*Symmetric Bending of Shallow Shells of Revolution**

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1. Introduction. The present paper is concerned with the problem of symmetrical bending of thin elastic shells of revolution for shells which at the same time may be considered as shallow shells. We assume a somewhat more general system of stress strain relations than is usually done in shell theory and for this system reduce the problem to two simultaneous second-order non-linear differential equations. For an isotropic shell, which is homogeneous in wall thickness direction, and without temperature strains, these equations become special cases of general shallow shell theory [1] and also of earlier results for non-shallow shells [2].

Of the many possible applications of the general equations obtained in this paper we consider first the problem of the edge effect in shallow spherical shells subjected to internal pressure. For this effect there exist two well known solutions, one within the framework of linear bending theory [3] and the other within the framework of non-linear membrane theory [4]. The connection between these two theories, and, in fact, a more general theory in which linear bending theory and non-linear membrane theory appear as limiting cases, has previously been considered by the writer [2]. The present treatment reproduces our earlier results for the edge effect in somewhat greater detail and supplements them by numerical calculations.

As a second problem we consider the effect of uniform surface pressure in accordance with linear theory, assuming different elastic behavior in meridional and circumferential direction. We find that the nature of the solution of the problem is altered in a fundamental way by going from an isotropic material to an orthotropic material. While linear membrane theory furnishes an appropriate solution for the interior of the isotropic shell, this solution becomes singular in an inadmissible way for the interior of the orthotropic shell. Accord-

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ingly, wall bending stiffness must be considered in order to predict the nature of the stress distribution in the orthotropic shell. Explicit results are obtained for two classes of orthotropic stiffness assumptions for which the differential equations of the problem can be solved in terms of Bessel functions.

2. Basic equations. We consider a material layer—subsequently referred to as a *shell*—bounded by two co-axial surfaces of revolution. In addition to the two bounding surfaces we introduce a third co-axial surface of revolution which will be referred to as the *middle surface* of the shell.

We designate by r the distance, before deformation, of points of the middle surface from its axis of symmetry. We designate by

$$z = z(r) \quad (2.1)$$

the equation of the middle surface in its undeformed state.

Analysis of strain. It is assumed that the shell is deformed in a rotationally symmetric manner, in such a way that its points move in radial as well as in axial direction, but not in circumferential direction. Radial and axial displacement components of points of the middle surface of the shell are designated by u and w , respectively. It is further assumed that deformation of the shell takes place in accordance with the Euler-Bernoulli hypothesis which states that “normals to the undeformed middle surface are deformed, without extension, into normals to the deformed middle surface.” Finally, we are limiting ourselves to the consideration of shells which are *shallow* in the sense that the difference between meridian slope and sloping angle may be disregarded.

Defining components of normal strain as relative changes of length, and indicating differentiation with respect to r by primes, we obtain, in accordance with Figure 1, the following expressions for circumferential strain ϵ_θ and

FIGURE 1.

meridional strain ϵ_r , for a surface with coordinate ζ relative to the middle surface:

$$\epsilon_\theta = \frac{u - \zeta w'}{r - \zeta z'}, \quad (2.2)$$

$$(1 + \epsilon_r) \frac{ds}{dr} = \frac{1 + u' - \zeta(z'' + w')}{\cos(z' + w')}. \quad (2.3)$$

With the thinness assumption $|\zeta z'| \ll r$, and the shallowness assumptions $ds = dr(1 - \zeta z'')$, $\cos(z' + w') = 1 - \frac{1}{2}(z' + w')^2$, we have that (2.2) and (2.3) can be written in the form

$$\epsilon_\theta = \epsilon_{\theta m} + \zeta \kappa_\theta, \quad \epsilon_r = \epsilon_{r m} + \zeta \kappa_r \quad (2.4)$$

where $\epsilon_{\theta m}$ and $\epsilon_{r m}$ are midsurface strains given by

$$\epsilon_{\theta m} = \frac{u}{r}, \quad \epsilon_{r m} = u' + z'w' + \frac{1}{2}(w')^2 \quad (2.5)$$

and κ_θ and κ_r are midsurface curvature changes given by

$$\kappa_\theta = -\frac{w'}{r}, \quad \kappa_r = -w''. \quad (2.6)$$

Stress strain relations. Corresponding to the components of strain ϵ_θ and ϵ_r we have circumferential and meridional components of normal stress σ_θ and σ_r . We take as relations between stresses, strains and temperature T the following:

$$\epsilon_r = \frac{\sigma_r}{E_r} - \frac{\sigma_\theta}{E_{r\theta}} + \eta_r T, \quad (2.7)$$

$$\epsilon_\theta = \frac{\sigma_\theta}{E_\theta} - \frac{\sigma_r}{E_{\theta r}} + \eta_\theta T. \quad (2.8)$$

We admit that E_r , E_θ , $E_{r\theta}$, $E_{\theta r}$, η_r and η_θ may be functions of r , ζ and T and we stipulate the symmetry relation $E_{r\theta} = E_{\theta r}$.

Equations (2.7) and (2.8) may be solved for σ_r and σ_θ in the form

$$\sigma_r = E_r^* \epsilon_r + E_{r\theta}^* \epsilon_\theta - (E_r^* \eta_r + E_{r\theta}^* \eta_\theta) T, \quad (2.9)$$

$$\sigma_\theta = E_\theta^* \epsilon_\theta + E_{\theta r}^* \epsilon_r - (E_\theta^* \eta_\theta + E_{\theta r}^* \eta_r) T \quad (2.10)$$

where

$$E_r^* = \frac{E_r}{1 - \nu^2}, \quad E_\theta^* = \frac{E_\theta}{1 - \nu^2}, \quad E_{r\theta}^* = E_{\theta r}^* = \frac{\nu^2 E_{r\theta}}{1 - \nu^2}, \quad \nu^2 = \frac{E_\theta E_r}{E_{r\theta}^2}. \quad (2.11)$$

The stress strain relations (2.9) and (2.10) are to be transformed into relations between strains and *stress resultants* and *stress couples*. Stress resultants N_r and N_θ are defined as follows:

$$(N_r, N_\theta) = \int_{-h_1}^{h_2} (\sigma_r, \sigma_\theta) d\zeta. \quad (2.12)$$

Stress couples M_r and M_θ are defined as follows:

$$(M_r, M_\theta) = \int_{-h_1}^{h_2} (\sigma_r, \sigma_\theta) \zeta d\zeta. \quad (2.13)$$

Introduction of (2.9) and (2.10) into (2.12) leads to expressions of the following form

$$N_r = A_{\epsilon_{rm}}^* + A_{\epsilon_{\theta m}}^* + B_{\kappa_r}^* + B_{\kappa_\theta}^* - N_{rT}, \quad (2.14)$$

$$N_\theta = A_{\epsilon_{\theta m}}^* + A_{\epsilon_{rm}}^* + B_{\kappa_\theta}^* + B_{\kappa_r}^* - N_{\theta T}. \quad (2.15)$$

Introduction of (2.9) and (2.10) into (2.13) leads to expressions of the form

$$M_r = B_{\epsilon_{rm}}^* + B_{\epsilon_{\theta m}}^* + C_{\kappa_r}^* + C_{\kappa_\theta}^* - M_{rT}, \quad (2.16)$$

$$M_\theta = B_{\epsilon_{\theta m}}^* + B_{\epsilon_{rm}}^* + C_{\kappa_\theta}^* + C_{\kappa_r}^* - M_{\theta T} \quad (2.17)$$

where the coefficients A^* , B^* , C^* and the terms N_T and M_T are defined as follows:

$$(A^*, B^*, C^*) = \int_{-h_1}^{h_2} (1, \zeta, \zeta^2) E^* d\zeta, \quad (2.18)$$

$$(N_{rT}, M_{rT}) = \int_{-h_1}^{h_2} (1, \zeta) (E_r^* \eta_r + E_{r\theta}^* \eta_\theta) T d\zeta, \quad (2.19)$$

$$(N_{\theta T}, M_{\theta T}) = \int_{-h_1}^{h_2} (1, \zeta) (E_\theta^* \eta_\theta + E_{\theta r}^* \eta_r) T d\zeta. \quad (2.20)$$

Equilibrium equations for resultants and couples. We introduce, in addition to resultants parallel to the middle surface, a transverse resultant Q and radial and axial resultants H and V (Figure 2). We also introduce components of surface load intensity p_H , p_V and m , per unit of middle surface area.

In order that every element of the shell is in equilibrium the following three differential equations must be satisfied:

$$(rH)' - N_\theta + rp_H = 0, \quad (2.21)$$

$$(rV)' + rp_V = 0, \quad (2.22)$$

$$(rM_r)' - M_\theta - rQ + rm = 0. \quad (2.23)$$

In (2.23) the transverse stress resultant Q and axial and radial resultants V and H are related as follows, in the range of applicability of shallow shell theory:

$$Q = V - (z' + w')H. \quad (2.24)$$

FIGURE 2.

A corresponding relation for N_r is taken in the further simplified form

$$N_r = H \quad (2.25)$$

which may be based on the *a posteriori* verifiable fact that V is of a smaller order of magnitude than H .

3. Reduction to two simultaneous second-order differential equations. We assume that the external load functions p and m are given functions of r and time t . With this assumption it becomes possible to reduce the problem as stated in Section 2 to two simultaneous equations for a deformation variable ϕ , defined by

$$\phi = w', \quad (3.1)$$

and, for a stress variable ψ , defined by

$$\psi = rH. \quad (3.2)$$

In this reduction, use is made of the moment equilibrium equation (2.23), which is written in the form

$$rM'_r + M_r - M_\theta + (z' + w')rH + r(m - V) = 0, \quad (3.3)$$

and of a compatability equation for strains which follows from (2.5) as

$$r\epsilon'_{\theta m} + \epsilon_{\theta m} - \epsilon_{rm} + (z' + \frac{1}{2}w')w' = 0. \quad (3.4)$$

In order to carry out this reduction it is necessary to rewrite the stress strain relations (2.14) to (2.17) in the form

$$\epsilon_{rm} = A_r N_r + A_{r\theta} N_\theta + B_r \kappa_r + B_{r\theta} \kappa_\theta + \epsilon_{rT}, \quad (3.5)$$

$$\epsilon_{\theta m} = A_{\theta r} N_r + A_{\theta\theta} N_\theta + B_{\theta r} \kappa_r + B_{\theta\theta} \kappa_\theta + \epsilon_{\theta T}, \quad (3.6)$$

$$M_r = C_r N_r + C_{r\theta} N_\theta + D_r \kappa_r + D_{r\theta} \kappa_\theta + M_{rT}^*, \quad (3.7)$$

$$M_\theta = C_{\theta r} N_r + C_\theta N_\theta + D_{\theta r} \kappa_r + D_\theta \kappa_\theta + M_{\theta T}^*. \quad (3.8)$$

The quantities A , B , C , D , E_T and M^* are given as follows:

$$(A_r, A_\theta, A_{r\theta}) = \frac{(A_\theta^*, A_r^*, -A_{r\theta}^*)}{A_r^* A_\theta^* - A_{r\theta}^{*2}} \quad (3.9)$$

$$\left. \begin{aligned} B_r &= \frac{B_{\theta r}^* A_{r\theta}^* - B_r^* A_\theta^*}{A_r^* A_\theta^* - A_{r\theta}^{*2}} \\ B_{r\theta} &= \frac{B_\theta^* A_{r\theta}^* - B_{r\theta}^* A_\theta^*}{A_r^* A_\theta^* - A_{r\theta}^{*2}} \\ B_\theta &= \frac{B_{\theta r}^* A_{r\theta}^* - B_\theta^* A_r^*}{A_r^* A_\theta^* - A_{r\theta}^{*2}} \end{aligned} \right\} \quad (3.10)$$

$$\left. \begin{aligned} \epsilon_{rT} &= \frac{A_\theta^* N_{rT} - A_{r\theta}^* N_{\theta T}}{A_r^* A_\theta^* - A_{r\theta}^{*2}} \\ \epsilon_{\theta T} &= \frac{A_r^* N_{\theta T} - A_{r\theta}^* N_{rT}}{A_r^* A_\theta^* - A_{r\theta}^{*2}} \end{aligned} \right\} \quad (3.11)$$

$$\left. \begin{aligned} C_r &= B_r^* A_r + B_{r\theta}^* A_{\theta r} \\ C_{r\theta} &= B_r^* A_{r\theta} + B_{r\theta}^* A_\theta \\ C_{\theta r} &= B_\theta^* A_{\theta r} + B_{\theta r}^* A_r \\ C_\theta &= B_\theta^* A_\theta + B_{\theta r}^* A_{\theta r} \end{aligned} \right\} \quad (3.12)$$

$$\left. \begin{aligned} D_r &= B_r^* B_r + B_{r\theta}^* B_{\theta r} \\ D_{r\theta} &= D_{\theta r} = B_r^* B_{r\theta} + B_{r\theta}^* B_\theta \\ D_\theta &= B_\theta^* B_\theta + B_{\theta r}^* B_{r\theta} \end{aligned} \right\} \quad (3.13)$$

$$\left. \begin{aligned} M_{rT}^* &= B_r^* \epsilon_{rT} + B_{r\theta}^* \epsilon_{\theta T} - M_{rT} \\ M_{\theta T}^* &= B_{\theta r}^* \epsilon_{rT} + B_\theta^* \epsilon_{\theta T} - M_{\theta T} \end{aligned} \right\}. \quad (3.14)$$

In view of (2.6), (2.21), (2.25), (3.1) and (3.2) we have as expressions for stress resultants and couples, in terms of the variables ψ and ϕ ,

$$N_r = \frac{\psi}{r}, \quad N_\theta = \psi' + r p_H, \quad (3.15)$$

$$M_r = C_r \frac{\psi}{r} + C_{r\theta} \psi' - D_r \phi' - D_{r\theta} \frac{\phi}{r} + M_{r\theta}, \quad (3.16)$$

$$M_\theta = C_{\theta r} \frac{\psi}{r} + C_\theta \psi' - D_{\theta r} \phi' - D_\theta \frac{\phi}{r} + M_{\theta p} \quad (3.17)$$

where

$$M_{rp} = M_{rT}^* + C_{r\theta} r p_H, \quad M_{\theta p} = M_{\theta T}^* + C_\theta r p_H. \quad (3.18)$$

Introduction of (3.16) and (3.17) into the moment equilibrium equation (3.3) gives as the first of two simultaneous equations for ψ and ϕ

$$\begin{aligned} D_r \phi'' + (D_r + r D_r') \frac{\phi}{r} - (D_\theta - r D_\theta') \frac{\phi}{r^2} - C_{r\theta} \psi'' \\ - (C_r - C_\theta + C_{r\theta} + r C_{r\theta}') \frac{\psi'}{r} + (C_{\theta r} - r C_r') \frac{\psi}{r^2} - \frac{z' + \phi}{r} \psi = q_\phi \end{aligned} \quad (3.19)$$

where

$$q_\phi = m - V + M'_{rp} + \frac{1}{r} (M_{rp} - M_{\theta p}). \quad (3.20)$$

In order to obtain the second equation of the system we introduce

$$\epsilon_{rm} = A_r \frac{\psi}{r} + A_{r\theta} \psi' - B_r \phi' - B_{r\theta} \frac{\phi}{r} + \epsilon_{rp}, \quad (3.21)$$

$$\epsilon_{\theta m} = A_{\theta r} \frac{\psi}{r} + A_\theta \psi' - B_{\theta r} \phi' - B_\theta \frac{\phi}{r} + \epsilon_{\theta p} \quad (3.22)$$

with

$$\epsilon_{rp} = \epsilon_{rT} + A_{r\theta} r p_H, \quad \epsilon_{\theta p} = \epsilon_{\theta T} + A_\theta r p_H \quad (3.23)$$

into the compatibility equation (3.4). This gives

$$\begin{aligned} A_\theta \psi'' + (A_\theta + r A_\theta') \frac{\psi'}{r} - (A_r - r A_r') \frac{\psi}{r^2} - B_{\theta r} \phi'' \\ - (B_\theta - B_r + B_{\theta r} + r B_{\theta r}') \frac{\phi'}{r} + (B_{r\theta} - r B_\theta') \frac{\phi}{r^2} + \frac{z' + \frac{1}{2}\phi}{r} \phi = q_\psi \end{aligned} \quad (3.24)$$

where

$$q_\psi = -\epsilon'_{\theta p} + \frac{1}{r} (\epsilon_{rp} - \epsilon_{\theta p}). \quad (3.25)$$

In many, but not all, cases it will be possible to choose the position of the middle surface of the shell in such a manner that the coefficients B and C in equations (3.19) and (3.24) all vanish. For such cases the form of (3.19) and (3.24) is essentially that of the usual equations of shallow shell theory except for the generalization associated with polar symmetrical orthotropy.

4. The edge effect for uniformly loaded, uniform homogeneous isotropic spherical shells. We consider a shell with surface equation

$$z = H - \frac{r^2}{2R} \quad (4.1)$$

and meridian slope

$$z' = -\frac{r}{R} \quad (4.2)$$

in the interval $0 \leq r \leq a$. Base radius a , rise H and curvature radius R of the shallow spherical shell are related as follows:

$$a^2 = 2HR. \quad (4.3)$$

The assumption of shallowness requires that we limit ourselves to shells for which $|z'|_{\max} \lesssim \frac{1}{3}$. According to (4.2) and (4.3) this means that the shell with surface equation (4.1) may be considered as shallow as long as

$$\frac{H}{a} \lesssim \frac{1}{6}. \quad (4.4)$$

If we assume that the middle surface of the shell is halfway between the outer and inner surfaces and denote the thickness of the shell by h , so that $h_1 = h_2 = \frac{1}{2}h$, we have for an isotropic homogeneous shell

$$D_r = D_\theta \equiv D = \frac{Eh^3}{12(1-\nu^2)}, \quad D_{r\theta} = \nu D, \quad (4.5)$$

$$A_r = A_\theta \equiv A = \frac{1}{Eh}, \quad A_{r\theta} = -\nu A. \quad (4.6)$$

The coefficients B and C all vanish.

We further assume the absence of thermal effects, a uniform axial load $p_r = p$, with $p_H = 0$, and uniform properties of the shell so that $D' = A' = 0$. There-with the differential equations (3.19) and (3.24) reduce to the following form:

$$D \left[\phi'' + \frac{\phi'}{r} - \frac{\phi}{r^2} \right] + \frac{\psi}{R} - \frac{\phi\psi}{r} = \frac{1}{2}pr, \quad (4.7)$$

$$A \left[\psi'' + \frac{\psi'}{r} - \frac{\psi}{r^2} \right] - \frac{\phi}{R} + \frac{1}{2} \frac{\phi^2}{r} = 0. \quad (4.8)$$

Expressions for stress resultants, stress couples and displacements are now

$$N_r = \frac{\psi}{r}, \quad N_\theta = \psi', \quad Q = -\frac{1}{2}pr - \frac{\psi}{R} - \frac{\phi\psi}{r}, \quad (4.9)$$

$$M_r = -D \left(\phi' + \nu \frac{\phi}{r} \right), \quad M_\theta = -D \left(\frac{\phi}{r} + \nu \phi' \right), \quad (4.10)$$

$$\frac{u}{r} = A \left(\psi' - \nu \frac{\psi}{r} \right), \quad w = \int \phi \, dr. \quad (4.11)$$

We shall assume that we are dealing with a simply supported shell segment so that ϕ and ψ must satisfy the following boundary conditions at the edge of the shell:

$$r = a: \quad \phi' + \frac{\nu}{r}\phi = 0, \quad \psi' - \frac{\nu}{r}\psi = 0. \quad (4.12)$$

In addition to this we have symmetry, or regularity, conditions at the apex of the shell, of the form

$$r = 0: \quad \phi = 0, \quad \psi' - \frac{\nu}{r}\psi = 0. \quad (4.13)$$

Linear membrane solution. A particular solution of the system (4.7) and (4.8), designated as linear membrane solution, is

$$\psi = \psi_M = \frac{1}{2}pRr, \quad \phi = \phi_M = 0. \quad (4.14)$$

Stress resultants and couples associated with the solution (4.14) are

$$N_{rM} = N_{\theta M} = \frac{1}{2}pR, \quad (4.15)$$

$$M_{rM} = M_{\theta M} = 0. \quad (4.16)$$

The solution (4.14) satisfies three of the four boundary conditions (4.12) and (4.13) but not the fourth condition of vanishing radial edge displacement. In view of (4.11) we have the following expression for radial displacements according to linear membrane theory:

$$u_M = \frac{1}{2}(1 - \nu)ApRr. \quad (4.17)$$

In order to remove the displacement u_M at the edge $r = a$ it is necessary to consider solutions of (4.7) and (4.8) which are more general than the solution (4.14). It will be shown that, subject to certain order of magnitude relations, these more general solutions effectively coincide with the linear membrane solution except in a relatively narrow zone adjacent to the edge of the shell. Taking account of the narrowness of this edge zone results in a considerable mathematical simplification of the problem of determining the stress distribution in the shell.

Edge zone solution. In order to separate out the results of linear membrane theory we write the solution of the differential equations (4.7) and (4.8) in the following form, in terms of a new independent variable ξ :

$$\psi = \psi_M + \psi_0 g(\xi), \quad \phi = \phi_0 f(\xi). \quad (4.18)$$

The relation between ξ and r is taken as

$$\xi = \lambda \left(1 - \frac{r}{a} \right) \quad (4.19)$$

and ψ_0 , ϕ_0 and λ are three as yet undetermined parameters.

Introduction of (4.18) and (4.19) into (4.7) and (4.8) results in the following two differential equations for the two functions f and g :

$$\frac{D\lambda^2}{a^2} \phi_0 Lf + \frac{\psi_0}{R} g - \frac{pR}{2} \phi_0 f - \frac{\lambda}{a} \psi_0 \phi_0 \frac{fg}{\lambda - \xi} = 0, \quad (4.20)$$

$$\frac{A\lambda^2}{a^2} \psi_0 Lg - \frac{\phi_0}{R} f + \frac{\lambda}{2a} \phi_0^2 \frac{f^2}{\lambda - \xi} = 0. \quad (4.21)$$

The operator L has the form

$$L = \frac{d^2}{d\xi^2} - \frac{1}{\lambda - \xi} \frac{d}{d\xi} - \frac{1}{(\lambda - \xi)^2}. \quad (4.22)$$

Introduction of (4.18) and (4.19) into the boundary conditions (4.12) and (4.13) gives the following boundary conditions for f and g :

$$\xi = 0: \quad \begin{cases} \frac{\lambda}{a} \psi_0 \left[\frac{dg}{d\xi} + \frac{\nu g}{\lambda - \xi} \right] = \frac{1 - \nu}{2} pR \\ \frac{df}{d\xi} - \frac{\nu f}{\lambda - \xi} = 0 \end{cases} \quad (4.23)$$

$$\xi = \lambda: \quad \frac{dg}{d\xi} + \frac{\nu g}{\lambda - \xi} = 0, \quad f = 0. \quad (4.24)$$

In as much as the solutions f and g are to serve the purpose of satisfying the first of the two boundary conditions (4.23) we dispose of the three parameters ϕ_0 , ψ_0 and λ in such a manner that $dg/d\xi$ in (4.23) and $d^2g/d\xi^2$ in (4.21) remain of importance for the boundary value problem. This means that we set

$$\frac{\lambda}{a} \psi_0 = \frac{pR}{2}, \quad \frac{A\lambda^2}{a^2} \psi_0 = \frac{\phi_0}{R}. \quad (4.25)$$

Introduction of ψ_0 and ϕ_0 , in terms of λ and the other parameters, into (4.20)–(4.24) leads to the following form of the boundary value problem:

$$DAR^2 \frac{\lambda^4}{a^4} Lf + g - \frac{1}{2} pR^3 A \frac{\lambda^2}{a^2} \left(f + \frac{fg}{\lambda - \xi} \right) = 0, \quad (4.26)$$

$$Lg - f + \frac{1}{2} pR^3 A \frac{\lambda^2}{a^2} \frac{\frac{1}{2} f^2}{\lambda - \xi} = 0, \quad (4.27)$$

$$\xi = 0: \quad \frac{dg}{d\xi} + \frac{\nu g}{\lambda - \xi} = 1 - \nu, \quad \frac{df}{d\xi} - \frac{\nu f}{\lambda - \xi} = 0, \quad (4.28)$$

$$\xi = \lambda: \quad \frac{dg}{d\xi} + \frac{\nu g}{\lambda - \xi} = 0, \quad f = 0. \quad (4.29)$$

We now distinguish two cases, (i) and (ii), in regard to the choice of the two

remaining parameters in the differential equations (4.26) and (4.27). Case (i) is given when

$$DAR^2 \frac{\lambda^4}{a^4} = 1. \quad (4.30)$$

From (4.30) follows, with suitably defined parameters ρ and κ ,

$$\lambda = \frac{a}{\sqrt[4]{DAR^2}} = 2\sqrt[4]{3(1-\nu^2)} \sqrt{\frac{H}{h}} = \rho \quad (4.31)$$

and

$$\frac{1}{2}pR^3A \frac{\lambda^2}{a^2} = pR^2 \sqrt{\frac{A}{D}} = \sqrt{12(1-\nu^2)} \frac{pR^2}{Eh^2} = 4\kappa. \quad (4.32)$$

The differential equations of the problem are now

$$Lf + g - 4\kappa \left(f + \frac{fg}{\rho - \xi} \right) = 0, \quad (4.33)$$

$$Lg - f + 4\kappa \frac{\frac{1}{2}f^2}{\rho - \xi} = 0 \quad (4.34)$$

and the boundary conditions (4.28) and (4.29) are

$$\xi = 0: \quad \frac{dg}{d\xi} + \frac{\nu g}{\rho - \xi} = 1 - \nu, \quad \frac{df}{d\xi} - \frac{\nu f}{\rho - \xi} = 0, \quad (4.35)$$

$$\xi = \rho: \quad \frac{dg}{d\xi} + \frac{\nu g}{\rho - \xi} = 0, \quad f = 0. \quad (4.36)$$

We assume that as long as ξ is of the order of magnitude unity—more briefly written as long as $\xi = O(1)$ —that f and g and their derivatives with respect to ξ are all $O(1)$. The form of the differential equations and boundary conditions indicates that this requires for the present Case (i) that κ is restricted by the condition

$$\kappa = O(1). \quad (4.37)$$

Now, as long as (4.37) is satisfied we shall have the existence of a narrow edge zone (which is the region given by $\xi = O(1)$) as long as

$$1 \ll \rho. \quad (4.38)$$

When (4.38) is satisfied in addition to (4.37), the boundary value problem (4.33)–(4.36) effectively reduces to the following simplified form:

$$\left. \begin{aligned} \frac{d^2 f}{d\xi^2} + g - 4\kappa f &= 0, & \frac{d^2 g}{d\xi^2} - f &= 0, \\ \xi = 0: & \quad \frac{dg}{d\xi} = 1 - \nu, & \frac{df}{d\xi} &= 0, \\ \xi = \infty: & \quad \frac{dg}{d\xi} = 0, & f &= 0. \end{aligned} \right\} \quad (4.39)$$

Turning now to Case (ii) we set in (4.26) and (4.27)

$$\frac{1}{2}pR^3A\frac{\lambda^2}{a^2} = 1. \quad (4.40)$$

From (4.40) follows, in terms of the parameters ρ and κ ,

$$\lambda = \frac{a\sqrt{2}}{\sqrt{pR^3A}} = \frac{\rho}{\sqrt{2\kappa}} \quad (4.41)$$

and

$$DAR^2\frac{\lambda^4}{a^4} = \frac{1}{4\kappa^2}. \quad (4.42)$$

Therewith the system (4.26)–(4.29) becomes

$$\frac{1}{4\kappa^2}Lf + g - f - \frac{fg}{\sqrt{\rho^2/2\kappa} - \xi} = 0, \quad (4.43)$$

$$Lg - f + \frac{\frac{1}{2}f^2}{\sqrt{\rho^2/2\kappa} - \xi} = 0, \quad (4.44)$$

$$\xi = 0: \quad \frac{dg}{d\xi} + \frac{\nu g}{\sqrt{\rho^2/2\kappa} - \xi} = 1 - \nu, \quad \frac{df}{d\xi} - \frac{\nu f}{\sqrt{\rho^2/2\kappa} - \xi} = 0, \quad (4.45)$$

$$\xi = \frac{\rho}{\sqrt{2\kappa}}: \quad \frac{dg}{d\xi} + \frac{\nu g}{\sqrt{\rho^2/2\kappa} - \xi} = 0, \quad f = 0. \quad (4.46)$$

Condition (4.37) for κ is now replaced by

$$\frac{1}{\kappa^2} = O(1). \quad (4.47)$$

Existence of a relatively narrow edge zone now requires validity of the following order of magnitude relation:

$$1 \ll \frac{\rho}{\sqrt{\kappa}} \quad (4.48)$$

which, in view of (4.47), is more restrictive than the corresponding relation (4.38) for Case (i).

When both (4.47) and (4.48) are satisfied then the system (4.43)–(4.46) simplifies to

$$\left. \begin{aligned} \frac{1}{4\kappa^2}\frac{d^2f}{d\xi^2} + g - f &= 0, & \frac{d^2g}{d\xi^2} - f &= 0, \\ \xi = 0: & \quad \frac{dg}{d\xi} = 1 - \nu, & \frac{df}{d\xi} &= 0, \\ \xi = \infty: & \quad \frac{dg}{d\xi} = 0, & f &= 0. \end{aligned} \right\} \quad (4.49)$$

A comparison of the forms of the systems (4.39) and (4.49) reveals that as long as one is solely concerned with those cases of the problem for which deviations from linear membrane theory are confined to a relatively narrow edge zone, one and the same system of *linear* differential equations and boundary conditions is appropriate, regardless of the magnitude of the load parameter κ . In view of equations (4.38) and (4.48) the condition for this unified system to be applicable can be written in the form

$$\text{Max} (1, \sqrt{\kappa}) \ll \rho. \quad (4.50)$$

As long as (4.50) is satisfied we have then, in terms of the original variables ϕ and ψ , and with $\psi_M = \frac{1}{2}pRr$, the following boundary value problem for the determination of the state of stress and deformation in the shell:

$$\left. \begin{aligned} D\phi'' + \frac{\psi}{R} - \psi_M \frac{\phi}{r} &= \frac{1}{2}pr \\ A\psi'' - \frac{\phi}{R} &= 0 \\ r = a: \quad \phi' &= 0, \quad \psi' = \nu \frac{\psi_M}{r} \\ r = -\infty: \quad \phi' &= 0, \quad \psi = \psi_M \end{aligned} \right\} \quad (4.51)$$

Returning to the dimensionless forms (4.49) and (4.39) of the system (4.51) the following further conclusions may be drawn:

(1) If in addition to $1 \ll \rho$ we have $\kappa \ll 1$ then deviations from linear membrane theory are effectively given by linear bending theory [3].

(2) If in addition to $\sqrt{\kappa} \ll \rho$ we have $1 \ll \kappa$ then deviations from linear membrane theory are effectively given by non-linear membrane theory (in which the boundary conditions on f are discarded) [4].

(3) If neither $\kappa \ll 1$ nor $1 \ll \kappa$ then both linear bending action and non-linear membrane action must simultaneously be considered [2].

Numerical calculations. In order to estimate the numerical magnitude of the edge effect we have calculated the maximum meridional bending stress $\sigma_{rB, \max}$ in terms of the corresponding stress $\sigma_{rBO, \max}$ according to linear bending theory as a function of the load parameter κ . We have also calculated the distance y from the edge of the shell where this maximum stress occurs in terms of the corresponding distance y_0 according to linear bending theory.

According to linear bending theory,

$$\begin{aligned} y_0 &= \frac{\pi}{4} \frac{\sqrt{Rh}}{\sqrt[4]{12(1-\nu^2)}}, \\ \sigma_{rBO, \max} &= \sqrt{\frac{3}{2} \frac{1-\nu}{1+\nu}} e^{-\frac{1}{2}\pi} \frac{pa}{2h}. \end{aligned} \quad (4.52)$$

Solution of equations (4.51) leads to the following formulas for the maximum bending stress and for the distance from the edge where this stress occurs:

$$\frac{y}{y_0} = \frac{2}{\pi} \frac{\ln(\kappa + \sqrt{\kappa^2 - 1})}{\sqrt{\kappa - 1}}, \quad (4.53)$$

$$\frac{\sigma_{rB, \max}}{\sigma_{rBO, \max}} = \frac{e^{\frac{1}{2}\pi}}{\sqrt{2(\kappa^2 - 1)}} \left\{ \exp \left[-\sqrt{\kappa - \sqrt{\kappa^2 - 1}} \frac{\ln(\kappa + \sqrt{\kappa^2 - 1})}{\sqrt{2(\kappa - 1)}} \right] \right. \\ \left. - \exp \left[-\sqrt{\kappa + \sqrt{\kappa^2 - 1}} \frac{\ln(\kappa + \sqrt{\kappa^2 - 1})}{\sqrt{2(\kappa - 1)}} \right] \right\}. \quad (4.54)$$

Equations (4.53) and (4.54) are derived by means of the system (4.51) or, equivalently, by means of the dimensionless system (4.39), with f and g related to ϕ and ψ through equations (4.18), (4.19), (4.25), (4.30), (4.31) and (4.32). The formula for the meridional bending stress in the narrow edge zone is of the form

$$\sigma_{rB} = \mp \frac{hE}{2(1 - \nu^2)} \frac{d\phi}{dr} = \pm \frac{Eh\phi_0\lambda}{2(1 - \nu^2)a} \frac{df}{d\xi} \quad (4.55)$$

and according to (4.32) the quantity κ is given by

$$\kappa = \frac{\sqrt{3(1 - \nu^2)}}{2} \frac{pR^2}{Eh^2}. \quad (4.56)$$

Evaluation of (4.53) and (4.54) in the range $\kappa < 1$ is effected by writing

FIGURE 3.

$\kappa = \cos \chi$ and $\sqrt{\kappa^2 - 1} = i\sqrt{1 - \kappa^2} = i \sin \chi$. In this way equations (4.53) and (4.54) are in real form in the range $\kappa < 1$:

$$\frac{y}{y_0} = \frac{\sqrt{2}}{\pi} \frac{\chi}{\sin \frac{1}{2}\chi}, \quad \frac{\sigma_{rB, \max}}{\sigma_{rBO, \max}} = \frac{\exp(\frac{1}{4}\pi - \frac{1}{2}\chi \cot \frac{1}{2}\chi)}{\sqrt{2} \cos \frac{1}{2}\chi}. \quad (4.57)$$

For sufficiently large values of κ the following approximate formulas hold:

$$\frac{y}{y_0} \sim \frac{2}{\pi} \frac{\ln 2\kappa}{\sqrt{\kappa}}, \quad \frac{\sigma_{rB, \max}}{\sigma_{rBO, \max}} \sim \frac{e^{\frac{1}{2}\pi}}{\sqrt{2\kappa}}. \quad (4.58)$$

Numerical values for distance ratio and stress ratio may be found in Figure 3. It is apparent that linear bending theory overestimates the maximum bending stress appreciably, even for relatively small values of κ . When $\kappa = 0.25$ the present theory gives a bending stress 17 per cent smaller than the corresponding stress of linear bending theory. On the other hand, this bending stress is still appreciable even for quite large values of κ . In the limit of very large values of κ non-linear membrane theory becomes applicable in the sense that bending stresses become negligible compared with direct stresses.*

5. Linear theory of orthotropic uniform shallow spherical shell with uniform axial load. We now consider a shell for which equations (3.19) and (3.24) reduce to the following form:

$$A_\theta \left[\psi'' + \frac{\psi'}{r} \right] - A_r \frac{\psi}{r^2} - \frac{\phi}{R} = 0, \quad (5.1)$$

$$D_r \left[\phi'' + \frac{\phi'}{r} \right] - D_\theta \frac{\phi}{r^2} + \frac{\psi}{R} = \frac{1}{2}pr. \quad (5.2)$$

Stress resultants N_r , N_θ and Q are again given by (4.9) except that the non-linear term in Q is absent. Stress couples M_r and M_θ are now

$$M_r = -D_r \phi' - D_{r\theta} \frac{\phi}{r}, \quad M_\theta = -D_\theta \frac{\phi}{r} - D_{\theta r} \phi' \quad (5.3)$$

and the radial displacement u is given by

$$\frac{u}{r} = A_\theta \psi' + A_{\theta r} \frac{\psi}{r}. \quad (5.4)$$

We may, as for isotropic shells, consider a shell with vanishing wall bending stiffness by setting in (5.2)

$$D_r = D_\theta = 0.$$

*It should be noted that this latter conclusion is of limited significance in the sense that it depends not only on the order of magnitude of the quantity κ but also on the nature of the boundary conditions which have been assumed. For instance, it may be shown that for the case in which the boundary condition of simple support is replaced by the condition of a built-in edge the maximum bending stress never becomes small compared to the maximum direct stress, no matter how large κ is.

If this is done we obtain from (5.2) the linear membrane solution

$$\psi_M = \frac{1}{2}pRr. \quad (5.5)$$

We may then introduce ψ_M in equation (5.1) in order to obtain the corresponding change of slope function

$$\phi_M = \frac{1}{2}pR^2 \frac{A_\theta - A_r}{r}. \quad (5.6)$$

In as much as ϕ_M becomes infinite as r approaches zero, unless $A_\theta = A_r$, we conclude that, in contrast to what is the case for isotropic shells, linear membrane theory may not always be adequate to describe the state of stress due to continuous surface load distributions in the interior of a polar orthotropic shell of revolution.

In order to establish the nature of the actual solution of a given boundary value problem in the interior of the shell it is necessary to consider the complete differential equations (5.1) and (5.2). In the following we discuss two cases of this problem which are mathematically somewhat more convenient than the general case.

Pseudo-membrane shell. We consider a class of shells which shares with the class of membrane shells the property of leading to one second-order differential equation rather than to two simultaneous second-order differential equations, and which we therefore call pseudo-membrane shells. We define a pseudo-membrane shell of revolution by the equation

$$D_r = 0. \quad (5.7)$$

This reduces the differential equation (5.2) from second-order to zeroth-order, as follows:

$$D_\theta \frac{\phi}{r^2} = \frac{\psi}{R} - \frac{1}{2}pr. \quad (5.8)$$

Introduction of (5.8) into (5.1) leads to a second-order equation for ψ , of the form

$$A_\theta \left[\psi'' + \frac{\psi'}{r} \right] - \left[A_r + \frac{r^4}{D_\theta R^2} \right] \frac{\psi}{r^2} = -\frac{pr^3}{2D_\theta R}. \quad (5.9)$$

The solution of the homogeneous part of (5.9) may be expressed in terms of modified Bessel functions:

$$\psi_h = c_1 I_q \left(\frac{r^2}{L^2} \right) + c_2 K_q \left(\frac{r^2}{L^2} \right) \quad (5.10)$$

where

$$q = \frac{1}{2} \sqrt{\frac{A_r}{A_\theta}}, \quad L^2 = 2 \sqrt{A_\theta D_\theta} R. \quad (5.11)$$

A particular solution of the complete equation (5.9) which is regular at the origin is as follows:

$$\psi_p = \frac{1}{2}pRL \left\{ \int_0^{r^2/L^2} \eta^3 I_q(\eta) d\eta K_q\left(\frac{r^2}{L^2}\right) + \int_{r^2/L^2}^{a^2/L^2} \eta^3 K_q(\eta) d\eta I_q\left(\frac{r^2}{L^2}\right) \right\}. \quad (5.12)$$

The solution (5.10)–(5.12) reduces to elementary form when $q = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \text{etc.}$ The simplest case is given when $q = \frac{1}{2}$, that is, when $A_r = A_\theta$. For this case the particular solution ψ_p becomes $\frac{1}{2}pRr$, which is the same as for the case $D_\theta = 0$. The next simplest case corresponds to $q = \frac{3}{2}$ which corresponds to $A_r = 9A_\theta$. For this case the particular solution becomes an elementary function and the general solution of (5.9) may be written in the form

$$q = \frac{3}{2}; \quad \psi = \frac{1}{2}pRL \left\{ \frac{r}{L} - 2 \frac{L^3}{r^3} + c_1 \frac{L}{r} \left(1 + \frac{L^2}{r^2} \right) e^{-r^2/L^2} + c_2 \frac{L}{r} \left(1 - \frac{L^2}{r^2} \right) e^{r^2/L^2} \right\}. \quad (5.13)$$

The following general observations may be made. In view of the behavior of the functions I_q and K_q for small r we have that the particular solution ψ_p in (5.12) behaves like r^5 for small values of r . The regular part of the homogeneous solution (5.10) behaves like r^{2q} for small values of r . Accordingly, the stress resultants N_r and N_θ associated with the homogeneous solution behave like r^{2q-1} for small values of r . This means that when $A_r > A_\theta$ we have vanishing N_r and N_θ at the origin and when $A_r < A_\theta$ we have that N_r and N_θ become infinite at the origin.

Homogeneous orthotropic shell. A second class of shells for which an explicit solution of the system (5.1) and (5.2) is possible is the class for which

$$\frac{A_r}{A_\theta} = \frac{D_\theta}{D_r} \equiv \rho^2. \quad (5.14)$$

Equation (5.14) is satisfied for homogeneous shells since for these $A_r = 1/E_r h$, $A_\theta = 1/E_\theta h$, $D_\theta = E_\theta h^3/12$, $D_r = E_r h^3/12$.

Writing (5.1) and (5.2) in the form

$$\psi'' + \frac{\psi'}{r} - \rho^2 \frac{\psi}{r^2} - \frac{\phi}{A_\theta R} = 0, \quad (5.15)$$

$$\phi'' + \frac{\phi'}{r} - \rho^2 \frac{\phi}{r^2} + \frac{\psi}{D_r R} = \frac{pr}{2D_r}, \quad (5.16)$$

we are led to introduce a complex function χ , containing both ϕ and ψ , given by

$$\chi = \phi + i\mu\psi. \quad (5.17)$$

If we choose

$$\mu = \sqrt{\frac{A_\theta}{D_r}} \quad (5.18)$$

and introduce a quantity λ defined by

$$\lambda^2 = \frac{1}{R\sqrt{A_\theta D_r}} \quad (5.19)$$

then χ is solution of the following complex second order equation

$$\chi'' + \frac{1}{r}\chi' - \left(i\lambda^2 + \frac{\rho^2}{r^2}\right)\chi = \frac{pr}{2D_r}. \quad (5.20)$$

The solution of (5.20) can be written in the form

$$\begin{aligned} \chi = c_1 I_\rho(\sqrt{i}\lambda r) + c_2 K_\rho(\sqrt{i}\lambda r) + \frac{p}{2D_r} \left\{ \int_0^r \eta^2 I_\rho(\sqrt{i}\lambda \eta) d\eta K_\rho(\sqrt{i}\lambda r) \right. \\ \left. + \int_r^a \eta^2 K_\rho(\sqrt{i}\lambda \eta) d\eta I_\rho(\sqrt{i}\lambda r) \right\}. \end{aligned} \quad (5.21)$$

The integrals in (5.21) can be evaluated explicitly when $\rho = 1$, which is the case of isotropic shells. This case, however, is more simply solved directly. When $\rho = \frac{1}{2}, \frac{3}{2}, \text{etc.}$ it is possible to reduce the integrals in (5.21) in such a way that they have the appearance of Fresnel integrals.

The following qualitative observations may be made on the basis of the differential equations (5.15) and (5.16) and their solutions as given by (5.17) and (5.21). For small values of r we have that the particular solutions ϕ_ρ and ψ_ρ behave like r^3 and r^5 , respectively. At the same time the regular part of the solution of the homogeneous equation behaves like r^ρ . Accordingly, when $\rho > 1$ we have that stress resultants and couples vanish for $r = 0$, while when $1 < \rho$ stress resultants and couples become infinite for $r = 0$. This means that the solution for the case $\rho = 1$ differs in a fundamental way from the solution for the other cases and the familiar notion of "membrane solution" in the interior is one which does not have the same significance for orthotropic shells which it has for isotropic shells.

Numerical example. In order to illustrate the nature of the effect which is encountered for orthotropic shells we consider a pseudo-membrane shell and assume the same boundary conditions which for an isotropic shell lead to the result of linear membrane theory. These boundary conditions are

$$\psi(0) = 0, \quad \psi(a) = \frac{1}{2}pRa. \quad (5.22)$$

The first of these conditions means that $c_2 = 0$ in (5.10). The second condition determines c_1 in the form

$$c_1 I_\rho\left(\frac{a^2}{L^2}\right) + \frac{1}{2}pRL \int_0^{a^2/L^2} \eta^{\frac{3}{2}} I_\rho(\eta) d\eta K_\rho\left(\frac{a^2}{L^2}\right) = \frac{1}{2}pRa. \quad (5.23)$$

In the actual evaluation we are limiting ourselves to the case $q = \frac{3}{2}$, in accordance with equation (5.13). For this case values of the stress resultants N_r and N_θ , in terms of the value $\frac{1}{2}pR$ of these resultants for an isotropic shell, are given in

FIGURE 4.

FIGURE 5.

Figures 4 and 5. It is found that the nature of the stress distribution depends on the value of the dimensionless parameter

$$\frac{a^2}{L^2} = \frac{a^2}{2R\sqrt{A_\theta D_\theta}}$$

the significance of which is best realized for the case that $A = 1/E_\theta h$ and $D = E_\theta h^3/12$. For this case $a^2/L^2 = \frac{1}{4}\sqrt{3} a^2/Rh$. As the values of a^2/L^2 increase the stress resultants N_r and N_θ more and more approach the values which they would have according to linear membrane theory, except in a neighborhood of the apex of the shell. While N_r approaches the value $\frac{1}{2}pR$ from below we have that N_θ approaches this value from above, with a localized overshoot of about 25 per cent, no matter how large the value of a^2/L^2 .

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