Sufficient Conditions for the Existence of Periodic Solutions of Systems of Weakly Nonlinear First and Second Order Differential Equations

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1. In previous papers existence theorems have been given for periodic solutions (cycles) of weakly nonlinear autonomous analytic systems [J. K. HALE, 7] and for harmonic and subharmonic solutions of weakly nonlinear periodic analytic systems of ordinary differential equations [R. A. GAMBILL & J. K. HALE, 5; L. CESARI & J. K. HALE, 2]. The method used in all these papers is the one based on a convergent process of successive approximations developed by L. CESARI, J. K. HALE and R. A. GAMBILL. The same method has been used also for the proof of theorems of boundedness of the solutions of linear periodic differential systems [L. CESARI, 1; J. K. HALE, 6, 8; R. A. GAMBILL, 3] and of criteria of unboundedness for the same system [R. A. GAMBILL, 4]. In the present paper we shall use the same method for the proof of new sufficient conditions for the existence of periodic solutions of systems of weakly nonlinear periodic or autonomous differential systems.

Consider the system of ordinary differential equations

$$(1) \quad \begin{aligned} y_i'' + \sigma_i^2 y_i &= \epsilon f_i(y_1, \dots, y_n, y_1', \dots, y_\mu', \epsilon, t), \quad j = 1, \dots, \mu, \\ y_k' + \lambda_{\mu+k} y_k &= \epsilon f_k(y_1, \dots, y_n, y_1', \dots, y_\mu', \epsilon, t), \quad k = \mu + 1, \dots, n, \end{aligned}$$

where ϵ is a small parameter, $\sigma_1, \dots, \sigma_\mu$ are real positive numbers, $\lambda_{2\mu+1}, \dots, \lambda_{n+\mu}$ are real numbers, and each f_i is a real-valued function in the real variable t of period $2\pi/\omega$, $\omega > 0$, and analytic in the remaining arguments. It is not necessary in the following discussion that t appear explicitly in (1). In such a case the system is autonomous and f is periodic of period $T = 2\pi/\omega$ for every ω . System (1) with $n = 2\mu$ has been considered in [2] where sufficient conditions have been given for the existence of periodic solutions. These sufficient con-

ditions may be satisfied even in cases where the Jacobian obtained by applying POINCARÉ's periodicity conditions is identically zero. The aim of the present paper is to prove a similar result for system (1) with $\lambda_k = 0$, $k = 2\mu + 1, \dots, n + \mu$; that is, when some of the characteristic roots of the linear system (1) with $\epsilon = 0$ are zero. This case has not yet been discussed in the previously mentioned papers and is of particular interest in questions of stability.

We remark that the m^{th} order equation

$$w^{(m)} + a_1 w^{(m-1)} + \dots + a_m w = \epsilon f(w^{(m-1)}, \dots, w, \epsilon, t)$$

is a special case of system (1) if the characteristic roots for $\epsilon = 0$ are either purely imaginary or real, and the systems discussed in this paper include the above equation if all the characteristic roots are purely imaginary except one which is zero.

2. Summary of results. A vector function $f(x, t)$, $f = (f_1, \dots, f_n)$, of the real vector $x = (x_1, \dots, x_N)$, of the real variable t will be said to belong to the class $A[\omega]$, $\omega > 0$, if for every t , $-\infty < t < +\infty$, each component f_i of f is analytic in a neighborhood U of $x = (0, \dots, 0)$ independent of t , the power series expansion of f_i in $x_1, \dots, x_N$ is convergent in U , and its coefficients are periodic functions of t of period $T = 2\pi/\omega$.

Consider the system of differential equations in matrix form

$$(2) \quad \begin{cases} y'' + Dy = \epsilon f(y, w, y', \epsilon, t), \\ w' = \epsilon g(y, w, y', \epsilon, t), \end{cases}$$

where (α) $y = (y_1, \dots, y_\mu)$, $w = (y_{\mu+1}, \dots, y_n)$, $f = (f_1, \dots, f_\mu)$, $g = (f_{\mu+1}, \dots, f_n)$, $D = \text{diag}(\sigma_1^2, \dots, \sigma_\mu^2)$, (β) ϵ is real, $\sigma_1, \dots, \sigma_\mu$ are positive numbers, and (γ) $f \in A[\omega]$.

Let $a = (a_1, \dots, a_\mu)$, $b = (b_1, \dots, b_\mu)$, where $a_j, b_j, j = 1, 2, \dots, \mu$, are positive integers, $\sigma = (\sigma_1, \dots, \sigma_\mu)$, $c = (c_1, \dots, c_n)$. Certain real functions $H_j(a, b, c, \sigma, \omega, \epsilon)$, $j = 1, 2, \dots, \mu$, analytic in ϵ , $|\epsilon| < \epsilon_0$, will be determined (§4) for which the following theorem holds.

Theorem 1. *If*

$$(A) \quad f(y, w, -y', \epsilon, -t) = f(y, w, y', \epsilon, t),$$

$$(B) \quad g(y, w, -y', \epsilon, -t) = -g(y, w, y', \epsilon, t),$$

and if, for ϵ sufficiently small, the system of equations

$$(3) \quad \frac{a_j}{b_j} \omega + \epsilon H_j(a, b, c, \sigma, \omega, \epsilon) = \sigma_j, \quad j = 1, 2, \dots, \mu,$$

has a solution for some real non-zero c, σ , and ω , then (2) has a solution $y(\epsilon, t)$, $w(\epsilon, t) \in A[\omega/b_1 \dots b_\mu]$ of the form

$$y_i(0, t) = c_i \cos(a_i \omega t / b_i), \quad y_i(\epsilon, -t) = y_i(\epsilon, t), \quad j = 1, 2, \dots, \mu,$$

$$y_i(0, t) = c_i, \quad y_i(\epsilon, -t) = y_i(\epsilon, t), \quad j = \mu + 1, \dots, n.$$

If $\sigma_j = a_j \omega / b_j$ and ω is fixed, then (2) represents a system of μ equations for the n (n may be greater than μ) unknowns $c_1, \dots, c_n$. If there exist $c_{j_0} = c_{j_0}(c_{\mu+1}, \dots, c_n)$, $j = 1, 2, \dots, \mu$, such that $H_j(a, b, c_0, \sigma, \omega, 0) = 0$, $j = 1, 2, \dots, \mu$, for every $c_{\mu+1}, \dots, c_n$ and the determinant

$$(4) \quad \left| \frac{\partial H_j(a, b, c_0, \sigma, \omega, 0)}{\partial c_i} \right|, j, l = 1, 2, \dots, \mu \neq 0,$$

then, for ϵ sufficiently small, system (3) certainly has a non-zero solution c_i , $j = 1, 2, \dots, n$. Notice that, in such a case, the numbers $c_{\mu+1}, \dots, c_n$ are arbitrary and, therefore, there is an $n - \mu$ parameter family of periodic solutions. It is also interesting to note that condition (4) may be satisfied even in cases where the Jacobian obtained by applying POINCARÉ'S periodicity conditions is zero.

The theorem above holds even when f is independent of t (autonomous case). Then system (3) determines relations between the basic "periods" $2\pi b_j / (a_j \omega)$ of the dominant terms of the solutions and their "amplitudes" c_k .

It may be pointed out that the functions $H_j(a, b, c, \sigma, \omega, 0)$ are given by

$$(5) \quad H_j(a, b, c, \sigma, \omega, 0) = \frac{1}{c_j T} \int_0^T f_j(y_0, w_0, y'_0, 0, t) \cos \frac{a_j \omega t}{b_j} dt,$$

$j = 1, \dots, \mu,$

where $T = 2\pi b_1 \dots b_\mu / \omega$, $y_0 = (c_1 \cos a_1 \omega t / b_1, \dots, c_\mu \cos a_\mu \omega t / b_\mu)$, and $w_0 = (c_{\mu+1}, \dots, c_n)$.

In §5 we state a generalization of Theorem 1 for the existence of periodic solutions which may not be even in t . Applications of these theorems are given for a general third order non-autonomous system in §6 and a general third order autonomous system in §7.

3. Review of the method. We will refer here, as in [5], to a system of first order ordinary equations of the form

$$(6) \quad z' = Az + \epsilon q(z, \epsilon, t)$$

where $A = \text{diag}(\rho_1, \dots, \rho_N)$, $\rho_1, \dots, \rho_N$ complex numbers, $z = (z_1, \dots, z_N)$, $q = (q_1, \dots, q_N)$, and $q \in A[\omega]$. First consider the auxiliary system

$$(7) \quad z' = Bz + \epsilon q(z, \epsilon, t)$$

where $B = \text{diag}(i\tau_1, \dots, i\tau_N)$ and each τ_i is a rational multiple of ω . Let $s_i^{(r)}$ denote the coefficient of ϵ^{r-1} , $r = 1, 2, \dots$, when z in $q_i(z, t)$ is replaced by $z = x^{(0)}(t) + \epsilon x^{(1)}(t) + \dots$, where each $x^{(j)}(t)$ is independent of ϵ . Since only the case in which each $s_i^{(r)}$ is a periodic function of t of some period T will be considered, define

$$(8) \quad \begin{cases} d_j S_j^{(r)} = M[e^{-i\tau_j t} s_j^{(r)}], & j = 1, 2, \dots, N, \\ S^{(r)} = \text{diag}(S_1^{(r)}, \dots, S_N^{(r)}), \end{cases}$$

where $d_1, \dots, d_N$ are non-zero complex constants and $M[\ ]$ denotes the ordinary mean value for periodic functions. The method of successive approximations is defined as follows:

$$(9) \quad \begin{cases} z^{(0)} = x^{(0)} = (d_1 e^{i\tau_1 t}, \dots, d_N e^{i\tau_N t}), \\ z^{(r)} = e^{Bt} \int e^{-Bt} \left[\sum_{k=1}^r \epsilon^k s^{(k)} - \left(\sum_{k=1}^r \epsilon^k S^{(k)} \right) z^{(r-1)} \right] dt \pmod{\epsilon^{r+1}}, \end{cases} \quad r = 1, 2, \dots,$$

where $s^{(k)} = (s_1^{(k)}, \dots, s_N^{(k)})$, $z^{(r)} = x^{(0)} + \epsilon x^{(1)} + \dots + \epsilon^r x^{(r)}$, $e^{Bt} = \text{diag}(e^{i\tau_1 t}, \dots, e^{i\tau_N t})$, and the above integrations are performed so as to obtain the unique primitive of mean value zero. It is clear that each $z^{(r)}$ defined by (9) is periodic in t , the period being determined by the numbers τ_i . If $z^{(r)}$ is replaced by its expression in terms of the $x^{(k)}$ and coefficients of powers of ϵ are equated, it follows that

$$(10) \quad \begin{cases} x^{(0)} = (d_1 e^{i\tau_1 t}, \dots, d_N e^{i\tau_N t}), \\ x^{(r)} = e^{Bt} \int e^{-Bt} [s^{(r)} - (S^{(r)} x^{(0)} + S^{(1)} x^{(r-1)} + \dots + S^{(r-1)} x^{(1)})] dt, \end{cases} \quad r = 1, 2, \dots.$$

It is then shown in [5] that the method of successive approximations defined above converges to a solution of the system of equations

$$(11) \quad z' = [B - \epsilon h(\tau, d, \epsilon)]z + \epsilon q(z, \epsilon, t),$$

where $\tau = (\tau_1, \dots, \tau_N)$, $d = (d_1, \dots, d_N)$, $h = \text{diag}(h_1, \dots, h_N)$, and

$$(12) \quad h(\tau, d, \epsilon) = S^{(1)}(d) + \epsilon S^{(2)}(\tau, d) + \epsilon^2 S^{(3)}(\tau, d) + \dots$$

where $S^{(r)}(\tau, d)$ is defined by (8). Consequently, the function z satisfying (11) will be a solution of (5) if the system of equations

$$(13) \quad i\tau_k - \epsilon h_k(\tau, d, \epsilon) = \rho_k, \quad k = 1, 2, \dots, N,$$

has a solution for some $\rho_1, \dots, \rho_N$ and non-zero $d_1, \dots, d_N$.

4. Proof of Theorem 1. By the transformations

$$y_i = v_{2i-1}, \quad y'_i = v_{2i}, \quad v_{2i-1} = \frac{1}{2i\sigma_i} (z_{2i-1} + z_{2i}),$$

$$v_{2i} = \frac{1}{2}(z_{2j-1} - z_{2i}), \quad j = 1, 2, \dots, \mu, \quad y_k = z_{\mu+k}, \quad k = \mu + 1, \dots, n + \mu,$$

system (2) is equivalent to the system of first order equations

$$(14) \quad z' = Az + \epsilon q(z, \epsilon, t)$$

where $A = \text{diag}(i\sigma_1, -i\sigma_1, \dots, i\sigma_\mu, -i\sigma_\mu, 0, \dots, 0)$ and

$$(15) \quad \left\{ \begin{array}{l} q_{2j-1}(z, \epsilon, t) = f_j \left[\frac{1}{2i\sigma_1} (z_1 + z_2), \dots, \frac{1}{2i\sigma_\mu} (z_{2\mu-1} + z_{2\mu}), \right. \\ \qquad \qquad \qquad \left. z_{2\mu+1}, \dots, z_{n+\mu}, \frac{1}{2}(z_1 - z_2), \dots, \frac{1}{2}(z_{2\mu-1} - z_{2\mu}), \epsilon, t \right], \\ q_{2j}(z, \epsilon, t) = -q_{2j-1}(z, \epsilon, t), \quad j = 1, 2, \dots, \mu, \\ q_{\mu+k}(z, \epsilon, t) = f_k[\dots], \quad k = \mu + 1, \dots, n + \mu. \end{array} \right.$$

System (14) with some of the characteristic exponents zero was not considered in [5], but by putting $\tau_{2j-1} = a_j \omega / b_j$, a_j, b_j positive integers, $\tau_{2j-1} = -\tau_{2j}$, $j = 1, 2, \dots, \mu$, $\tau_k = 0$, $k = 2\mu + 1, \dots, n + \mu$, and taking the zeroth approximation in (10) to be

$$(16) \quad x^{(0)} = (d_1 e^{i\tau_1 t}, \dots, d_{2\mu} e^{i\tau_{2\mu} t}, d_{2\mu+1}, \dots, d_{n+\mu}),$$

it is easy to see that the proof of convergence in [5] is the same. Furthermore, by following the proof in [5, p. 365-368], it is seen that if $d_{2j-1} = -\bar{d}_{2j}$ ($\bar{}$ is the complex conjugate), $j = 1, 2, \dots, \mu$, $d_k = \bar{d}_k$, $k = 2\mu + 1, \dots, n + \mu$, the function z defined by the above algorithm (10) leads to real functions v_k , $k = 1, 2, \dots, n + \mu$, which are periodic in t of period $T = 2\pi b_1 \dots b_\mu / \omega$. Furthermore, $S_{2j-1}^{(r)} = \bar{S}_{2j}^{(r)}$ for every r and, therefore, from (12), $h_{2j-1} = \bar{h}_{2j}$, $j = 1, 2, \dots, \mu$, $h_k = \bar{h}_k$, $k = 2\mu + 1, \dots, n + \mu$. Consequently there will be a periodic solution of (2) if there exist non-zero complex numbers $d_1, d_3, \dots, d_{2\mu-1}$ and real numbers $d_{2\mu+1}, \dots, d_{n+\mu}$, $\sigma_1, \dots, \sigma_\mu, \omega$ which satisfy the system of equations

$$(17) \quad \left\{ \begin{array}{l} ia_j \omega / b_j - \epsilon h_{2j-1} = i\sigma_j, \quad j = 1, 2, \dots, \mu, \\ h_k = 0, \quad k = 2\mu + 1, \dots, n + \mu. \end{array} \right.$$

In general, one can solve these $n + \mu$ equations (h_{2j-1} may be complex) for combinations of the d 's, σ 's and ω to obtain periodic solutions of (2). However, to obtain solutions of the type specified in the theorem, further restrictions must be imposed on the numbers $d_1, d_3, \dots, d_{2\mu-1}$. We shall need the following

Lemma: Suppose f and g satisfy conditions (A) and (B) of Theorem 1. If the preceding algorithm (10) with zeroth approximation given by (16) is applied with $d_{2j-1} = ic_j = -\bar{d}_{2j}$, $j = 1, 2, \dots, \mu$, $d_{k+\mu} = c_k$, $k = \mu + 1, \dots, n$, where each c_l , $l = 1, 2, \dots, n$ is a non-zero real number, then

$$x_{2j-1}^{(r)}(-t) = x_{2j}^{(r)}(t), \quad S_{2j-1}^{(r)} = -S_{2j}^{(r)}, \quad j = 1, 2, \dots, \mu, \quad x_k^{(r)}(-t) = x_k^{(r)}(t), \\ S_k^{(r)} = 0, \quad k = 2\mu + 1, \dots, n + \mu, \quad r = 0, 1, 2, \dots.$$

Proof: We first prove by induction that $x_{2j-1}^{(r)}(-t) = x_{2j}^{(r)}(t)$, $j = 1, 2, \dots, \mu$, $x_k^{(r)}(-t) = x_k^{(r)}(t)$, $k = 2\mu + 1, \dots, n + \mu$ and all r . From the choice of the numbers d_l , the assertion is true for $r = 0$. Assume the assertion true for $r = 0, 1, \dots, v - 1$ and all j, k . Then

$$\begin{aligned}
x_{2j-1}^{(r)}(-t) + x_{2j}^{(r)}(-t) &= x_{2j-1}^{(r)}(t) + x_{2j}^{(r)}(t), \\
x_{2j-1}^{(r)}(-t) - x_{2j}^{(r)}(-t) &= -[x_{2j-1}^{(r)}(t) - x_{2j}^{(r)}(t)], \quad j = 1, 2, \dots, \mu, \\
x_k^{(r)}(-t) &= x_k^{(r)}(t), \quad k = 2\mu + 1, \dots, n + \mu.
\end{aligned}$$

Consequently conditions (A) and (B) and formula (15) imply that

$$\begin{aligned}
q_{2j-1}[z^{(r)}(-t), \epsilon, -t] &= q_{2j-1}[z^{(r)}(t), \epsilon, t], \quad j = 1, 2, \dots, \mu, \\
(18) \quad q_k[z^{(r)}(-t), \epsilon, -t] &= -q_k[z^{(r)}(t), \epsilon, t], \quad k = 2\mu + 1, \dots, n + \mu, \\
r &= 0, 1, 2, \dots, v - 1.
\end{aligned}$$

Since $s_i^{(r)}(t)$ is the coefficient of ϵ^{r-1} in the expansion of $q_i(z^{(r-1)}, \epsilon, t)$,

$$\begin{aligned}
(19) \quad s_{2j-1}^{(r)}(-t) &= s_{2j-1}^{(r)}(t), \quad j = 1, 2, \dots, \mu, \\
s_k^{(r)}(-t) &= -s_k^{(r)}(t), \quad k = 2\mu + 1, \dots, n + \mu, r = 1, 2, \dots, v.
\end{aligned}$$

Furthermore,

$$q_{2j-1} = -q_{2j}, \quad j = 1, 2, \dots, \mu,$$

implies that

$$s_{2j-1}^{(r)} = -s_{2j}^{(r)}, \quad j = 1, 2, \dots, \mu.$$

From the preceding relations,

$$\begin{aligned}
S_{2j-1}^{(r)} &= \frac{1}{d_{2j-1}T} \int_0^T e^{-i\tau_{2j-1}t} s_{2j-1}^{(r)}(t) dt = -\frac{1}{d_{2j}T} \int_0^T e^{i\tau_{2j}t} s_{2j}^{(r)}(t) dt \\
&= \frac{1}{d_{2j}T} \int_0^{-T} e^{-i\tau_{2j}t} s_{2j}^{(r)}(t) dt = -\frac{1}{d_{2j}T} \int_0^T e^{-i\tau_{2j}t} s_{2j}^{(r)}(t) dt \\
&= -S_{2j}^{(r)}, \quad j = 1, 2, \dots, \mu, r = 1, 2, \dots, v, \\
S_k^{(r)} &= \frac{1}{d_kT} \int_0^T s_k^{(r)}(t) dt = -\frac{1}{d_kT} \int_0^{-T} s_k^{(r)}(-t) dt \\
&= -\frac{1}{d_kT} \int_0^T s_k^{(r)}(t) dt = -S_k^{(r)},
\end{aligned}$$

or $S_k^{(r)} = 0$, $k = 2\mu + 1, \dots, n + \mu$, $r = 1, 2, \dots, v$. From (10),

$$\begin{aligned}
x_{2j-1}^{(v)}(-t) &= -e^{-i\tau_{2j-1}t} \int e^{i\tau_{2j-1}t} [s_{2j-1}^{(v)}(-t) \\
&\quad - \{S_{2j-1}^{(1)} x_{2j-1}^{(v-1)}(-t) + \dots + S_{2j-1}^{(v)} x_{2j-1}^{(0)}(-t)\}] dt \\
&= e^{i\tau_{2j}t} \int e^{-i\tau_{2j}t} [s_{2j}^{(v)}(t) - \{S_{2j}^{(1)} x_{2j}^{(v-1)}(t) + \dots + S_{2j}^{(v)} x_{2j}^{(0)}(t)\}] dt \\
&= x_{2j}^{(v)}(t), \quad j = 1, 2, \dots, \mu.
\end{aligned}$$

Furthermore, since $S_k^{(r)} = 0$, $k = 2\mu + 1, \dots, n + \mu$, $r = 1, 2, \dots, v$,

$$x_k^{(v)}(-t) = - \int s_k^{(v)}(-t) dt = \int s_k^{(v)}(t) dt = x_k^{(v)}(t), \quad k = 2\mu + 1, \dots, n + \mu,$$

and the induction on the $x_i^{(r)}$ is completed. If the assertion is true for $x_i^{(r)}$ for all r , then the other relations must also hold for all r and the lemma is proved.

Using the above lemma and the fact that $S_{2i-1}^{(r)} = \bar{S}_{2i}^{(r)}$ for $j = 1, 2, \dots, \mu$ and all r , it follows that each $S_{2i-1}^{(r)}$ and, therefore, from (12), each h_{2i-1} , is purely imaginary, $j = 1, 2, \dots, \mu$. Further, each $h_k = 0$, $k = 2\mu + 1, \dots, \eta + \mu$. By using the above transformation formulas, it follows that the solution obtained in this manner is of the form (3). Furthermore, if we put $H_j(a, b, c, \sigma, \omega, \epsilon) = I(h_{2j-1})$, $j = 1, 2, \dots, \mu$, and apply (8), then the H_j satisfy (5), and the theorem is proved. The details are as in [5], pages 368, 375, and we refer to this paper for the sake of brevity.

5. Generalizations. In this section Theorem I is generalized to the case where some of the components of the periodic solution of (2) may be even in t and some odd in t . Let m be any integer, $0 \leq m \leq \mu$, and let $y = (u, v)$, $u = (y_1, \dots, y_m)$, $v = (y_{m+1}, \dots, y_\mu)$. In a manner analogous to the preceding discussion one can determine real functions $H_j(a, b, c, \sigma, \omega, \epsilon)$, $j = 1, 2, \dots, \mu$, analytic in ϵ , $|\epsilon| < \epsilon_0$, such that the following theorem holds:

Theorem 2. *If*

$$(A_1) \quad f_j(u, -v, w, -u', v', \epsilon, -t) = f_j(u, v, w, u', v', \epsilon, t), \quad j = 1, 2, \dots, m;$$

$$(A_2) \quad f_j(u, -v, w, -u', v', \epsilon, -t) = -f_j(u, v, w, u', v', \epsilon, t), \quad j = m+1, \dots, \mu;$$

$$(B') \quad f_j(u, -v, w, -u', v', \epsilon, -t) = -f_j(u, v, w, u', v', \epsilon, t), \quad j = \mu+1, \dots, n;$$

and if, for ϵ sufficiently small, the system of equations

$$(20) \quad \frac{a_j}{b_j} \omega + \epsilon H_j(a, b, c, \sigma, \omega, \epsilon) = \sigma_j, \quad j = 1, 2, \dots, \mu,$$

has a solution for some real non-zero c , σ and ω , then (2) has a solution $y(\epsilon, t) \in A[\omega/b_1 \dots b_\mu]$ of the form

$$y_i(0, t) = c_i \cos \frac{a_i \omega t}{b_i}, \quad y_i(\epsilon, -t) = y_i(\epsilon, t), \quad j = 1, 2, \dots, m,$$

$$y_i(0, t) = c_i \sin \frac{a_i \omega t}{b_i}, \quad y_i(\epsilon, -t) = -y_i(\epsilon, t), \quad j = m+1, \dots, \mu,$$

$$y_i(0, t) = c_i, \quad y_i(\epsilon, -t) = y_i(\epsilon, t), \quad j = \mu+1, \dots, n.$$

The functions H_j are given by

$$(21) \quad \begin{cases} H_j = \frac{1}{c_j T} \int_0^T f_j(y_0, w_0, y'_0, \epsilon, t) \cos \frac{a_j \omega t}{b_j} dt, & j = 1, 2, \dots, m, \\ H_j = \frac{1}{c_j T} \int_0^T f_j(y_0, w_0, y'_0, \epsilon, t) \sin \frac{a_j \omega t}{b_j} dt, & j = m+1, \dots, \mu, \end{cases}$$

where $T = 2\pi b_1 \cdots b_\mu / \omega$, $y_0 = (c_1 \cos a_1 \omega t / b_1, \dots, c_m \cos a_m \omega t / b_m, c_{m+1} \sin a_{m+1} \omega t / b_{m+1}, \dots, c_\mu \sin a_\mu \omega t / b_\mu)$ and $w_0 = (c_{\mu+1}, \dots, c_n)$.

The proof of this theorem is exactly the same as the proof of Theorem 1 except in the zeroth approximation the numbers d_i are chosen such that $d_{2i-1} = ic_i = -\bar{d}_{2i}$, $j = 1, 2, \dots, m$, $d_{2i-1} = c_i = -\bar{d}_{2i}$, $j = m+1, \dots, \mu$, $d_{i+\mu} = c_i$, $j = \mu+1, \dots, n$, where each c_l , $l = 1, 2, \dots, n + \mu$, is a non-zero real number. Some other obvious modifications must also be made in the statement of the lemma.

As for Theorem 1, this theorem also is true in the autonomous case. Then system (20) determines relations between the basic "periods" $2\pi b_i / (a_i \omega)$ of the dominant terms of the solutions and their "amplitudes" c_k .

6. Example for a non-autonomous third order equation. Consider the third order equation

$$(22) \quad y''' - a_1 y'' - a_2 y' - a_3 y = \epsilon f(y, y', y'', \epsilon, t)$$

where the a_i are real and $f \in A[\omega]$. In this case the order of the equation is odd and we assume the characteristic roots for $\epsilon = 0$ are such that λ_1 real, $\lambda_2 = \bar{\lambda}_3$. The general techniques described in [5] can be applied to determine the periodic solutions of (22), but to apply the more specific theorems mentioned above we must require $\lambda_1 = 0$, $\lambda_2 = i\sigma$, $\lambda_3 = -i\sigma$, i.e., we consider the equation

$$(23) \quad y''' + \sigma^2 y' = \epsilon f(y, y', y'', \epsilon, t).$$

By the transformation $z_1 = y'$, $z_2 = \sigma^{-2} y'' + y$, system (23) is equivalent to the system of equations

$$(24) \quad \begin{cases} z_1' + \sigma^2 z_1 = \epsilon f(-\sigma^{-2} z_1' + z_2, z_1, z_1', \epsilon, t), \\ z_2' = \epsilon \sigma^{-2} f(-\sigma^{-2} z_1' + z_2, z_1, z_1', \epsilon, t). \end{cases}$$

Furthermore, if $f(y, -y', y'', \epsilon, -t) = -f(y, y', y'', \epsilon, t)$, then system (24) is a special case of the system discussed in Theorem 2 with $m = 0$, $\mu = 1$, $n = 3$. Consequently the following theorem holds:

Theorem 3. *If in (23), $f(y, -y', y'', \epsilon, -t) = -f(y, y', y'', \epsilon, t)$, and if, for ϵ sufficiently small, the equation*

$$(25) \quad \frac{a\omega}{b} - \epsilon H(a, b, c_1, c_2, \omega, \sigma, \epsilon) = \sigma,$$

where a, b are positive integers, has a solution for some real non-zero c_1, c_2, ω, σ , then there is a periodic solution $y(\epsilon, t) \in A[\omega/b]$ of the form

$$(26) \quad y(0, t) = -\frac{c_1 a \omega}{\sigma^2 b} \cos \frac{a \omega t}{b} + c_2, \quad y(\epsilon, -t) = y(\epsilon, t).$$

Furthermore, if $T = 2\pi b / \omega$,

$$(27) \quad H = \frac{1}{c_1 T} \int_0^T f[y(0, t), y'(0, t), y''(0, t), 0, t] \sin \frac{a \omega t}{b} dt + o(\epsilon).$$

If $\sigma = a\omega/b$ and there exists a $c_{10} = c_{10}(c_2)$ such that $H = 0$ and $\partial H/\partial c_1 \neq 0$ for $c_1 = c_{10}$, then (25) certainly has a non-zero solution for every c_2 ; i.e., there is a one parameter family of periodic solutions of (23), each with the same period.

7. Example for a third autonomous system. Consider the equation

$$(28) \quad y''' + \sigma^2 y' = \epsilon f(y, y', y'', \epsilon)$$

where $f(y, -y', y'', \epsilon) = -f(y, y', y'', \epsilon)$. It is clear that in the autonomous case the above theory can be applied but the unknown parameters are the frequency $\tau = a\omega/b$ for the periodic solution and the "basic" amplitude c_1 and c_2 . From Theorem 3 and the implicit function theorem, equation (25) always has a solution $\tau = \tau(\sigma, c_1, c_2, \epsilon) = \sigma + o(\epsilon)$ for every real number c_1, c_2 , i.e. there is a two parameter family of periodic solutions $y(c_1, c_2, \epsilon, t) \in A[\tau]$ of the form

$$(29) \quad \begin{cases} y(c_1, c_2, 0, t) = -\frac{c_1}{\sigma} \cos \sigma t + c_2, \\ y(c_1, c_2, \epsilon, -t) = y(c_1, c_2, \epsilon, t). \end{cases}$$

It is easy to see in the three dimensional phase space that every point on the y -axis is a center for a one parameter family of periodic solutions and the periodic solutions (29) fill up the phase space. These results are summarized in the following

Theorem 4. If $f(y, -y', y'', \epsilon) = -f(y, y', y'', \epsilon)$ in (28) and y_0, y'_0, y''_0 are any numbers such that $|y_0|, |y'_0|, |y''_0| < C$, where C is a constant, then there exists, for ϵ sufficiently small, a periodic solution $y(\epsilon, t)$ of (28) such that $y(\epsilon, 0) = y_0, y'(\epsilon, 0) = y'_0, y''(\epsilon, 0) = y''_0$. Furthermore, every point $(y, y', y'') = (y, 0, 0)$ in the three dimensional phase space is stable in the sense of Liapounoff.

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