On the Velocity of Formation of Lüders Bands

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1. Introduction. In the following article¹ it is assumed that the Lüders band is initiated by the formation of an indefinitely small plastic region within the elastic material at its yield point. This plastic region then grows or is propagated with great rapidity into the elastic material resulting in the appearance of the band as observed, for example, in tensile tests on flat plates.

In Fig. 1 the region BFGC represents the portion of the plastic band already formed in a thin rectangular bar while AEFB and CGHD are the associated elastic regions which will pass into the plastic state as a result of the propagation. The angle θ in Fig. 1, which determines the direction of the final band relative to the applied load, is considered to have its theoretical value $35^{\circ}16'$. This idealized concept of the formation of a Lüders band in a thin rectangular bar, with ABCD and EFGH in Fig. 1 representing planes perpendicular to the flat sides of the bar, will furnish the basis of the following investigation.

We shall be concerned with (1) the determination of the portion CG (or BF), hereafter denoted by Σ for brevity, of the surface of separation of the plastic material of the band and the elastic medium, and (2) the normal velocity G of propagation of Σ . On the basis of the continuity assumption in §2 we have shown in §4 that Σ must be a plane normal to the flat sides of the bar and either parallel or perpendicular to the direction of the applied load. Moreover, the normal velocity G of Σ will be equal to the velocity of propagation of an equivoluminal disturbance in the elastic medium (§4). From these results we immediately deduce the *velocity of formation* V of the Lüders band, defined as the velocity of propagation of Σ in the direction of the lateral sides ABCD and EFGH of the band. Formulæ for V are given in §5.

The Prandtl-Reuss stress-strain relations for plastic flow are used in this investigation and we have availed ourselves of the results on Lüders bands, valid under these relations, in our previous paper² in THIS JOURNAL.

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² *The Lüders band problem*, Jour. of Math. and Mech. 7 (1958), pp. 17-27.

FIG. 1

2. Continuity assumption and compatibility conditions. In our discussion of Lüders bands² we were concerned with the surface of separation BC (or FG) of the band and the surrounding elastic material of the medium. We shall now apply certain of the results of this discussion to the question of the variation of the components of the stress tensor over the plane BC. The reader is referred to our foregoing paper for the meaning of the symbols employed as well as for the definition of the x and y coordinate systems which will enter into the discussion of the present article.

Let us begin by considering the previous relations (14), namely

$$(1) \quad \mu_{12} = \bar{\sigma}_{12} ; \quad \mu_{22} = \bar{\sigma}_{22} ,$$

over BC, relative to the y coordinate system. Now it was shown that $\bar{\sigma}_{11}^* = 0$ along BC, relative to the y system, for the plastic flow governed by the Prandtl-Reuss equations; from this equation and the second equation (1) we readily find

$$(2) \quad 2\bar{\sigma}_{11} = \mu_{22} .$$

But

$$\mu_{11} = \tau \sin^2 \theta = \frac{\tau}{3} ; \quad \mu_{22} = \tau \cos^2 \theta = \frac{2\tau}{3} ,$$

where τ is the yield stress, and hence we have

$$(3) \quad \bar{\sigma}_{11} = \mu_{11}$$

on account of (2). Now $\mu_{i3} = 0$ for $i = 1, 2, 3$. Assuming the plane stress condition in consequence of which all components σ_{ik} vanish whenever the index i or k has the value 3, it follows from (1) and (3) therefore that

$$\bar{\sigma}_{ik} = \mu_{ik}, \quad (i, k = 1, 2, 3).$$

In other words the stress tensor is continuous over the plane BC. A similar remark of course applies to the plane FG in Fig. 1.

We assume that the stress tensor has components τ_{ik} with $\tau_{22} = k$ (yield stress) and $\tau_{ik} = 0$ otherwise, relative to the x coordinate system, over the entire elastic region including the regions AEFB and CGHD in Fig. 1. Hence it follows from the above italicized result that the stress tensor must be continuous over Σ , *i.e.* the surface CG in Fig. 1, at its end points C and G . This suggests that the stress tensor should properly be taken to be continuous over the entire surface Σ in this consideration. *We therefore make the assumption that the stress tensor is continuous over Σ .*

Now consider the general dynamical conditions

$$(4) \quad \rho G = \hat{\rho}(G - \hat{v}_n); \quad [\sigma_{ik}]v_k = -\rho G[v_i],$$

which must hold over Σ where ρ and $\hat{\rho}$ denote the densities on the elastic and plastic sides of Σ respectively, $\hat{v}_n$ is the normal component of velocity on the plastic side of Σ and the v_k are the components of the unit normal vector ν to Σ , assumed to be directed from the plastic to the elastic side of this surface. Also

$$(5) \quad [\sigma_{ik}] = \hat{\sigma}_{ik} - \sigma_{ik}; \quad [v_i] = \hat{v}_i,$$

where the σ_{ik} and $\hat{\sigma}_{ik}$ are the components of the stress tensor on the elastic and plastic sides of Σ . Finally, the quantities $\hat{v}_i$ are the velocity components on the plastic side of Σ . Account is taken of the fact that the velocity of the material particles is zero in the elastic medium by assumption in writing the conditions (4) and (5).

The quantities $[\sigma_{ik}]$ must vanish from the above continuity assumption. Assuming $G \neq 0$ it follows from the second condition (4) that $[v_i] = 0$ or, in other words, that *the velocity is continuous over Σ* . Hence $\hat{v}_n = 0$ from the second condition (5) and we see from the first condition (4) that $\rho = \hat{\rho}$, *i.e. the density is continuous over Σ* .

In view of the continuity of the velocity and the stress tensor over Σ we can now write

$$(6) \quad [v_{i,k}] = \hat{v}_{i,k} = \lambda_i v_k; \quad \left[\frac{\partial v_i}{\partial t} \right] = \frac{\partial \hat{v}_i}{\partial t} = -G\lambda_i,$$

$$(7) \quad [\sigma_{ij,k}] = \hat{\sigma}_{ij,k} = \xi_{ij} v_k; \quad \left[\frac{\partial \sigma_{ij}}{\partial t} \right] = \frac{\partial \hat{\sigma}_{ij}}{\partial t} = -G\xi_{ij},$$

relative to the x or y coordinate systems, where the quantities λ_i and ξ_{ij} are suitable functions defined over the moving surface Σ . The relations (6) and (7) constitute *compatibility conditions* for the problem under consideration. There is, of course, a corresponding compatibility condition for the density ρ but this will not be written since it will not be needed in the following discussion.

It will be assumed that the surface Σ is singular of order 1 relative to the functions v_i and σ_{ik} . We mean precisely by this that not all the quantities λ_i and ξ_{ij} vanish at any point of Σ . In fact, if $\lambda_i = 0$ and $\xi_{ij} = 0$ the continuity of the first derivatives of the components of velocity and stress over Σ would follow from (6) and (7); hence the equations (6) and (7) could be replaced by analogous equations involving second derivatives of the components of the velocity and the stress tensor.

3. Application of the compatibility conditions. Let us now consider the following system of equations

$$(8) \quad \frac{d\hat{\sigma}_{ij}^*}{dt} = 2\mu(\hat{\epsilon}_{ij} - \hat{\psi}\hat{\sigma}_{ij}^*), \quad \text{Prandtl-Reuss equations,}$$

$$(9) \quad \hat{\sigma}_{ij,i} = \rho \frac{d\hat{\theta}_i}{dt}, \quad \text{equations of motion,}$$

$$(10) \quad \hat{\nu}_{i,i} = 0, \quad \text{incompressibility condition,}$$

$$(11) \quad \hat{\sigma}_{ij}^* \hat{\sigma}_{ij}^* = K, \quad \text{quadratic yield condition,}$$

where the symbol $\hat{}$ in (8), $\dots$, (11) is used, as in §2, to denote evaluation on the plastic side of the surface Σ . This symbol has not been placed over the function ρ in the equations (9) since $\rho = \hat{\rho}$ over Σ by the result in §2. Hence the quantity ρ in (9) is actually the material density in the elastic medium. An expression for the function $\hat{\psi}$ in (8) is given by

$$\hat{\psi} = \frac{\hat{\epsilon}_{ij} \hat{\sigma}_{ij}^*}{K},$$

which follows readily from the Prandtl-Reuss equations in conjunction with the quadratic yield condition.

From (10) and the first relation (6) we immediately have

$$(12) \quad \lambda_i \nu_i = 0.$$

Similarly from (6), (7) and (9) we obtain

$$(13) \quad \xi_{ij} \nu_j + \rho G \lambda_i = 0.$$

Finally, when we expand the left member of (8) and make use of the conditions (6) and (7) we are led to relations of the form

$$(14) \quad (\xi_{ij} - \frac{1}{3} \xi_{kk} \delta_{ij}) G + \mu(\lambda_i \nu_j + \lambda_j \nu_i - \frac{2}{K} \hat{\sigma}_{ab} \lambda_a \nu_b \hat{\sigma}_{ij}^*) = 0.$$

We observe immediately that not all of the quantities ξ_{ij} can vanish since it would then follow from (13) that all λ_i must likewise vanish and this would contradict the assumption in §2 that the surface Σ is singular of order 1. Conversely, all of the λ_i cannot vanish. In fact, if $\lambda_i = 0$ for $i = 1, 2, 3$ the conditions (13) and (14) become

$$\xi_{ij}\nu_j = 0; \quad \xi_{ij} = \frac{1}{3}\xi_{kk}\delta_{ij}.$$

But if we multiply the second of these relations by ν_i and take account of the first relation it follows that $\xi_{kk} = 0$. Hence $\xi_{ij} = 0$ from the second set of relations and we again have a contradiction with the assumption that Σ is singular of order 1.

Putting $j = 3$ in the first relation (7) we see that $\xi_{i3} = 0$ on account of the plane stress condition. Hence, putting $i = 3$ in (13), it follows that $\lambda_3 = 0$. Now take $i = 1, 2$ and $j = 3$ in (14). This gives $\lambda_i\nu_3 = 0$ because of the vanishing of the quantities ξ_{i3} and λ_3 and the fact that $\sigma_{i3}^* = 0$ in view of the condition of plane stress. Hence $\nu_3 = 0$ since otherwise we would have $\lambda_i = 0$ for $i = 1, 2, 3$ and this, as we have seen, would violate the hypothesis that Σ is singular of order 1. *In other words the surface Σ must be perpendicular to the flat sides of the bar.*

Since $\lambda_3 = \nu_3 = 0$ the equation (12) becomes

$$\lambda_1\nu_1 + \lambda_2\nu_2 = 0.$$

Between the values of $\lambda_1, \lambda_2, \nu_1$ and ν_2 we must therefore have the proportionality

$$(15) \quad \lambda_1, \lambda_2 \propto -\nu_2, \nu_1.$$

4. Velocity of the wave front. To deduce the normal velocity G of the wave front Σ we turn to the relations (14). Multiplying these relations by ν_i and giving K its value $\frac{2}{3}\tau^2$ we obtain

$$(16) \quad (\mu - \rho G^2)\lambda_i - \frac{1}{3}G\xi_{kk}\nu_i - \frac{3\mu}{\tau^2}\sigma_{ab}\lambda_a\nu_b\sigma_{ij}^*\nu_j = 0,$$

when use is made of (12) and (13). Also, putting $i = j = 3$ in (14), we have

$$(17) \quad \frac{1}{3}\xi_{kk}G + \frac{3\mu}{\tau^2}\sigma_{ab}\lambda_a\nu_b\sigma_{33}^* = 0.$$

Let us now consider the relations (16) and (17) in the x coordinate system for which the components τ_{ik} of the stress tensor are $\tau_{22} = \tau$ (yield stress) and $\tau_{ik} = 0$ otherwise. But due to the continuity of the stress components over Σ we have $\sigma_{ik} = \tau_{ik}$. Hence

$$(18) \quad \sigma_{ab}\lambda_a\nu_b = \tau\lambda_2\nu_2.$$

Also

$$(19) \quad \sigma_{11}^* = -\frac{\tau}{3}; \quad \sigma_{22}^* = \frac{2\tau}{3}; \quad \sigma_{33}^* = -\frac{\tau}{3},$$

while $\dot{\sigma}_{ik}^* = 0$ for $i \neq k$. Hence

$$(20) \quad \frac{1}{3}\xi_{kk}G = \mu\lambda_2\nu_2$$

from (17). Making the substitutions (18) and (20) the relations (16) become

$$(21) \quad (\mu - \rho G^2)\lambda_i - \mu\lambda_2\nu_2\nu_i - \frac{3\mu}{\tau}\lambda_2\nu_2\dot{\sigma}_{ii}^*\nu_i = 0.$$

Because of the plane stress condition and the fact that $\lambda_3 = \nu_3 = 0$ the relations (21) are satisfied automatically when $i = 3$. However, taking $i = 1, 2$ and using (19) and the proportionality relation (15), we find that

$$(\rho G^2 - \mu)\nu_2 = 0; \quad (\rho G^2 - \mu)\nu_1 + 3\mu\nu_1\nu_2^2 = 0.$$

From these two equations we can immediately deduce that we must have one or the other of the following two possibilities, namely

$$(22) \quad G^2 = \frac{\mu}{\rho}, \quad \nu_1 = 1,$$

$$(23) \quad G^2 = \frac{\mu}{\rho}, \quad \nu_2 = 1.$$

We see from (22) and (23) that the normal velocity G of Σ is the same as the velocity of an equivoluminal wave in the elastic medium. Moreover, since $\nu_1 = 1$ or $\nu_2 = 1$, relative to the x coordinate system, and since $\nu_3 = 0$, we can also state the following result. *The surface Σ is a plane perpendicular to the flat sides of the bar and is either (1) perpendicular or (2) parallel to the direction of the applied load.*

5. Velocity of formation of the Lüders band. The two cases corresponding to the formulæ (22) and (23) are illustrated in Fig. 2 and Fig. 3 respectively.

FIG. 2

FIG. 3

In Fig. 2 the wave surface Σ is represented by the vertical line AB and in Fig. 3 by the horizontal line AB. The velocity of propagation of these surfaces in the direction of the Lüders band, *i.e.* the velocity of formation V of the band, is given by

$$(24) \quad V = \frac{G}{\cos \theta} = \sqrt{\frac{3}{2}} G = \sqrt{\frac{3}{2}} \sqrt{\frac{\mu}{\rho}}, \quad \text{for } \nu_1 = 1,$$

$$(25) \quad V = \frac{G}{\sin \theta} = \sqrt{3} G = \sqrt{3} \sqrt{\frac{\mu}{\rho}}, \quad \text{for } \nu_2 = 1.$$

These formulæ determine the velocity V when the velocity G of an equivoluminal wave in the elastic medium is known.

To express the relation between the velocity V and the velocity G_* of an irrotational wave in the elastic solid, we have

$$G_*^2 = \frac{\lambda + 2\mu}{\rho} = \frac{\mu}{\rho} \left(2 + \frac{\lambda}{\mu} \right).$$

But

$$\frac{\lambda}{\mu} = \frac{2\nu}{1 - 2\nu},$$

where ν is Poisson's ratio. Hence

$$G_*^2 = \frac{2(1 - \nu)}{1 - 2\nu} \frac{\mu}{\rho} = \frac{2(1 - \nu)}{1 - 2\nu} G^2,$$

and from (24) and (25) we obtain

$$V = \frac{\sqrt{3}}{2} \sqrt{\frac{1-2\nu}{1-\nu}} G_*, \quad \text{for } \nu_1 = 1,$$

$$V = \sqrt{\frac{3}{2}} \sqrt{\frac{1-2\nu}{1-\nu}} G_*, \quad \text{for } \nu_2 = 1.$$

The above analysis does not disclose which of the two possible situations, namely that illustrated by Fig. 2 or the one shown in Fig. 3, will arise when the band is produced by tension of the thin rectangular bar under consideration. Possibly one of these cases is associated *in practice* with the tension problem and the other with the analytically equivalent problem of compression for which the algebraic value of the yield stress τ is negative.

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