Representation of Functions by Convolutions

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1. Introduction. To state the problem with which we are concerned in this paper in its simplest form, consider the class $L^1(R^1)$, *i.e.*, the set of all complex-valued functions on the real line R^1 which are integrable on R^1 in the sense of Lebesgue. If g and h belong to $L^1(R^1)$, it is well known that the integral

$$(1.1) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)h(t) dt$$

exists for almost all $x \in R^1$ and represents a member f of $L^1(R^1)$ which is called the convolution of g and h ; in symbols

$$(1.2) \quad f = g * h.$$

The Fourier transform of $f \in L^1(R^1)$ will be denoted by $\hat{f}$:

$$(1.3) \quad \hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iyx} dx \quad (y \in R^1).$$

The relation (1.2) is then equivalent to

$$(1.4) \quad \hat{f}(y) = \hat{g}(y)\hat{h}(y) \quad (y \in R^1).$$

The problem under consideration is the following: Which functions $f \in L^1(R^1)$ can be represented in the form (1.2) (for almost all x) with g and h in $L^1(R^1)$? Equivalently, which Fourier transforms are products of Fourier transforms, as in (1.4)? The answer is that *every function in $L^1(R^1)$ is a convolution*. That is to say, every element of the convolution algebra $L^1(R^1)$ can be factored in $L^1(R^1)$, although this algebra lacks a unit. There are no primes in $L^1(R^1)$.

This problem can of course be considered in a wider setting. If G is any locally compact abelian group (we write the group operation as addition) and $\hat{G}$ is the group of all continuous characters of G , there is a translation invariant measure (Haar measure) on G , and we denote by $L^1(G)$ the set of all complex functions on G which are integrable with respect to this measure. Convolution

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in $L^1(G)$ is defined by

$$(1.5) \quad (g * h)(x) = \int_G g(x - t)h(t) dt,$$

where dt denotes the Haar measure. The integral again exists for almost all x . The Fourier transform of any $f \in L^1(G)$ is given by

$$(1.6) \quad \hat{f}(y) = \int_G f(x)y(-x) dx \quad (y \in \hat{G}).$$

The relation $f = g * h$ is again equivalent to $\hat{f}(y) = \hat{g}(y)\hat{h}(y)$. (For the theory of Fourier analysis on groups see [7] or [3]; only the simplest results of this theory will be needed in the present paper.)

Let us note in passing that if $G = R^1$ then $\hat{G} = R^1$, and the usual way of writing $y(x)$ is e^{ix} .

We will be concerned with the following groups:

- (a) n -dimensional Euclidean space R^n , with vector addition as group operation.
- (b) The n -dimensional torus T^n , which is a homomorphic image of R^n , obtained by identifying any two points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ of R^n such that $x_i - y_i$ is an integral multiple of 2π for all i ($1 \leq i \leq n$).
- (c) More generally, locally Euclidean groups, i.e., topological groups in which the identity element has a neighborhood which is homeomorphic to R^n , for some n .

Definition. For any G , let $P(G)$ be the following proposition: *Every member of $L^1(G)$ is the convolution of two members of $L^1(G)$.*

The principal result of the present paper is now easily stated:

Theorem 1. *$P(G)$ is true for every locally Euclidean abelian group.*

The case $G = R^1$ of this theorem has been published previously [5]. It is presented again in Part II, since the idea of the proof appears here in its simplest form. The details are considerably more involved in the case $G = R^n$; this is due partly to the fact that it is advantageous to replace the Fejér kernel by the Riesz kernel in the study of Fourier integrals on R^n (see, for instance, [1]). The relevant lemmas concerning functions of one variable are derived in Part III, and Fourier transforms on R^n are discussed in Part IV; here Theorem 3 might be of independent interest. Once $P(R^n)$ is proved, $P(G)$ follows fairly easily for any locally Euclidean abelian group.

2. The Case $G = R^1$. Let $K_t(x)$ be the Fejér kernel

$$(2.1) \quad K_t(x) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos tx}{tx^2} \quad (t > 0, x \in R^1),$$

whose Fourier transform is

$$(2.2) \quad \hat{K}_t(y) = \begin{cases} 1 - \frac{|y|}{t} & (|y| \leq t), \\ 0 & (|y| > t). \end{cases}$$

Fix an $f \in L^1(R^1)$ and put

$$(2.3) \quad \epsilon(t) = \|f * K_t - f\| \quad (t > 0),$$

where the norm is defined by

$$(2.4) \quad \|f\| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| dx \quad (f \in L^1(R^1)).$$

It is well known and easily proved (see Theorem 2 below) that $\epsilon(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence there is a function α , positive and continuous on $(0, \infty)$, such that

$$(2.5) \quad \int_0^{\infty} \left\{ \epsilon(t) + \frac{1}{t} \right\} \alpha(t) dt < \infty$$

but

$$(2.6) \quad \int_0^{\infty} \alpha(t) dt = \infty.$$

Put

$$(2.7) \quad \phi(t) = 1 + \int_0^t \alpha(s) ds + t \int_t^{\infty} \frac{\alpha(s)}{s} ds \quad (t \geq 0).$$

By (2.6), $\phi(t) \rightarrow \infty$ as $t \rightarrow \infty$. By (2.5), $\phi(t) < \infty$.

Differentiation of (2.7) leads to

$$(2.8) \quad \phi''(t) = -\frac{\alpha(t)}{t} < 0,$$

so that ϕ is concave.

Now define

$$(2.9) \quad g(x) = f(x) + \int_0^{\infty} \{f(x) - (f * K_t)(x)\} \alpha(t) dt$$

The Fubini theorem, together with (2.3) and (2.5), shows that $g \in L^1(R^1)$ and that

$$(2.10) \quad g(y) = f(y) \left\{ 1 + \int_0^{\infty} [1 - \hat{K}_t(y)] \alpha(t) dt \right\}.$$

Substitution of (2.2) into (2.10) and comparison with (2.7) gives

$$(2.11) \quad g(y) = f(y) \phi(|y|),$$

so that

$$(2.12) \quad \hat{f}(y) = g(y)\lambda(|y|) \quad (y \in R^1),$$

where λ is a positive convex function on $(0, \infty)$ which tends to 0 at infinity. If

$$(2.13) \quad h(x) = \int_0^\infty K_t(x) t \lambda''(t) dt \quad (x \in R^1)$$

then $h \in L^1(R^1)$ and

$$(2.14) \quad \hat{h}(y) = \int_0^\infty \hat{K}_t(y) t \lambda''(t) dt = \int_{|y|}^\infty (t - |y|) \lambda''(t) dt = \lambda(|y|).$$

Thus (2.12) becomes

$$(2.15) \quad \hat{f}(y) = g(y) \hat{h}(y),$$

and we have proved that $P(R^1)$ is true.

The idea of this proof was quite simple: Find an even function ϕ whose reciprocal is a Fourier transform and whose behavior at infinity is such that $\hat{f}\phi$ is a Fourier transform; in particular, ϕ must not tend to infinity too rapidly.

3. The Differential Operator Δ . Let Δ be the operator defined by

$$(3.1) \quad \Delta f(t) = t^{-1} Df(t) = \frac{1}{t} \frac{df}{dt}.$$

As usual, Δ^k is the operator obtained by k -fold iteration of Δ , and Δ^0 is the identity operator.

Definition 1. For any non-negative integer p , M_p denotes the class of all real functions f on $(0, \infty)$ such that

- (i) $\lim_{t \rightarrow \infty} f(t) = 0$,
- (ii) $D^{p+1}f$ is continuous on $(0, \infty)$,
- (iii) $(-1)^k \Delta^k f(t) > 0$ on $(0, \infty)$ for $k = 0, 1, \dots, p+1$.

Definition 2. If $f \in M_p$ and if

- (iv) $\lim_{t \rightarrow 0} f(t)$ exists as a finite number,

then we say that $f \in M_p^*$.

Lemma 1. Let p be a non-negative integer. If α is a positive continuous function which is integrable on (δ, ∞) for every $\delta > 0$, and if f is defined by

$$(3.2) \quad f(t) = \int_t^\infty \left(1 - \frac{t^2}{s^2}\right)^p \alpha(s) ds \quad (0 < t < \infty),$$

then $f \in M_p$.

Conversely, if $f \in M_p$, then there exists a function α with the above properties such that f is represented by (3.2).

Proof. The first part of the lemma is easily established by direct computation, since differentiation of (3.2) gives

$$(-1)^k \Delta^k f(t) = \frac{2^k p!}{(p-k)!} \int_t^\infty \left(1 - \frac{t^2}{s^2}\right)^{p-k} \frac{\alpha(s)}{s^{2k}} ds$$

for $k = 0, \dots, p$, and thus

$$(-1)^{p+1} \Delta^{p+1} f(t) = 2^p p! \alpha(t) t^{-2p-1}.$$

The converse is trivial if $p = 0$: Simply put $\alpha(t) = -f'(t)$. We proceed by induction, assuming the converse to be proved for $p = q - 1$. If now $f \in M_q$, then $-\Delta f \in M_{q-1}$, so that

$$(3.3) \quad f'(t) = -t \int_t^\infty \left(1 - \frac{t^2}{s^2}\right)^{q-1} \beta(s) ds,$$

where β satisfies the conditions imposed on α in the statement of the lemma.

Since $f(t) \rightarrow 0$ as $t \rightarrow \infty$, we have

$$(3.4) \quad f(t) = -\int_t^\infty f'(u) du = \int_t^\infty \beta(s) ds \int_t^s \left(1 - \frac{u^2}{s^2}\right)^{q-1} u du.$$

The interchange in the order of integration is justified by the positivity of the integrand. The last integral in (3.4) can be evaluated explicitly and yields

$$(3.5) \quad f(t) = \frac{1}{2q} \int_t^\infty \left(1 - \frac{t^2}{s^2}\right)^q s^2 \beta(s) ds \quad (t > 0),$$

that is, (3.2) with $p = q$ and $\alpha(s) = s^2 \beta(s)/2q$. Integrability of α on (δ, ∞) follows from (3.5), and the induction is complete.

Lemma 2. Suppose ψ is a positive bounded continuous function on $(0, \infty)$ such that

$$(3.6) \quad \int_0^\infty \psi(s) ds = \infty.$$

If g is defined on $(0, \infty)$ by

$$(3.7) \quad g(t) = 1 + \int_0^t \psi(s) ds + \int_t^\infty \left\{1 - \left(1 - \frac{t^2}{s^2}\right)^p\right\} \psi(s) ds$$

for some positive integer p , and if $h(t) = 1/g(t)$, then $h \in M_p^*$.

Proof For any fixed t the integrand in the second integral of (3.7) is $O(s^{-2})$, and the boundedness of ψ implies that $g(t) < \infty$. By (3.6), $g(t) \rightarrow \infty$ as $t \rightarrow \infty$. Differentiation of (3.7) shows that

$$g'(t) = 2pt \int_t^\infty \left(1 - \frac{t^2}{s^2}\right)^{p-1} \frac{\psi(s)}{s^2} ds \quad (t > 0)$$

so that

$$(3.8) \quad \Delta g(t) = 2p \int_t^\infty \left(1 - \frac{t^2}{s^2}\right)^{p-1} \frac{\psi(s)}{s^2} ds \quad (t > 0).$$

By Lemma 1, $\Delta g \in M_{p-1}$. Thus $g(t) > 0$ and

$$(3.9) \quad (-1)^{k-1} \Delta^k g(t) > 0 \quad (1 \leq k \leq p+1; t > 0).$$

To complete the proof we have to show that

$$(3.10) \quad (-1)^k \Delta^k h(t) > 0 \quad (1 \leq k \leq p+1; t > 0).$$

Note that Δ satisfies the product law

$$(3.11) \quad \Delta(uv) = u \cdot \Delta v + v \cdot \Delta u.$$

Since $gh = 1$, $\Delta(gh) = 0$. Take $u = h$ and $v = g$ in (3.11); since g , Δg , and h are positive, Δh is negative, and (3.10) holds for $k = 1$.

Assume that (3.10) holds for $1 \leq k \leq r-1$, where $r \leq p+1$. Apply Δ^r to both sides of the equation $gh = 1$. By (3.11), we obtain

$$\sum_{i=0}^r \binom{r}{i} \Delta^i g \Delta^{r-i} h = 0$$

or

$$(3.12) \quad (-1)^r g \Delta^r h = \sum_{i=1}^r \binom{r}{i} [(-1)^{i-1} \Delta^i g][(-1)^{r-i} \Delta^{r-i} h].$$

By (3.9) and our induction hypothesis, all terms on the right of (3.12) are positive, so that (3.10) holds with $k = r$.

This completes the proof of the lemma.

The following result is not needed for our main purpose but may be useful in some applications.

Lemma 3. Suppose f is a real function on $(0, \infty)$ such that $f(t) \rightarrow 0$ as $t \rightarrow \infty$, $D^{p+1}f$ is continuous on $(0, \infty)$ and

$$(3.13) \quad (-1)^k D^k f(t) > 0 \quad (t > 0; k = 0, 1, \dots, p+1).$$

Then $f \in M_p$.

Proof. One shows easily by induction that

$$\Delta^k f(t) = \sum_{i=1}^k (-1)^{k+i} a_i t^{-2k+i} D^i f(t),$$

where $a_1, \dots, a_k$ are positive integers. Hence (3.13) implies that

$$(-1)^k \Delta^k f(t) > 0 \quad (k = 0, \dots, p+1).$$

4. Fourier Transforms on R^n . In this section n will be a fixed positive integer. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are two points of R^n , we write

$x \cdot y = \sum x_i y_i$. Also $|x| = (x \cdot x)^{\frac{1}{2}}$ will be the notation for the distance of x from 0, and dx denotes n -dimensional Lebesgue measure.

If $f \in L^1(R^n)$, its Fourier transform $\hat{f}$ is defined by

$$(4.1) \quad \hat{f}(y) = (2\pi)^{-\frac{1}{2}n} \int_{R^n} f(x) e^{-ix \cdot y} dx \quad (y \in R^n).$$

If also $\hat{f} \in L^1(R^n)$, then the inversion formula

$$(4.2) \quad f(x) = (2\pi)^{-\frac{1}{2}n} \int_{R^n} \hat{f}(y) e^{ix \cdot y} dy \quad (x \in R^n)$$

is valid. Convolution of two members of $L^1(R^n)$ is defined by

$$(4.3) \quad (g * h)(x) = (2\pi)^{-\frac{1}{2}n} \int_{R^n} g(x - u) h(u) du.$$

Now choose a fixed integer p such that

$$2p > n - 1,$$

put $r = \frac{1}{2}n + p$, and consider the function

$$(4.4) \quad K(x) = C \cdot |x|^{-r} J_r(|x|) \quad (x \in R^n),$$

where J_r is the Bessel function of the first kind, of order r , and C is a constant which will be determined presently. Since

$$\int_{R^n} |K(x)| dx = A \int_0^\infty s^{-r} |J_r(s)| s^{n-1} ds,$$

where A is another constant, and since

$$J_r(s) = \begin{cases} O(s^r) & (s \rightarrow 0) \\ O(s^{-\frac{1}{2}}) & (s \rightarrow \infty), \end{cases}$$

the inequality $r > n - \frac{1}{2}$ shows that $K \in L^1(R^n)$. Hence C in (4.4) can be chosen so that $\hat{K}(0) = 1$; then

$$(4.5) \quad \hat{K}(y) = \begin{cases} (1 - |y|^2)^p & (|y| \leq 1), \\ 0 & (1 < |y|). \end{cases}$$

The easiest way to see that the transform of K is indeed given by (4.5) is to apply the inverse transform (4.2) to (4.5); by formula (15) on p. 187 of [2], the inverse transform of (4.5) is

$$(4.6) \quad |x|^{-\frac{1}{2}n+1} \int_0^1 (1 - s^2)^p s^{\frac{1}{2}n} J_{\frac{1}{2}(n-2)}(|x|s) ds,$$

and formula (1) on p. 373 of [6] shows that (4.6) is equal to (4.4), with $C = p!2^p$.

The Riesz kernel mentioned in the introduction is defined by

$$(4.7) \quad Q_t(x) = t^n K(tx) \quad (x \in R^n; t > 0).$$

By (4.1),

$$(4.8) \quad \hat{Q}_t(y) = \hat{K}(y/t) \quad (y \in R^n; t > 0).$$

The following result is probably known, but I am unable to give a reference.

Theorem 2. *If $f \in L^1(R^n)$, then*

$$\lim_{t \rightarrow \infty} \|f * Q_t - f\| = 0,$$

where the norm is defined by

$$\|f\| = \frac{1}{(2\pi)^{1/2n}} \int_{R^n} |f(x)| dx \quad (f \in L^1(R^n)).$$

Proof. For any $x \in R^n$,

$$\begin{aligned} (f * Q_t)(x) - f(x) &= \frac{1}{(2\pi)^{1/2n}} \int_{R^n} \{f(x-u) - f(x)\} Q_t(u) du \\ &= \frac{1}{(2\pi)^{1/2n}} \int_{R^n} \left\{ f\left(x - \frac{v}{t}\right) - f(x) \right\} K(v) dv, \end{aligned}$$

where we put $tu = v$; consequently

$$\|f * Q_t - f\| \leq \frac{1}{(2\pi)^{1/2n}} \int_{R^n} |K(v)| dv \int_{R^n} \left| f\left(x - \frac{v}{t}\right) - f(x) \right| dx.$$

The inner integral is a bounded function of t and v and tends to 0 as $t \rightarrow \infty$ for every $v \in R^n$. The theorem now follows from Lebesgue's theorem on dominated convergence.

Theorem 3. *If $\lambda \in M_p^*$, then there exists a function $h \in L^1(R^n)$ such that $h(y) = \lambda(|y|)$ for all $y \in R^n$.*

Proof. By Lemma 1, there exists a positive continuous function α on $(0, \infty)$ such that

$$\lambda(|y|) = \int_{|y|}^{\infty} \left(1 - \frac{|y|^2}{s^2}\right)^p \alpha(s) ds = \int_0^{\infty} \hat{Q}_s(y) \alpha(s) ds,$$

using (4.8) and (4.5). As $y \rightarrow 0$, $\hat{Q}_s(y) \rightarrow 1$ monotonically; thus α is integrable on $(0, \infty)$, since

$$\int_0^{\infty} \alpha(s) ds = \lambda(0) < \infty.$$

If we now define

$$h(x) = \int_0^{\infty} Q_s(x) \alpha(s) ds$$

and note that $\|Q_s\| = \|K\|$ for all $s > 0$, the Fubini theorem shows that $h \in$

$L^1(R^n)$ and that

$$\hat{h}(y) = \int_0^\infty \hat{Q}_s(y) \alpha(s) ds = \lambda(|y|).$$

5. Proof of Theorem 1. It will be convenient to break the proof up into four steps.

Step 1. *The case $G = R^n$.*

Instead of proving $P(R^n)$, we shall prove the following slightly stronger result, which will simplify the work in Step 3.

Theorem 4. *Suppose $f_i \in L^1(R^n)$ for $i = 1, 2, 3, \dots$ and*

$$\sum_1^\infty \|f_i\| < \infty.$$

Then there exist functions g_i and h in $L^1(R^n)$ such that

$$\sum_1^\infty \|g_i\| < \infty \quad \text{and} \quad f_i = h * g_i \quad (i = 1, 2, 3, \dots).$$

(The norm used here was defined in Theorem 2.)

Proof. Put

$$\epsilon_i(t) = \|f_i * Q_t - f_i\| \quad (i = 1, 2, 3, \dots; t > 0),$$

where Q_t is the kernel defined by (4.7). Then

$$\epsilon_i(t) \leq \|f_i\| \cdot \{\|Q_t\| + 1\} = \|f_i\| \cdot \{\|K\| + 1\},$$

so that the series

$$\epsilon(t) = \sum_1^\infty \epsilon_i(t)$$

converges uniformly on $(0, \infty)$. It then follows from Theorem 2 that $\epsilon(t) \rightarrow 0$ as $t \rightarrow \infty$, so that there exists a function ψ on $(0, \infty)$ which satisfies the hypotheses of Lemma 2, as well as the additional condition

$$\int_0^\infty \psi(s) \epsilon(s) ds < \infty.$$

Define

$$(5.1) \quad g_i(x) = f_i(x) + \int_0^\infty \{f_i(x) - (f_i * Q_s)(x)\} \psi(s) ds \quad (x \in R^n).$$

Then

$$\|g_i - f_i\| \leq \int_0^\infty \epsilon_i(s) \psi(s) ds,$$

so that

$$\sum_1^{\infty} \|g_i\| \leq \sum_1^{\infty} \|f_i\| + \int_0^{\infty} \epsilon(s) \psi(s) ds < \infty.$$

We still have to show that $f_i = h * g_i$ for some $h \in L^1(R^n)$. By (5.1),

$$(5.2) \quad g_i(y) = f_i(y) \left\{ 1 + \int_0^{\infty} [1 - \hat{Q}_s(y)] \psi(s) ds \right\} \quad (y \in R^n).$$

The coefficient of $f_i(y)$ in (5.2) is equal to $\phi(|y|)$, where

$$\phi(t) = 1 + \int_0^t \psi(s) ds + \int_t^{\infty} \left\{ 1 - \left(1 - \frac{t^2}{s^2} \right)^p \right\} \psi(s) ds,$$

by (4.8) and (4.5). If $\lambda(t) = 1/\phi(t)$, Lemma 2 shows that $\lambda \in M_p^*$, and by Theorem 3 there exists $h \in L^1(R^n)$ such that $\hat{h}(y) = \lambda(|y|)$. By (5.2),

$$\hat{f}_i(y) = \hat{g}_i(y) \lambda(|y|) = \hat{g}_i(y) \hat{h}(y),$$

and Theorem 4 follows.

Corollary. *If $f_i \in L^1(R^n)$ for $i = 1, 2, 3, \dots$, there exist functions g_i and h in $L^1(R^n)$ such that $f_i = h * g_i$ ($i = 1, 2, 3, \dots$). I.e., each countable collection of elements of the algebra $L^1(R^n)$ has a common divisor (namely, h).*

Proof. Choose numbers $C_i > 0$ such that $\sum C_i \|f_i\| < \infty$, and apply Theorem 4 to the set $\{C_i f_i\}$ in place of $\{f_i\}$.

Step 2. *The case $G = T^n$.*

Let C^n be the set of all $x = (x_1, \dots, x_n) \in R^n$ such that

$$0 \leq x_i < 2\pi \quad (i = 1, \dots, n),$$

and let Λ^n be the set of all lattice points in R^n , i.e., the set of all $r = (r_1, \dots, r_n)$ with integral coordinates r_i . Every $x \in R^n$ has a unique representation of the form

$$x = v + 2\pi r \quad (v \in C^n, r \in \Lambda^n).$$

Now suppose $f \in L^1(T^n)$. We may regard f as a function defined on R^n which has period 2π in each of the n variables. Put

$$f_1(x) = \begin{cases} f(x) & (x \in C^n), \\ 0 & (x \notin C^n). \end{cases}$$

Then $f_1 \in L^1(R^n)$. The result of Step 1 shows that there are functions g_1 and $h_1 \in L^1(R^n)$ such that

$$f_1(x) = \int_{R^n} g_1(x - u) h_1(u) du$$

for almost all $x \in R^n$. Define, for $x \in R^n$,

$$g(x) = \sum_r g_1(x + 2\pi r), \quad h(x) = \sum_s h_1(x + 2\pi s),$$

the sums being extended over all lattice points r . If we replace g_1 by $|g_1|$ in the first of these sums and integrate the resulting series over C^n , we obtain a finite number, namely the integral of $|g_1|$ over R^n . Consequently, the series which defines g converges absolutely for almost all x , and $g \in L^1(C^n)$. Since g has period 2π in each of the n variables, g may be regarded as being defined on T^n ; then $g \in L^1(T^n)$. The same reasoning applies to the function h .

To prove $P(T^n)$, we shall show that the equation

$$(5.3) \quad f(x) = \int_{T^n} g(x - u)h(u) \, du$$

holds for almost all $x \in T^n$; here du denotes n -dimensional Lebesgue measure. In (5.3), T^n may be replaced by C^n , and $P(T^n)$ is then a consequence of the following computation, which is valid for almost all x :

$$\begin{aligned} \int_{C^n} g(x - u)h(u) \, du &= \int_{C^n} \sum_r g_1(x - u + 2\pi r) \sum_s h_1(u + 2\pi s) \, du \\ &= \sum_{r,s} \int_{C^n} g_1[x + 2\pi(r + s) - (u + 2\pi s)]h_1(u + 2\pi s) \, du \\ &= \sum_{s,t} \int_{C^n + 2\pi s} g_1(x + 2\pi t - u)h_1(u) \, du \\ &= \sum_t \int_{R^n} g_1(x + 2\pi t - u)h_1(u) \, du \\ &= \sum_t f_1(x + 2\pi t) = f(x). \end{aligned}$$

In the above summations, r , s , and t range independently over the lattice points in R^n .

Step 3. Let G_0 be the component of the identity in a locally Euclidean abelian group G . We shall show that $P(G)$ is true if G_0 is isomorphic to R^n .

Since G_0 is an open and closed subgroup of G , the quotient group G/G_0 is discrete. Hence, if $f \in L^1(G)$, there are only a countable number of cosets modulo G_0 , say $H_1, H_2, H_3, \dots$, on which f does not vanish almost everywhere. Let k_i be the characteristic function of H_i , and put

$$(5.4) \quad f_i(x) = f(x)k_i(x) \quad (x \in G).$$

Choose points $x_i \in H_i$, and define functions F_i on G_0 by setting

$$(5.5) \quad F_i(x) = f_i(x + x_i) \quad (x \in G_0).$$

Since $|f| = \sum |f_i|$ and $f \in L^1(G)$, we see that

$$(5.6) \quad \sum_1^\infty \|F_i\| < \infty.$$

The norm in (5.6) is that of $L^1(G_0)$, i.e., $L^1(R^n)$. By Theorem 4, there are functions g_i and h in $L^1(G_0)$ such that

$$(5.7) \quad \sum_1^\infty \|g_i\| < \infty$$

and

$$(5.8) \quad F_i(x) = \int_{G_0} g_i(x - u)h(u) du$$

for almost all $x \in G_0$; we assume that the Haar measure on G is so chosen that it coincides with Lebesgue measure on G_0 .

Extend the functions g_i and h to G by defining $g_i(x) = h(x) = 0$ outside G_0 , and put

$$(5.9) \quad g(x) = \sum_1^\infty g_i(x - x_i) \quad (x \in G).$$

By (5.7), $g \in L^1(G)$. For almost all $x \in H_i$, (5.8) shows that

$$(5.10) \quad f_i(x) = F(x - x_i) = \int_G g_i(x - x_i - u)h(u) du.$$

Since $f = \sum_1^\infty f_i$, addition of the equations (5.10) gives

$$(5.11) \quad f(x) = \int_G g(x - u)h(u) du$$

for almost all $x \in G$, by (5.9). This proves $P(G)$.

Step 4. *The general case of Theorem 1.*

Let G_0 be the component of the identity of a locally Euclidean abelian group G . Then G_0 is an abelian, connected, locally Euclidean group, hence is isomorphic to the direct sum $T^n \oplus R^k$ for some integers $n \geq 0$, $k \geq 0$ [4; p. 170]. The quotient group G/G_0 is discrete. It follows [7; p. 95] that G is isomorphic to the direct sum

$$T^n \oplus R^k \oplus D,$$

where D is discrete. Let $H = R^{n+k} \oplus D$. If $f(x, y, z)$ is defined on G , where $x \in T^n$, $y \in R^k$, $z \in D$, then f may be regarded as a function defined on $H((x, y) \in R^{n+k}, z \in D)$ which has period 2π in each of the variables $x_1, \dots, x_n$.

By Step 3, $P(H)$ is true. The proof of $P(G)$ now proceeds exactly as in Step 2, where $P(T^n)$ was deduced from $P(R^n)$.

REFERENCES

1. SALOMON BOCHNER, Summation of multiple Fourier series by spherical means, *Trans. Amer. Math. Soc.* **40** (1936) pp. 175–207.
2. ———, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932.
3. L. H. LOOMIS, *An Introduction to Abstract Harmonic Analysis*, New York, 1953.
4. L. PONTRJAGIN, *Topological Groups*, Princeton, 1939.
5. WALTER RUDIN, Factorization in the group algebra of the real line, *Proc. Nat. Acad. Sci.* **43** (1957) pp. 339–340.
6. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2d. ed., Cambridge, 1944.
7. ANDRÉ WEIL, *L'intégration dans les groupes topologiques*, Paris, 1938.

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