean there are points $\lambda_n^{(1)}$ in $(n, n+1)$ such that $\sum |f'(\lambda_n^{(1)})|^p$ converges. This is not enough to let us infer that $\int_1^\infty |f'(x)|^p dx$ is finite, since we do not know that the $\lambda_n^{(1)}$ satisfy the separation condition $|\lambda_n^{(1)} - \lambda_{n+1}^{(1)}| \geq \delta > 0$. To overcome this difficulty, we consider the function $g(z) = f(z+1) - f(z)$, which is of exponential type less than π . We have $\sum |g(n)|^p$ convergent, hence by Theorem 7 we have $\int_1^\infty |g(x)|^p dx$ convergent. Therefore $[1] \sum |g(\frac{1}{2}n)|^p$ converges, so, in particular, $\sum |f(n + \frac{1}{2}) - f(n + \frac{3}{2})|^p$ converges. Therefore there are points $\lambda_n^{(2)}$ in $(n + \frac{1}{2}, n + \frac{3}{2})$ such that $\sum |f'(\lambda_n^{(2)})|^p$ converges. Similarly there are points $\lambda_n^{(3)}$ in $(n + \frac{1}{3}, n + \frac{4}{3})$ such that $\sum |f'(\lambda_n^{(3)})|^p$ converges, and so on. From the sequences $\{\lambda_n^{(1)}\}, \dots, \{\lambda_n^{(k)}\}$ we can select a single sequence $\{\sigma_n\}$ such that $|\sigma_{n+1} - \sigma_n| \geq \delta > 0$ and $|\sigma_n - n(1 + k^{-1})| < 1$ for all n . For, choose $\sigma_1 = \lambda_1^{(1)}$ in $(1, 2)$; $\sigma_2 = \lambda_2^{(k)}$ in $(2 + k^{-1}, 3 + k^{-1})$; $\sigma_3 = \lambda_3^{(k-1)}$ in $(3 + (k-1)^{-1}, 4 + (k-1)^{-1})$; $\dots$; $\sigma_k = \lambda_k^{(2)}$ in $(k + 2, k + 3)$; and so on. Then $|\sigma_n - \sigma_{n+1}| \geq \{k(k-1)\}^{-1}$ and $|\sigma_n - n(1 + k^{-1})| < 1$. If k is large enough so that $k/(k+1) > \tau/\pi$, we have $\sum |f'(\sigma_n)|^p$ convergent, $|\sigma_{n+1} - \sigma_n| > \delta > 0$, and $|\sigma_n - n(k+1)/k| < 1$. Consequently, by Theorem 7 (with a change of variable) $\int_1^\infty |f'(x)|^p dx$ converges, or equivalently $\sum |f'(n)|^p$ converges.

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