*Linear Partial Differential Systems with an Additional Differential Equation at One Point*¹

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Introduction. In this paper we prove that all the solutions of a linear differential system of elliptic or parabolic type and of order m cannot satisfy an additional linear partial differential equation of order m at a fixed point. That such a theorem is true for linear hyperbolic systems is a simple consequence of the fact that the Cauchy problem can be solved for these systems (see, for instance, [9], [12]). We also note that for any linear partial differential system with analytic coefficients, that theorem will follow as a consequence of the Cauchy-Kowalewski theorem. However, for nonanalytic systems, such a theorem cannot be expected to be generally true, since it might happen that there exist no solutions to a given linear differential system (see [10]).

In Part I of this paper we are concerned with the elliptic case. The above mentioned theorem is stated in Section 1 (Theorem 1) and its proof is given in Sections 2, 3 and 4. In Section 2 we prove it for a special case of constant coefficients. In Section 3 we prove some properties concerning fundamental solutions, and, finally, in Section 4 we complete the proof of Theorem 1 by reducing it, with the aid of fundamental solutions, to the special case of Section 2. Theorem 1 allows us to solve the coefficients of a linear differential system in terms of the solutions. The proof of that fact is included in the proof of Theorem 2, Section 5. In that section, and also in Section 6, we consider some properties satisfied by all the solutions of a linear elliptic system, and show that also the coefficients satisfy, up to proportional factors, the same properties. In Section 7 an interpolation problem related to Theorem 1 is considered.

In Part II we consider the parabolic case. In Section 8 we state the analogue of Theorem 1 (Theorem 4), whose proof is given in Sections 9, 10 and 11. Applications of Theorem 4, analogous to the applications of Theorem 1, are discussed in Section 12.

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PART I

1. All functions and numbers in this paper are real. We shall use the following notation:

$$x = (x_1, \dots, x_n), \quad x^i = x_1^{i_1} \cdots x_n^{i_n}, \quad i! = i_1! \cdots i_n!,$$

$$D_x^i = \frac{\partial^i}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \quad a_i^{\lambda\mu}(x) = a_{i_1 \dots i_n}^{\lambda\mu}(x), \quad \alpha_i^\mu = \alpha_{i_1 \dots i_n}^\mu.$$

We shall consider the system of partial differential equations

$$(1) \quad \sum_{\mu=1}^N \sum_{i \leq m} a_i^{\lambda\mu}(x) D_x^i u_\mu(x) = 0 \quad (\lambda = 1, \dots, N)$$

defined in a domain D . The inner sum is understood to be taken over all the $i_1, \dots, i_n$ from 0 to m such that $i = \sum i_k \leq m$. The system (1) is called *elliptic* if $\det(\sum a_m^{\lambda\mu}(x) \xi^m) \neq 0$ for all real $\xi = (\xi_1, \dots, \xi_n) \neq 0$. Denote by b the number of the terms $D_x^i u_\mu(x)$ when $(i_1, \dots, i_n)$ runs over all possible different vectors with $i \leq m$ and $\mu = 1, \dots, N$. For each λ , denote by a^λ the b -dimensional vector $(a_i^{\lambda\mu})$ when $i = 0, 1, \dots, m$ and $\mu = 1, \dots, N$. Similarly define $\alpha = (\alpha_i^\mu)$, $v = (D_x^i u_\mu)$. For a set of solutions u_μ^i , we denote $v^i = (D_x^i u_\mu^i)$. The scalar product of two vectors a and c is denoted by $a \cdot c$. Thus (1) can be written in the form $a^\lambda \cdot v = 0$.

Together with the system (1), we consider a differential relation of the form

$$(2) \quad \sum_{\mu=1}^N \sum_{i \leq m} \alpha_i^\mu D_x^i u_\mu(x^0) = 0,$$

where x^0 belongs to D . Trivially, (2) is satisfied whenever

$$(3) \quad \alpha_i^\mu = \sum_{\lambda=1}^N c_\lambda a_i^{\lambda\mu}(x^0) \quad (\mu = 1, \dots, N; 0 \leq i \leq m)$$

for some numbers c_λ . We can now state the principal result of Part I.

Theorem 1. Assume that (1) is elliptic and that the $a_i^{\lambda\mu}(x)$ belong to $C^i(D)$. If every solution u_μ of (1) which exists in a neighborhood D_u of x^0 , depending on u_μ , satisfies (2) then, for appropriate c_λ , (3) holds.

The proof of Theorem 1 is given in Sections 2, 3 and 4. Obviously we may assume in the forthcoming proof that $x^0 = 0$.

2. In this section we prove Theorem 1 in the special case where $a_m(x) \equiv \text{const.} = a_m$ and $a_i(x) \equiv 0$ for $i < m$. Substituting $u_\lambda(x) = \sum \gamma_m^\lambda x^m / m!$ in (1) we find that if $\sum_{\mu=1}^N \sum a_m^{\lambda\mu} \gamma_m^\mu = 0$ then $\sum_{\mu=1}^N \sum \alpha_m^\mu \gamma_m^\mu = 0$. This means that every vector $\gamma = (\gamma_m^\mu)$ orthogonal to the vectors a^λ is necessarily orthogonal to α . Consequently, α is linearly dependent on the vectors a^λ , and (3) follows for $i = m$. Since every polynomial of degree smaller than m satisfies (1), and therefore also satisfies (2), it follows that $\alpha_i^\mu = 0$ for $i < m$. We conclude that (3) holds also for $i < m$ and the proof is completed.

3. In this section we shall prove some properties concerning the fundamental solutions of (1). For simplicity we shall consider only the case $N = 1$; the general case can be treated in exactly the same way. (1) and (2) take the simpler forms

$$(4) \quad Lu(x) \equiv \sum_{i \leq m} a_i(x) D_x^i u(x) = 0,$$

$$(5) \quad \sum_{i \leq m} \alpha_i D_x^i u(0) = 0.$$

Denote by $K_z(x)$ the fundamental solution of the equation

$$(6) \quad L_z u(x) \equiv \sum a_m(z) D_x^m u(x) = 0,$$

with singularity at $x = z$, constructed in [6] (see also [7], [8]). Then a fundamental solution $K(x, z)$ of (4), with singularity at $x = z$, can be constructed in the form [7]

$$(7) \quad K(x, z) = K_z(x) + \int_B K_z(x) u(y, z) dy,$$

where B is any domain with a sufficiently small diameter, and $u(x, y)$ is a solution of the integral equation

$$(8) \quad 0 = \alpha(x, z) + u(x, z) + \int_B \alpha(x, y) u(y, z) dy.$$

$\alpha(x, z)$ is defined by

$$(9) \quad \alpha(x, z) = L[K_z(x)].$$

The fundamental solution of the equation $\sum a_m D_x^m u(x) = 0$ with singularity at $x = z$ has the form

$$(10) \quad K(x - z) = |x - z|^{m-n} [\psi(\zeta, a_m) \lg |x - z| + \varphi(\zeta, a_m)],$$

where $\zeta = (x - z)/|x - z|$, $\psi(\zeta, a_m) \equiv 0$ for odd n or for $m < n$, and $|x - z|^{m-n} \psi(\zeta, a_m)$ is a homogeneous polynomial in $x - z$ of degree $m - n$. Both functions $\psi(x, a_m)$ and $\varphi(x, a_m)$ are analytic in x and in the a_m provided the a_m are the coefficients of a definite form and, in the case of φ , provided $x \neq 0$. This follows from the explicit formulas for ψ and φ [6], [7], [8]. In what follows, B_i will be used to denote appropriate constants.

Lemma 1. For any $\epsilon > 0$, $D_x^i K_0(x - z) - D_x^i K_z(x) |_{x=0} = O(|z|^{m-n-i+1-\epsilon})$ ($0 \leq i \leq m$; $|z| \rightarrow 0$).

Proof. We shall give the proof only in the case $i = 0$; the proof in the other cases is obtained in a similar way. Using (10) with $a_m = a_m(z)$ and with $a_m = a_m(0)$, we get

$$\begin{aligned} I(x) \equiv |K_0(x - z) - K_z(x)| &\leq |x - z|^{m-n} [|\psi(\zeta, a_m(z)) - \psi(\zeta, a_m(0))| \lg \frac{1}{|x - z|} \\ &\quad + |\varphi(\zeta, a_m(z)) - \varphi(\zeta, a_m(0))|]. \end{aligned}$$

Since ψ and φ are analytic in the variables a_m , and since the $a_m(z)$ have first bounded derivatives, we get

$$I(x) \leq B_1 |x - z|^{m-n} (|z| |x - z|^{-\epsilon} + |z|) \leq B_2 |x - z|^{m-n-\epsilon} |z|.$$

Taking $x = 0$, the proof is completed.

Lemma 2. *If $|x - x'| \leq |z - x|^{1+\delta}$, $0 < \delta < 1$, and if $|z - x| \leq \rho < 1$ then, for any $\epsilon > 0$,*

$$(11) \quad |D_x^i K_z(x) - D_x^i K_z(x')| \leq B_3 |x - x'|^\gamma |z - x|^{m-n-i-\epsilon} \quad (0 \leq i \leq m),$$

where $\gamma = \delta/(1 + \delta)$.

Proof. It will be enough to give the proof for the case $i = m$ and $\psi \equiv 0$. Then

$$(12) \quad \begin{aligned} I &\equiv |D_x^m K_z(x) - D_x^m K_z(x')| \\ &= ||x - z|^{-n} \varphi_1(\zeta, a_m(z)) - |x' - z|^{-n} \varphi_1(\zeta', a_m(z))|, \end{aligned}$$

where $\zeta' = (x' - z)/|x' - z|$, and $\varphi_1(\zeta, a_m)$ involves derivatives of $\varphi(\zeta, a_m)$ with respect to ζ . Since the ζ -derivatives of $\varphi(\zeta, a_m(z))$ are uniformly bounded with respect to z , we have

$$(13) \quad |\varphi_1(\zeta, a_m(z)) - \varphi_1(\zeta', a_m(z))| \leq B_4 |\zeta' - \zeta|.$$

Noting that

$$||x' - z|^{-n} - |x - z|^{-n}| \leq B_5 |x - z|^{-n} \frac{|x' - x|}{|x - z|} \leq B_5 |x - z|^{-n} |x' - x|^\gamma$$

and, similarly, that

$$|\zeta' - \zeta| \leq B_6 |x' - x|^\gamma,$$

we conclude from (13) and (12) that (11) holds for $i = m$.

Note that

$$(14) \quad \alpha(x, z) = (L - L_z)K_z(x) = O(|x - z|^{1-n}).$$

With the aid of Lemma 2 we shall now prove an appropriate Hölder condition for $\alpha(x, z)$.

Lemma 3. *If $|x' - x| \leq |x - z|^{1+\delta}$, $0 < \delta < 1$, and if $|x - z| \leq \rho < 1$, then, for any $\epsilon > 0$,*

$$(15) \quad |\alpha(x, z) - \alpha(x', z)| \leq B_7 |x' - x|^\gamma |x - z|^{1-n-\epsilon} \quad (\gamma = \delta/(1 + \delta)).$$

Proof. We can write

$$\begin{aligned} I &= \alpha(x, z) - \alpha(x', z) \\ &= \sum_{i \leq m} [a_i(x) - a_i(x')] D_x^i K_z(x) + \sum_{i \leq m-1} a_i(x') [D_x^i K_z(x) - D_x^i K_z(x')] \\ &\quad + \sum_m [a_m(x') - a_m(z)] [D_x^m K_z(x) - D_x^m K_z(x')] = \sum_1 + \sum_2 + \sum_3, \end{aligned}$$

where $\sum_1$ and $\sum_2$ denote the first and second sums respectively. Clearly,

$$|\sum_1| \leq B_8 |x' - x| |x - z|^{-n} \leq B_8 |x' - x|^\gamma |x - z|^{1-n}.$$

Using Lemma 2, we obtain

$$\begin{aligned} |\sum_2| &\leq B_9 |x' - x|^\gamma |x - z|^{1-n-\epsilon}, \\ |\sum_3| &\leq B_{10} |x' - z| |x' - x|^\gamma |x - z|^{-n-\epsilon} \leq B_{11} |x' - x|^\gamma |x - z|^{1-n-\epsilon}. \end{aligned}$$

In the last inequality we used the fact that $|x' - z| \leq B_{12} |x - z|$. Combining the estimates for the $\sum_i$ ($i = 1, 2, 3$), (15) follows.

Since $\alpha(x, z)$, as a function of x , is integrable (see (14)), one can transform (8) into an integral equation with a continuous kernel (the n -times iterated kernel of $\alpha(x, z)$) and thus prove that Fredholm's theory applies to (8). If the diameter of B is sufficiently small, there would be no nontrivial solutions of the homogeneous equation; moreover, the solution $u(x, z)$ of (8) can be developed into a Neumann's series operating on $\alpha(x, z)$. Using this series representation of $u(x, z)$ and using Lemma 3, we shall prove

Lemma 4. *If $|x' - x| \leq |x - z|^{1+\delta}$, $0 < \delta < 1$, and $|x - z| \leq \rho < 1$ then, for any $\epsilon > 0$,*

$$(16) \quad |u(x, z) - u(x', z)| \leq B_{13} |x' - x|^\gamma |x - z|^{1-n-\epsilon} \quad (\gamma = \delta/(1 + \delta)).$$

Proof. In what follows we shall need the inequalities

$$(17) \quad \int_B \frac{dy}{|x - y|^\alpha |y - z|^\beta} \leq \frac{B_{14}}{|x - z|^{\alpha+\beta-n}} \quad (0 < \alpha < n, 0 < \beta < n, \alpha + \beta > n).$$

The proof is elementary and will be omitted (see also [1]). Let

$$(18) \quad u(x, z) = \alpha(x, z) + \sum_{k=1}^{\infty} \int_B \alpha_k(x, y) \alpha(y, z) dy.$$

Consider the first function $\alpha_2(x, z) = \int_B \alpha_1(x, y) \alpha(y, z) dy$ ($\alpha_1 = \alpha$).

$$\begin{aligned} |\alpha_2(x, z) - \alpha_2(x', z)| &\leq \int_{B'} |\alpha(x, y) - \alpha(x', y)| |\alpha(y, z)| dy \\ &\quad + \int_{B''} |\alpha(x, y) - \alpha(x', y)| |\alpha(y, z)| dy = \int_1 + \int_2, \end{aligned}$$

where $B' = \{y; |y - x|^{1+\delta} \geq |x' - x|\}$ and $B'' = B - B'$. Using Lemma 3, (14) and (17), we get

$$\int_1 \leq B_{15} |x' - x|^\gamma \int_{B'} \frac{dy}{|x - y|^{n-1+\epsilon} |y - z|^{n-1}} \leq B_{14} B_{15} |x' - x|^\gamma |x - z|^{2-n-\epsilon}.$$

Using (14) and (17), we get

$$\begin{aligned} \int_2 &\leq B_{16} \int_{B''} \left(\frac{1}{|x-y|^{n-1}} + \frac{1}{|x'-y|^{n-1}} \right) \frac{dy}{|y-z|^{n-1}} \\ &\leq 2B_{16} \int_{B''} \frac{|x-y|^\delta dy}{|x-y|^{n-1+\delta} |y-z|^{n-1}} \leq 2B_{16} |x'-x|^\gamma B_{14} |x-z|^{2-n-\delta}. \end{aligned}$$

We have thus proved that

$$|\alpha_2(x, z) - \alpha_2(x', z)| \leq B_{17} |x' - x|^\gamma |x - z|^{1-n}.$$

The above considerations apply also to the functions $\int \alpha_k(x, y) \alpha(y, z) dy$, $2 \leq k \leq n$. If $k > n$, we may assume that $|D_x \alpha_k(x, z)| \leq \eta^k$, $0 < \eta < 1$. Then

$$\int_B |\alpha_k(x, y) - \alpha_k(x', y)| |\alpha(y, z)| dy \leq B_{18} |x' - x| \eta^k,$$

and, using (18), the proof of Lemma 4 is easily completed.

In a similar way, and on using (14), we can prove

Lemma 5. $u(x, z) = O(|x - z|^{1-n})$.

With the aid of Lemmas 4 and 5 we shall prove

Lemma 6. For any $\epsilon > 0$,

$$D_x^i K(x, z) - D_x^i K_z(x)|_{x=0} = O(|z|^{m-n-i+1-\epsilon}) \quad (0 \leq i \leq m).$$

Proof. We first remark that the construction of $K(x, z)$ by (7) depends on the domain B , which can be chosen arbitrarily as long as its diameter is sufficiently small. Since z is fixed in (7), we can take $B = \{x; |x| \leq 2|z|\}$. This does not affect the preceding lemmas provided $|z| \leq \epsilon_0$ and ϵ_0 is sufficiently small. If $i < m$ then, on using (7), Lemma 5 and (17), we get

$$\begin{aligned} |I(x)| &\equiv |D_x^i K(x, z) - D_x^i K_z(x)| \leq B_{19} \int_B \frac{dy}{|x-y|^{n-m+i+\epsilon} |y-z|^{n-1}} \\ &\leq B_{20} |x-z|^{m-n-i+1-\epsilon}, \end{aligned}$$

provided $i \geq m - n$. Substituting $x = 0$, the proof of Lemma 6 is completed for such i .

If $i < m - n$ (this implies $m > n$), then

$$|I(0)| \leq B_{21} \int_B \frac{|y|^{m-n-i-\epsilon}}{|y-z|^{n-1}} dy \leq B_{22} |z|^{m-n-i+1-\epsilon}.$$

Here we used our special choice of the domain B .

It remains to consider the case $i = m$. Write

$$\begin{aligned} (19) \quad I(x) &= D_x^m \int_B K_y(x) u(y, z) dy = D_x^m \int_B K_y(x) u(y, z) dy \\ &\quad + D_x^m \int_{B''} K_y(x) u(y, z) dy = I_1(x) + I_2(x), \end{aligned}$$

where $B' = \{y; |y - x^0| \leq \frac{1}{4} |x^0 - z|^{1+\delta}\}$ and $B'' = B - B'$. In the first integral u satisfies an Hölder condition, namely,

$$|u(y', z) - u(y'', z)| \leq B_{23} |x^0 - z|^{1-n} |y' - y''|^\gamma;$$

this follows from

$$|y'' - y'| \leq \frac{1}{2} |x^0 - z|^{1+\delta} \leq |y' - z|^{1+\delta}$$

(if $|z|$ and $|x|$ are small enough) and from Lemma 4. Therefore, by a well known potential-theoretical lemma (see, for instance, [1; 482]),

$$\begin{aligned} I_1(x^0) = u(x^0, z) & \left[D_x^m \int_{B'} K_\nu(x) dy \right]_{x=x^0} \\ & + \left[\int_{B'} D_x^m K_\nu(x) [u(y, z) - u(x, z)] dy \right]_{x=x^0}. \end{aligned}$$

The function in the first square brackets is bounded. The second integral is bounded by

$$B_{24} \int_{B'} \frac{|y - x^0|^\gamma dy}{|y - x^0|^n |z - y|^{n-1+\epsilon}} \leq B_{25} |x^0 - z|^{-n+1-\epsilon+\gamma}.$$

We have thus proved that

$$(20) \quad I_1(x^0) \leq B_{26} |x^0 - z|^{1-n-\epsilon+\gamma}.$$

It remains to estimate $I_2(x^0)$. Using (17) we obtain

$$\begin{aligned} I_2(x^0) & \leq B_{27} \int_{B''} \frac{dy}{|y - x^0|^n |y - z|^{n-1}} \leq \frac{4B_{27}}{|x^0 - z|^{1+\delta}} \int_{B''} \frac{dy}{|y - x^0|^{n-1} |y - z|^{n-1}} \\ & \leq B_{28} |x^0 - z|^{-n+1-\delta}. \end{aligned}$$

Combining the last inequalities with (20), using (19) and taking $x = x^0 = 0$, we find that if we take $\delta \leq \epsilon$ at the beginning, then Lemma 6 holds also for $i = m$.

4. In this section we shall complete the proof of Theorem 1. For simplicity we shall consider only the case $N = 1$; the proof of the general case is analogous. Since $K(x, z)$, for each z , is a solution of (4), we have

$$\sum_{i \leq m} \alpha_i D_x^i K(x, z)|_{x=0} = 0.$$

Using Lemma 6 we get

$$\sum \alpha_m D_x^m K_z(x)|_{x=0} = O(|z|^{1-n-\epsilon}).$$

Using Lemma 1 with $i = m$, we get

$$(21) \quad \sum \alpha_m D_z^m K_0(z) = O(|z|^{1-n-\epsilon}).$$

On the other hand, it follows easily from (10) that $\sum \alpha_m D_x^m K_0(z)$ has the form $\varphi_0(\zeta) |z|^{-n}$, where $\zeta = z/|z|$. Comparing this with (21) we conclude that $\varphi_0(\zeta) \equiv 0$. Hence

$$(22) \quad \sum \alpha_m D_x^m K_0(x - z) = 0,$$

for all x and z in a neighborhood of the origin ($x \neq z$). Differentiating (22) i times with respect to z and substituting $x = 0$, we conclude that

$$(23) \quad \sum \alpha_m D_x^m D_z^i K_0(x - z)|_{x=0} = 0.$$

Every solution of

$$(24) \quad L_0 u \equiv \sum a_m(0) D_x^m u(x) = 0$$

can be represented in a sufficiently small domain B in the form

$$(25) \quad u(x) = \sum_{i=1}^n \int_S M_i[K_0(x - z), u(z)] \cos(z_i, \nu) dS_z$$

where ν is the outwardly directed normal to the surface S of B , and the $M_i[v, w]$ are bilinear functions in $D^i v$ and $D^k w$ ($j + k = m - 1$) and satisfy Green's identity

$$v L_0 w - w L_0 v = \sum_{i=1}^N \frac{\partial}{\partial x_i} M_i[v, w].$$

Applying the operator $\sum \alpha_m D_x^m$ to both sides of (25) and using (23), we conclude that

$$(26) \quad \sum \alpha_m D_x^m u(x)|_{x=0} = 0.$$

We have thus proved that all the solutions of (24) satisfy (26). From Section 2 it follows that the α_m are proportional to the $a_m(0)$, namely, for some c , $\alpha_m = c a_m(0)$.

Multiplying (4) by c and subtracting from (5), we find that all the solutions of (4) satisfy the condition

$$(27) \quad \sum_{i \leq m-1} \alpha_i^* D_x^i u(0) = 0,$$

where $\alpha_i^* = c a_i(0) - \alpha_i$. We can now repeat the arguments described above (in this section), making use of Lemma 6, Lemma 1 (with $i = m - 1$) and Section 2. We then conclude that $\alpha_{m-1}^* = 0$. Consequently, all the solutions of (4) will satisfy (27) with $\alpha_{m-1}^* = 0$. Continuing in this way, we can prove that $\alpha_{m-2}^* = \dots = \alpha_0^* = 0$, and the proof of Theorem 1 is completed.

5. Let $\{M_a\}$ be a sequence of nonnegative numbers, and denote by $C\{M_{a-a}; D\}$ ($a \geq 0$) the class of all indefinitely differentiable functions $f(x)$ in a domain D with the following property: To each closed subset D_0 of D there correspond constants H_0 and H such that for all $x \in D_0$

$$\left| \frac{\partial^q f(x)}{\partial x_1^{q_1} \dots \partial x_n^{q_n}} \right| < H_0 H^q M_{a-a} \quad (q = a, a + 1, a + 2, \dots).$$

For $M_q = q!$ we get the class of all functions analytic in D . The classes $C\{M_q; D\}$ were first introduced in the study of quasi-analytic functions [11].

In [3] we have essentially proved that *if all the coefficients of*

$$(28) \quad \sum_{\mu=1}^N \sum_{i \leq m} a_i^{\lambda\mu}(x) D_x^i u_\mu(x) = f^\lambda(x) \quad (\lambda = 1, \dots, N)$$

belong to $C\{M_{q-2}; D\}$, then all the solutions of (28) belong to $C\{M_{q-m-2}; D\}$, provided, for some $A > 0$, the M_q satisfy the monotonicity conditions

$$(29) \quad \left(\frac{q}{i}\right) M_i M_{q-i} \leq A M_q \quad (i = 0, 1, \dots, q; q = 1, 2, \dots).$$

Actually, the theorem proved in [3] has a slightly different form; the result stated here is taken from [5], in which the more general case of nonlinear systems was treated.

We shall now prove the inverse of the above theorem.

Theorem 2. *Assume that (28) is elliptic and that the $a_i^{\lambda\mu}(x)$ belong to $C^i(D)$. For any $x^0 \in D$, if every solution u_μ of (28) which exists in a neighborhood D_u of x^0 (depending on u_μ) belongs to $C\{M_{q-m}; D_u\}$, then there exists a neighborhood $D(x^0)$ of x^0 and a regular matrix $(e_{\lambda\tau}(x))$ ($\lambda, \tau = 1, \dots, N$) defined in $D(x^0)$ such that the functions*

$$(30) \quad \tilde{a}_i^{\tau\mu}(x) = \sum_{\lambda=1}^N e_{\lambda\tau}(x) a_i^{\lambda\mu}(x), \quad \tilde{f}^\tau(x) = \sum_{\lambda=1}^N e_{\lambda\tau}(x) f^\lambda(x)$$

belong to $C\{M_q; D(x^0)\}$.

Proof. We consider first the homogeneous case, where $f^\lambda(x) \equiv 0$. Using the notation of Section 1, there exist, in some neighborhood of x^0 , $b - N$ linearly independent vectors $v^i = (D_x^i u_\mu^i)$ (where $u^i = (u_\mu^i)$ is a solution of (1)) orthogonal at the point $x = x^0$ to the N vectors $a_\lambda(x^0) = (a_i^{\lambda\mu}(x^0))$ since in the contrary case we can find an additional vector $\alpha = (\alpha_i^*)$, linearly independent of the $a^\lambda(x^0)$ ($1 \leq \lambda \leq N$), such that, for all the solutions $u = (u_\mu)$ of (1), $\alpha \cdot v(x^0) = 0$; this is, however, in contradiction with Theorem 1.

The $b - N$ vectors $v^i(x)$, which are linearly independent at $x = x^0$, must also be linearly independent in some neighborhood $D(x^0)$ of x^0 . We have

$$(31) \quad \sum_{\lambda=1}^N \sum_{i \leq m} a_i^{\lambda\mu}(x) D_x^i u_\mu^j(x) = 0 \quad (\lambda = 1, \dots, N; j = 1, \dots, b - N).$$

This is a system of $N(b - N)$ equations for the bN functions $a_i^{\lambda\mu}(x)$. Since the equations (1) are linearly independent (that is, the vectors $a^\lambda(x)$ are linearly independent), we can find N columns in the matrix $(a^\lambda(x))$ which form a non-singular matrix $A(x)$. Denote by $1', 2', \dots, N'$ the ordinal numbers of these columns and denote by $1'', 2'', \dots, (b - N)''$ the ordinal numbers of the complementary set. We claim that, for every x , the matrix $v''(x)$ composed of the $b - N$ columns $1'', 2'', \dots, (b - N)''$ of the matrix $(v^i(x)) = (D^i u_\mu^j(x))$

is a regular matrix. Indeed, if the contrary is true, then we can make a linear transformation of the $v^i(x)$ into another set of $b - N$ independent vectors $w^i(x)$ such that the components $1'', 2'', \dots, (b - N)''$ of $w^1(x)$ vanish. Denote by $c(x) = (c_1(x), \dots, c_N(x))$ the remaining components of $w^1(x)$, then $c(x) \neq 0$ since the $w^i(x)$ are linearly independent. On the other hand, $A(x)c(x) = 0$, which is a contradiction since $A(x)$ is a regular matrix.

Since $A(x)$ is a regular matrix for every x we can find a regular linear transformation

$$(32) \quad \tilde{a}_i^{\tau\mu}(x) = \sum_{\lambda=1}^N e_{\lambda\tau}(x) a_i^{\lambda\mu}(x) \quad (\tau, \mu = 1, \dots, N)$$

which transforms the k' th column ($k = 1, \dots, N$) of $(a_\lambda(x))$ into the vector $\delta_k(x) = (\Delta(x)\delta_{\mu k})$, where $\Delta(x)$ is the determinant of $v''(x)$. Multiplying (31) by $e_{\lambda\tau}(x)$ and summing up, we can write the resulting system in the form

$$(33) \quad \sum_{\mu=1}^N \sum'' \tilde{a}_i^{\tau\mu}(x) D_x^i u_\mu^j(x) = g^\tau(x) \quad (\tau = 1, \dots, N; j = 1, \dots, b - N)$$

where

$$g^\tau(x) = - \sum_{\mu=1}^N \sum' \tilde{a}_i^{\tau\mu}(x) D_x^i u_\mu^i(x) = h^\tau(x) \Delta(x).$$

Here $\sum''$ and $\sum'$ denote summations taken over the indices k'' and k' respectively. Since $D_x^i u_\mu^j(x)$ belongs to $C\{M_a; D(x^0)\}$, $h^\tau(x)$ also belongs to $C\{M_a; D(x^0)\}$. From the remarks above it follows that, for each τ , (33) is a linear system of order $b - N$ in the $\tilde{a}_i^{\tau\mu}(x)$ with a regular matrix of coefficients ($\Delta(x) = \det v''(x) \neq 0$). Noting that the product of two functions of the class $C\{M_a; D(x^0)\}$ (with the M_a satisfying (29)) belongs to the same class, we conclude that all the $\tilde{a}_i^{\tau\mu}(x)$ which appear in $\sum''$ belong to $C\{M_a; D(x^0)\}$. Since the remaining $\tilde{a}_i^{\tau\mu}(x)$ are of the form $\Delta(x)\delta_{\mu k}$ and, consequently, belong to $C\{M_a; D(x^0)\}$, Theorem 2 is completely proved for homogeneous systems.

To prove Theorem 2 in the general nonhomogeneous case, let $u^0(x) = (u_\lambda^0(x))$ be a particular solution of (28). Then $u(x)$ is a solution of (28) if and only if $u(x) - u^0(x)$ is a solution of the corresponding homogeneous system. Applying Theorem 2 for homogeneous systems (proved above), the assertion of Theorem 2 concerning the $a_i^{\lambda\mu}(x)$ follows. Writing

$$\tilde{f}^\tau(x) = \sum_{\mu=1}^N \sum_{i \leq m} \tilde{a}_i^{\tau\mu}(x) D_x^i u_\mu^0(x),$$

the proof is easily completed.

6. The proof of Theorem 2 can be used to prove more general results. Denote by $B(D)$ a set of functions defined in D , closed under addition and multiplication in the usual sense. We then have

Theorem 2'. Assume that (28) is elliptic and that the $a_i^{\lambda\mu}(x)$ belong to $C^i(D)$. For

any $x^0 \in D$, if every solution u_μ of (28) which exists in a neighborhood D_u of x^0 together with its partial derivatives of orders $\leq m$ belongs to $B(D_u)$, then there exists a neighborhood $D(x^0)$ of x^0 and a regular matrix $(e_{\lambda\tau}(x))$ defined in $D(x^0)$ such that the functions $\tilde{a}_i^{\tau\mu}(x)$ and $\tilde{f}^\tau(x)$ defined in (30) belong to $B(D(x^0))$.

As an example, we can take $B(D)$ to be a class of functions analytic or $C\{M_a\}$ with respect to one or several independent variables x_i . One cannot take for $B(D)$ a class of entire functions, since it can be easily shown that it cannot happen that all the solutions will be, locally, entire functions.

7. Consider the following interpolation problem: Given k differential relations

$$(34) \quad \sum_{\mu=1}^N \sum_{i \leq m} \alpha_i^{\sigma\mu} D_x^i u_\mu(x^0) = 0 \quad (\sigma = 1, \dots, k),$$

to determine the solutions of (1) which satisfy (34).

Theorem 3. Assume that (1) is elliptic and that the $a_i^{\lambda\mu}(x)$ belong to $C^i(D)$. Then, in a sufficiently small neighborhood $D(x^0)$ of x^0 , there exist at least $b - N - k$ solutions $\tilde{u}^j(x)$ ($j = 1, \dots, b - N - k$) of (1) which satisfy (34) and such that the vectors $\tilde{v}^j(x) = (D_x^i \tilde{u}_\mu^j(x))$ are linearly independent.

Proof. For the general solution $u(x)$ of (1) there exist functions $\lambda_i(x)$ such that

$$D_x^i u_\mu(x) = \sum_{j=1}^{b-N} \lambda_j(x) D_x^i u_\mu^j(x),$$

where the $b - N$ vectors $v^j(x) = (D_x^i u_\mu^j(x))$ are linearly independent and the $u^j(x)$ are solutions of (1). Taking, in particular, solutions with $\lambda_j(x) \equiv \text{const.} = \lambda_j$ and substituting in (34), we obtain a linear algebraic system of equations for the λ_j , from which the assertion of Theorem 3 follows.

Remark. Theorems 2 and 3 can very easily be generalized to nonhomogeneous systems, that is, systems in which (1) is replaced by (28), and the right sides of (2) and (34) are replaced by arbitrary numbers c and c^σ , respectively.

PART II

8. We shall make use of the notation introduced in Part I. Consider the system

$$(1) \quad L_\lambda u(x, t) \equiv \sum_{\mu=1}^N \sum_{i \leq m} a_i^{\lambda\mu}(x, t) D_x^i u_\mu(x, t) - D_t u_\lambda(x, t) = 0 \quad (\lambda = 1, \dots, N)$$

in an $(n + 1)$ -dimensional domain D . The system (1) is called *parabolic* if, for any real ξ , the real parts of the characteristic roots of the matrix $((-1)^{i^m} \sum_m a_m^{\lambda\mu}(x, t) \xi^m)$ are negative. For parabolic systems, m is necessarily an even number.

Together with the system (1) we consider a differential relation

$$(2) \quad \sum_{\mu=1}^N \sum_{i \leq m} \alpha_i^\mu D_x^i u_\mu(x^0, t^0) - \sum_{\mu=1}^N \beta^\mu D_t u_\mu(x^0, t^0) = 0,$$

where (x^0, t^0) belongs to D and α_i^μ, β^μ are constants. Trivially, (2) is satisfied by all of the solutions of (1) if

$$\alpha_i^\mu = \sum_{\lambda=1}^N \beta^\lambda a_i^{\lambda\mu}(x^0, t^0).$$

Theorem 4. Assume that (1) is parabolic in D , that the $a_m^{\lambda\mu}(x, t)$ belong to $C^{m+1}(D)$ and that the $a_i^{\lambda\mu}(x, t)$ ($0 \leq i \leq m-1$) belong to $C^1(D)$. If, for a given point (x^0, t^0) of D , all the solutions $u(x, t)$ of (1) which exist in some neighborhood of (x^0, t^0) (depending on u) satisfy (2), then

$$\alpha_i^\mu = \sum_{\lambda=1}^N \beta^\lambda a_i^{\lambda\mu}(x^0, t^0).$$

The proof of Theorem 4 is given in Sections 9, 10, 11 and is patterned on the proof of Theorem 1. Obviously we may assume in the forthcoming proof that $x^0 = 0, t^0 = 0$.

9. In this section we prove Theorem 4 in the special case where $a_m^{\lambda\mu}(x, t) \equiv \text{const} = a_m^{\lambda\mu}$ and $a_i^{\lambda\mu}(x, t) \equiv 0$ for $i < m$. Taking solutions of the form $u_\lambda(x, t) = \sum \gamma_m^\lambda x^m/m!$ we find, as in Part I, that

$$(3) \quad \alpha_i^\mu = \sum_{\lambda=1}^N c_\lambda a_i^{\lambda\mu}.$$

Comparing (1) with (2) and using (3) we conclude that all the solutions of (1) must satisfy the relation

$$(4) \quad \sum_{\lambda=1}^N \beta^\lambda D_t u_\lambda(0, 0) = \sum_{\lambda=1}^N c_\lambda D_t u_\lambda(0, 0).$$

For every $\gamma^1, \dots, \gamma^N$ we can find solutions of the form $u_\lambda(x, t) = \sum \gamma_m^\lambda x^m/m! + \gamma^\lambda t$. Substituting in (4) we get $\sum \beta^\lambda \gamma^\lambda = \sum c_\lambda \gamma^\lambda$. Hence $c_\lambda = \beta^\lambda$ and the proof is completed.

10. In this section we shall prove some properties concerning the fundamental solutions of (1) in an $(n+1)$ -dimensional cylinder D_0 ($(0, 0) \in D_0 \subset \overline{D_0} \subset D$) which is the product of a domain D' with $\{t_1 < t < t_2\}$. We shall make extensive use of some results of EIDELMAN [2]. Denote by $P_0(x, t, (1/2\pi i)\partial/\partial x)$ the matrix $(\sum_m a_m^{\lambda\mu}(x, t) D_x^m)$ and by $P_1(x, t, (1/2\pi i)\partial/\partial x)$ the matrix $(\sum_{i \leq m-1} a_i^{\lambda\mu}(x, t) D_x^i)$. Let $V(t, \tau, y, \beta)$ be the solution of the ordinary differential system

$$(5) \quad \frac{dV}{dt} = P_0(y, \tau, \beta) V, \quad V(\tau, \tau, y, \beta) = E,$$

where E is the unit matrix. (5) is a system with constant coefficients, and the solution V is continuous in the parameters τ , y , β . A fundamental matrix $G(t, \tau, x - \xi, y)$ for the system

$$(6) \quad \frac{\partial u}{\partial t} = P_0\left(y, \tau, \frac{1}{2\pi i} \frac{\partial}{\partial x}\right)u,$$

with a singularity at $x = \xi$, $t = \tau$, is given by [2; p. 60]

$$(7) \quad G(t, \tau, x - \xi, y) = \int_{-\infty}^{\infty} e^{2\pi i(\sigma, x - \xi)} V(t, \tau, y, \sigma) d\sigma.$$

Substituting $\sigma = \beta/(t - \tau)^{1/m}$ and differentiating with respect to x , we get

$$(8) \quad \begin{aligned} & D_x^k G(t, \tau, x - \xi, y) \\ &= \frac{1}{(t - \tau)^{(n+k)/m}} \int_{-\infty}^{\infty} e^{2\pi i(\beta, \zeta)} (2\pi i)^k \beta_1^{k_1} \cdots \beta_n^{k_n} V\left(t, \tau, y, \frac{\beta}{(t - \tau)^{1/m}}\right) d\beta, \end{aligned}$$

where $\zeta = (x - \xi)/(t - \tau)^{1/m}$.

Lemma 7.

$$(a) \quad D_x^k G(t - \tau, 0, x - \xi, 0)|_{x=0, t=0} = \varphi_k(\zeta)/(-\tau)^{(n+k)/m} \quad (0 \leq k \leq m)$$

$$(b) \quad D_t G(t - \tau, 0, x - \xi, 0)|_{x=0, t=0} = \varphi(\zeta)/(-\tau)^{(n+m)/m},$$

where $\zeta = -\xi/(-\tau)^{1/m}$ ($\tau < 0$). The functions $\varphi_k(\zeta)$, $\varphi(\zeta)$ are bounded if ζ is bounded.

Proof. From (8) we get

$$(9) \quad \begin{aligned} & D_x^k G(t - \tau, 0, x - \xi, y)|_{x=0, t=0} \\ &= \frac{1}{(-\tau)^{(n+k)/m}} \int_{-\infty}^{\infty} e^{2\pi i(\beta, \zeta)} (2\pi i)^k \beta_1^{k_1} \cdots \beta_n^{k_n} V\left(-\tau, 0, 0, \frac{\beta}{(-\tau)^{1/m}}\right) d\beta. \end{aligned}$$

Since the solutions of (5) are uniquely determined, $V(\alpha t, 0, 0, \beta/\alpha^{1/m}) = V(t, 0, 0, \beta)$ for every fixed α . Taking $t = 1$, $\alpha = -\tau$ we conclude that the right side of (9) is independent of τ ; consequently it is a function, say $\varphi_k(\zeta)$, of ζ alone. The boundedness of $\varphi_k(\zeta)$, for bounded ζ , is a consequence of the estimates [2; p. 60] for V .

The proof of (b) follows in the same way, if we differentiate (7) with respect to t .

Lemma 8. If $k = 0, 1, \dots, m$, then

$$(a) \quad \begin{aligned} D_x^k G(t, \tau, x - \xi, y) - D_x^k G(t - \tau, 0, x - \xi, 0)|_{x=0, t=0} \\ = \psi_k(\zeta, \tau, y)/(-\tau)^{(n+k)/m} \end{aligned}$$

$$(b) \quad \begin{aligned} D_t G(t, \tau, x - \xi, y) - D_t G(t - \tau, 0, x - \xi, 0)|_{x=0, t=0} \\ = \psi(\zeta, \tau, y)/(-\tau)^{(n+m)/m}. \end{aligned}$$

ψ_k and ψ tend to zero if ζ is bounded and if y and τ tend to zero.

Proof. From (8) and from the estimate for V it follows that in order to prove (a) it is sufficient to prove that the function

$$f(\tau, y, \beta) = V(0, \tau, y, \beta/(-\tau)^{1/m}) - V(-\tau, 0, 0, \beta/(-\tau)^{1/m})$$

becomes arbitrarily small if $(\tau, y) \rightarrow (0, 0)$ and if β remains bounded. To prove it, let

$$W(t) = V(-\tau t, \tau, y, \beta/(-\tau)^{1/m}), \quad U(t) = V(-\tau(t+1), 0, 0, \beta/(-\tau)^{1/m}).$$

Then

$$\frac{dW}{dt} = P_0(y, \tau, \beta)W, \quad W(-1) = E,$$

$$\frac{dU}{dt} = P_0(0, 0, \beta)U, \quad U(-1) = E.$$

Noting that $f(\tau, y, \beta) = W(0) - U(0)$, the proof of (a) is easily completed. The proof of (b) is similar and will be omitted.

It is important to note [2; p. 61] that

$$(10) \quad |D_y^{k_1} D_x^{k_2} G(t, \tau, x - \xi, y)| \leq \frac{C_{k_1 k_2}}{(t - \tau)^{(n+k_2)/m}} \exp \left\{ -H_{k_1 k_2} \frac{|x - \xi|^{m/(m-1)}}{(t - \tau)^{1/(m-1)}} \right\}$$

where $C_{k_1 k_2}, H_{k_1 k_2}$ are constants.

Using this fact and Lemma 8, we conclude

Lemma 8'. *If ξ is bounded ($\xi = -\xi/(-\tau)^{1/m}$) then, as $\tau \rightarrow 0$,*

$$|D_x^k G(t, \tau, x - \xi, x) - D_x^k G(t - \tau, 0, x - \xi, 0)|_{x=0, t=0} = o((-\tau)^{(n+k)/m})$$

$$(0 \leq k \leq m),$$

$$|D_t G(t, \tau, x - \xi, x) - D_t G(t - \tau, 0, x - \xi, 0)|_{x=0, t=0} = o((-\tau)^{(n+m)/m}).$$

The fundamental solution Z of (1) is given in the form

$$(11) \quad Z(t, \tau, x, \xi) = G(t, \tau, x - \xi, x) + W(t, \tau, x, \xi),$$

where

$$W(t, \tau, x, \xi) = \int_{\tau}^t d\beta \int_{D'} G(t, \beta, x - y, x) \varphi(\beta, \tau, y, \xi) dy.$$

φ satisfies the integral equation

$$K(t, \tau, x - \xi, x) + \varphi(t, \tau, x, \xi) + \int_{\tau}^t d\beta \int_{D'} K(t, \beta, x - y, x) \varphi(\beta, \tau, y, \xi) dy,$$

where

$$K(t, \tau, x - \xi, x) = \left[\frac{\partial}{\partial t} - P \left(x, t, \frac{1}{2\pi i} \frac{\partial}{\partial x} \right) \right] G(t, \tau, x - \xi, x) \quad (P = P_0 + P_1).$$

Lemma 9. *If $k = 0, 1, \dots, m$, then*

$$(12) \quad |D_x^k W(t, \tau, x, \xi)| \leq \frac{B_k}{(t - \tau)^{(n+k-1)/m}} \exp \left\{ -\frac{\bar{B}_k |x - \xi|^{m/(m-1)}}{(t - \tau)^{1/(m-1)}} \right\},$$

$$(12') \quad |D_t W(t, \tau, x, \xi)| \leq \frac{B}{(t - \tau)^{(n+m-1)/m}} \exp \left\{ -\frac{\bar{B} |x - \xi|^{m/(m-1)}}{(t - \tau)^{1/(m-1)}} \right\},$$

where $B_k, \bar{B}_k, B, \bar{B}$ are constants independent of (ξ, τ) and (x, t) , provided (ξ, τ) and (x, t) vary in a closed subset of D .

Proof. For $k < m$ (12) was proved in [2; p. 69]. To consider the case $k = m$, we use the following formula [2; p. 70]:

$$\begin{aligned} D_x^m W(t, \tau, x, \xi) &= \int_{\tau}^{\tau + \frac{1}{2}(t - \tau)} d\beta \int_{D'} D_x^m G(t, \beta, x - y, x) \varphi(\beta, \tau, y, \xi) dy \\ &+ C_1 \int_{\tau + \frac{1}{2}(t - \tau)}^t d\beta \int_{D'} [D_x^{m-1} G(t, \beta, x - y, x)|_{x=y=\alpha}] D_x \varphi(\beta, \tau, x - \alpha, \xi) d\alpha \\ &+ C_2 \int_{\tau + \frac{1}{2}(t - \tau)}^t d\beta \int_{D'} D_x [D_x^{m-1} G(t, \beta, x - y, x)|_{x=y=\alpha}] \varphi(\beta, \tau, x - \alpha, \xi) d\alpha \\ &= I_1 + C_1 I_2 + C_2 I_3, \end{aligned}$$

where I_k denotes the k^{th} integral on the right side of the last equality. Using the estimates [2; p. 69] for φ and $D_x \varphi$, using (10) in estimating I_3 , and making use of the elementary inequality (see also [2; pp. 63–64])

$$\begin{aligned} (13) \quad &\int_{\tau}^t d\beta \int_{D'} \frac{1}{(t - \beta)^{(n+m-1)/m}} \exp \left\{ -A_1 \frac{|x - y|^{m/(m-1)}}{(t - \beta)^{1/(m-1)}} \right\} \frac{1}{(\beta - \tau)^{(n+m-1)/m}} \\ &\cdot \exp \left\{ -A_2 \frac{|y - \xi|^{m/(m-1)}}{(\beta - \tau)^{1/(m-1)}} \right\} dy \\ &\leq \frac{A_3}{(t - \tau)^{(n+m-2)/m}} \exp \left\{ -A_4 \frac{|x - \xi|^{m/(m-1)}}{(t - \tau)^{(n+m-2)/m}} \right\}, \end{aligned}$$

where A_3, A_4 are constants depending on the constants A_1, A_2 , we finally conclude that (12) holds also for $k = m$. Note that in using the necessary inequalities for φ and $D_x \varphi$, we make use of the assumptions concerning the regularity of the coefficients of (1) (see [2]). The proof of (12') is similar and will be omitted.

11. In this section we shall complete the proof of Theorem 4, by reducing it to the case of constant coefficients which was considered in Section 9. The fundamental solution $Z(t, \tau, x, \xi)$ satisfies

$$\left[\sum_{\mu=1}^N \sum_{i \leq m} \alpha_i^{\mu} D_x^i Z_{\mu\lambda}(t, \tau, x, \xi) - \sum_{\mu=1}^N \beta^{\mu} D_t Z_{\mu\lambda}(t, \tau, x, \xi) \right]_{x=0, t=0} = 0,$$

where Z is the matrix $(Z_{\mu\lambda})$. Using Lemma 9 and (11), we get

$$\left[\sum_{\mu=1}^N \sum_m \alpha_m^\mu D_x^m G_{\mu\lambda}(t, \tau, x - \xi, x) - \sum_{\mu=1}^N \beta^\mu D_t G_{\mu\lambda}(t, \tau, x - \xi, x) \right]_{x=0, t=0} = O((- \tau)^{-(n+m-1)/m}),$$

provided $|\zeta| = |\xi|/|\tau|^{1/m}$ is bounded. Here $G = (G_{\mu\lambda})$.

Using Lemma 8' we conclude that if ζ is bounded, then

$$(14) \quad \sum_{\mu=1}^N \sum_m \alpha_m^\mu D_x^m G_{\mu\lambda}(t - \tau, 0, x - \xi, 0) - \sum_{\mu=1}^N \beta^\mu D_t G_{\mu\lambda}(t - \tau, 0, x - \xi, 0) \Big|_{x=0, t=0} = o((- \tau)^{-(n+m)/m}).$$

On the other hand, from Lemma 7 it follows that the left side of (14) is equal to $\vartheta(\zeta) (- \tau)^{-(n+m)/m}$, where $\vartheta(\zeta)$ is a function of ζ alone. Since for every ζ_0 we can take $\tau \rightarrow 0$ such that $\zeta = \zeta_0$, we conclude that $\vartheta(\zeta) \equiv 0$. Therefore, $G(t - \tau, 0, x - \xi, 0)$ satisfies the differential relation (4) for all ξ and τ sufficiently small, $\tau < 0$. Obviously the derivatives of G also satisfy the same relation.

We now use Green's identity in order to represent the solutions of the equation

$$\frac{\partial u}{\partial t} = P_0 \left(0, 0, \frac{1}{2\pi i} \frac{\partial}{\partial x} \right) u$$

in the form (see, for instance [2; p. 58]):

$$(15) \quad u(x, t) = \int_{V_1} G(t - \tau, 0, x - \xi, 0) u(\xi, \tau) d\xi + \int_\tau^t d\gamma \int_{\Gamma_1} \sum_{k=1}^n B^k[G(t - \gamma, 0, x - \xi, 0), u(\xi, \gamma)] \frac{d\xi_1 \cdots d\xi_n}{d\xi_k},$$

where $B^i[G, u]$ is a bilinear form in $D_x^k G$ and $D_x^i u$ ($k + j = m - 1$). From (15) we conclude that all the solutions of

$$\frac{\partial u}{\partial t} = P_0 \left(0, 0, \frac{1}{2\pi i} \frac{\partial}{\partial x} \right) u$$

also satisfy the differential relation (4). Using the result of Section 9, it follows that $\alpha_m^\mu = \sum_{\lambda=1}^N \beta^\lambda a_m^{\lambda\mu}(0, 0)$. Consequently all of the solutions of (1) satisfy

$$\sum_{\mu=1}^N \sum_{i \leq m-1} \alpha_i^\mu D_x^i u_\mu(0, 0) = 0.$$

We can now proceed as before, using Lemma 7 with $k = m - 1$ and Lemmas 8' and 9 with $k \leq m - 1$. Again the general case is reduced to the case of constant coefficients (with $a_i^{\lambda\mu} \equiv 0$ if $i \leq m - 1$) and, by the method of Section 9,

we can get $\alpha_{m-1}^\mu = 0$. Continuing in this way, we obtain $\alpha_{m-2}^\mu = \dots = \alpha_0^\mu = 0$, and the proof of Theorem 4 is completed.

12. With the aid of Theorem 4 we can now solve the coefficients of (1) in terms of the solutions. The method is similar to that of Part I.

Let $B(D)$ be a set of functions $g(x, t)$ defined in D and closed under addition and multiplication in the usual sense. Denote by (1') the system obtained from (1) on replacing the right side of (1) by $f^\lambda(x, t)$.

Theorem 5. Assume that (1) is parabolic in D , that the $a_i^\lambda(x, t)$ belong to $C^{m+1}(D)$ and that the $a_i^\mu(x, t)$ ($0 \leq i < m$) belong to $C^1(D)$. Given a point $(x^0, t^0) \in D$, assume that all the solutions $u(x, t)$ of (1'), which exist in some neighborhood D_u of (x^0, t^0) (depending on u), together with their derivatives $D_i u(x, t)$, $D_x^k u(x, t)$ ($0 \leq k \leq m$) belong to $B(D_u)$. Then there exists, in a neighborhood D_0 of (x^0, t^0) , a function $\Delta(x, t) \neq 0$, which belongs to $B(D_0)$, such that all the functions

$$\Delta(x, t)a_i^\lambda(x, t), \quad \Delta(x, t)f^\lambda(x, t)$$

belong to $B(D_0)$.

The proof proceeds as in Part I. The only difference is that here we can take the columns $1', 2', \dots, N'$ to be the N columns corresponding to $D_i u_1^i, \dots, D_i u_N^i$. Consequently the matrix $(e_{\lambda r}(x, t))$ will take the form $(\delta_{\lambda r} \Delta(x, t))$, where $\Delta(x, t)$ is the determinant of the matrix $v''(x, t) \equiv (D_x^i u_\mu^j(x, t))$ ($j = 1, \dots, b - N; b - N$ is the length of the vector $(D_x^i u_\mu(x, t))$, $\mu = 1, \dots, N$, $0 \leq i \leq m$).

In particular, we can take $B(D) = C\{M_a; D\}$ or $B(D) = C\{M_a; D; m\}$ (see [4]) and thus obtain converses to the theorems which we proved in [4].

An interpolation problem, analogous to that of Section 7, can be considered also for the present parabolic case.

We finally remark that since an analogue to Theorem 4 holds for linear hyperbolic systems (see the Introduction), also an analogue of Theorem 5 is true for such systems.

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