Properties of Solutions of Certain Differential Equations in Three Variables[†]

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1. Integral operators in the theory of partial differential equations. The possibility of mapping the linear space of harmonic functions of two variables onto the algebra of analytic functions of a complex variable is used successfully in many investigations. As has been shown previously, similar mappings can be defined for linear partial differential equations in two variables and they yield various results. See [1, 3, 4, 8]. It is of interest to generalize this approach when studying solutions of differential equations

$$(1) \quad \mathbf{L}_N(\psi) = \Delta_N \psi + \sum_{n=1}^N A_n \psi_{x_n} + \tilde{\mathbf{F}} \psi = 0,$$

$$\Delta_N \psi \equiv \sum_{n=1}^N \frac{\partial^2 \psi}{\partial x_n^2}, \quad \psi_{x_n} \equiv \frac{\partial \psi}{\partial x_n}$$

in three and more variables with *entire* coefficients A_n , $\tilde{\mathbf{F}}$.

When “translating” various results in the theory of functions into theorems in the theory of partial differential equations it is important to show that various relations which one obtains for ψ are independent of the coefficients A_n and $\tilde{\mathbf{F}}$ of the equation (1), or depend only on some properties of A_n and $\tilde{\mathbf{F}}$, as is the case for differential equations in two variables.

Most of these considerations use integral operators transforming analytic functions into solutions ψ of $\mathbf{L}(\psi) = 0$. When considering partial differential equations in two variables it is of interest to derive *various* integral operators *for the same equation*, since different operators permit us to derive *different* properties of solutions. In the theory of differential equations in three variables,

$$(1a) \quad \mathbf{L}_3(\psi) \equiv \Delta_3 \psi + \tilde{\mathbf{F}} \psi \equiv \psi_{x_1 x_1} + \psi_{x_2 x_2} + \psi_{x_3 x_3} + \tilde{\mathbf{F}}(x_1, x_2, x_3) \psi \\ \equiv \psi_{xx} + \psi_{zz^*} + F(X, Z, Z^*) \psi = 0,$$

$$X = x_1, \quad Z = \frac{1}{2}(x_3 + ix_2), \quad Z^* = \frac{1}{2}(x_3 - ix_2), \quad \psi_{xx} \equiv \frac{\partial^2 \psi}{\partial X^2}, \dots,$$

only that case of equation (1a) has been considered where $\tilde{\mathbf{F}}$ is an entire function of $(x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$. See [2, 3, 5].

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In the present paper the case is investigated where $\tilde{F} \equiv \tilde{F}(x_2, x_3) \equiv F(Z, Z^*)$ is a function of Z, Z^* only and is entire when continued to arbitrary complex values of Z and Z^* ; *i.e.* we shall consider the equation

$$(2) \quad \mathbf{L}(\psi) \equiv \Delta_3 \psi + F(x_2, x_3) \psi \equiv \psi_{xx} + \psi_{zz^*} + F(Z, Z^*) \psi = 0.$$

In §2 an integral operator generating solutions $\psi(X, Z, Z^*)$ of (2) is derived. Using this result in §3 *real* solutions $\psi(X, Z, \bar{Z})$ of (2) are constructed and their properties in the characteristic space $Z^* = 0$ are established. In §4 and §5 two applications of the operators obtained in §3 are discussed. In §4 a real solution

$$(3) \quad \psi(X, Z, Z^*) = \sum_{n=0}^N \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} A_{nmp} X^n Z^m Z^{*p}$$

is considered, and properties of $\{A_{nm0}\}$, $n = 0, 1, \dots, N$, $m = 0, 1, 2, \dots$, are determined insuring that ψ has line singularities on the cylinder $|Z| = r$. In §5 conditions are given in terms of $\{A_{nm0}\}$ in order that ψ have finitely many singularities (which are denoted as pole-like singularity lines). Finally in §6 a generalization of the Hadamard multiplication theorem is discussed.

In the present and in the preceding papers various theorems in the theory of analytic functions have been “translated” into corresponding results for solutions of (2) and other linear partial differential equations. It is clear that, using integral operators, various other results of the theory of functions yield corresponding results in the theory of partial differential equations in two and three variables. In this connection it is not of interest to derive further examples of our procedure but rather to consider various types of operators and analyze their usefulness for the study of partial differential equations. In doing so there arise the following problems:

1. The determination of various types of operators generating solutions of partial differential equations.
2. The classification of these operators and the investigation of properties of analytic functions which can be translated into theorems about partial differential equations by this or other types of operators. (It should be noted that often the same property of analytic functions yields completely different properties of the solutions when different operators are used.)
3. The study of operators generating pairs (or triples) of (real) solutions and the investigation of mappings of this type which can be considered to some extent as generalizations of conformal mappings.

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2. An integral operator generating solutions of the equation (1.2). In this section an operator generating solutions of the partial differential equation†

†(1.2) indicates formula (2) of Section 1.

(1.2) will be determined.

Theorem. Let $g(Z)$ be an analytic function of the complex variable Z which is regular in a simply connected domain G of the Z -plane, which is starshaped with respect to the origin ‡ . Let further ††

$$(1) \quad \phi_n(X, Z, Z^*; g) = \sum_{\nu=0}^{\lfloor \frac{1}{2}n \rfloor} X^{n-2\nu} \psi^{(n, n-2\nu)}(Z, Z^*; g),$$

$$(2a) \quad \psi^{(n, n)}(Z, Z^*; g) = g(Z) + \sum_{p=1}^{\infty} q^{(n, n, p)}(Z, Z^*) L_p(g),$$

$$(2b) \quad \psi^{(n, n-2\nu)}(Z, Z^*; g) = \sum_{p=\nu}^{\infty} q^{(n, n-2\nu, p)}(Z, Z^*) L_p(g), \quad \nu = 1, 2, \dots, \lfloor \tfrac{1}{2}n \rfloor,$$

$$(2c) \quad L_p(g) = \int_0^Z \int_0^{Z_1} \dots \int_0^{Z_{p-1}} g(Z_p) dZ_p \dots dZ_1$$

$$(2d) \quad = \int_0^Z \sum_{\nu=0}^{p-1} a_{\nu}^{(p)} Z^{p-\nu-1} \zeta^{\nu} g(\zeta) d\zeta,$$

where $q^{(n, n-2\nu, p)}(Z, Z^*)$ are solutions of the system

$$(3.0) \quad q^{(n, n, 0)} = 1, \quad q_{ZZ^*}^{(n, n, p)} + q_{Z^*Z}^{(n, n, p+1)} + F q^{(n, n, p)} = 0, \quad p = 0, 1, 2, \dots,$$

$$(3.\nu) \quad \begin{cases} (n+2-2\nu)(n+1-2\nu) q^{(n, n+2-2\nu, \nu-1)} + q_{Z^*Z}^{(n, n-2\nu, \nu)} = 0, \\ \nu = 1, 2, \dots, \lfloor \tfrac{1}{2}n \rfloor, \\ (n+2-2\nu)(n+1-2\nu) q^{(n, n+2-2\nu, p)} + q_{ZZ^*}^{(n, n-2\nu, p)} \\ + q_{Z^*Z}^{(n, n-2\nu, p+1)} + F q^{(n, n-2\nu, p)} = 0, \\ \nu = 1, 2, \dots, \lfloor \tfrac{1}{2}n \rfloor, \quad p = \nu, \nu+1, \nu+2, \dots, \end{cases}$$

$$(4.\nu) \quad q^{(n, n-2\nu, p)}(Z, 0) = 0, \quad \nu = 0, 1, 2, \dots, \lfloor \tfrac{1}{2}n \rfloor, \\ p = \nu, \nu+1, \nu+2, \dots,$$

and for the constants $a_{\nu}^{(p)}$, $\nu = 0, 1, \dots, p-1$, the following recursion formulæ hold

$$(5a) \quad a_{\nu}^{(p)} = \frac{a_{\nu}^{(p-1)}}{p - \nu + 1}, \quad \nu = 0, 1, \dots, p-2,$$

$$(5b) \quad a_{p-1}^{(p)} = - \sum_{\nu=0}^{p-2} \frac{a_{\nu}^{(p-1)}}{p - \nu + 1}.$$

Then $\phi_n(X, Z, Z^*; g)$, $n = 0, 1, \dots$, are solutions of the differential equation

‡ In reality we need only that the domain G have the property that the distance from the origin to boundary points is uniformly bounded.

$^{\dagger\dagger} \lfloor \tfrac{1}{2}n \rfloor = \tfrac{1}{2}n$ if n is even, $= \tfrac{1}{2}(n-1)$ if n is odd.

(1.2) which are regular in the product domain $D = [|X| < \infty] \times G$ of the (three-dimensional) x_1, x_2, x_3 space[†]. The functions ϕ_n possess the property that

$$(6a) \quad \phi_n(X, Z, 0; g) = X^n g(Z), \quad (6b) \quad \phi_n(X, 0, Z^*; g) = X^n g(0).$$

Notation. The function $g(Z)$ will be denoted as the associate of the integral operator $\phi_n(X, Z, Z^*; g(Z))$.

Proof: I. Substituting (1) into (1.2) one obtains the following relations

$$(7) \quad \sum_{\nu=0}^{[\frac{1}{2}n]} (n-2\nu)(n-2\nu-1)X^{n-2\nu-2}\psi^{(n,n-2\nu)} + \sum_{\nu=0}^{[\frac{1}{2}n]} X^{n-2\nu}(\psi_{ZZ^*}^{(n,n-2\nu)} + F\psi^{(n,n-2\nu)}) = 0.$$

For the coefficients of the powers of $X^{n-2\nu}$, $\nu = 0, 1, \dots, [\frac{1}{2}n]$, to vanish, the $\psi^{(n,n-2\nu)}$ must satisfy the following set of equations:

$$(8.0) \quad \psi_{ZZ^*}^{(n,n)} + F\psi^{(n,n)} = 0,$$

$$(8.\nu) \quad (n-2\nu+2)(n-2\nu+1)\psi^{(n,n-2\nu+2)} + \psi_{ZZ^*}^{(n,n-2\nu)} + F\psi^{(n,n-2\nu)} = 0, \\ \nu = 1, 2, \dots, [\frac{1}{2}n].$$

We consider at first the equation (8.0). When one substitutes (2a) into (8.0) (with $L_p(g)$ given by (2c)) one obtains

$$(9a) \quad \psi_{ZZ^*}^{(n,n)} = q_{ZZ^*}^{(n,n,1)}g + \sum_{p=1}^{\infty} (q_{ZZ^*}^{(n,n,p)} + q_{ZZ^*}^{(n,n,p+1)})L_p(g),$$

$$(9b) \quad F\psi^{(n,n)} = Fg + \sum_{p=1}^{\infty} Fq^{(n,n,p)}L_p(g).$$

In accordance with (3.0), (9a) formally equals (9b) and it remains to show that the series for the function $\psi^{(n,n)}$ and its derivatives converge uniformly in every closed domain

$$D_r = [|Z| \leq \frac{1}{3}r, |Z^*| \leq \frac{1}{3}r], \quad 1 \leq r < \infty.$$

Since F is an entire function, we have[‡]

$$(10) \quad F(Z, Z^*) \ll \frac{\tilde{F}}{(1-Z/r)(1-Z^*/r)} \equiv \tilde{F} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{Z^m Z^{*n}}{r^{m+n}}$$

where $\tilde{F}$ is a conveniently chosen constant.

[†]As follows from the proof, the $\phi_n(X, Z, Z^*)$ are regular in $|X| < \infty$, $Z \in G$, $Z^* \in G^*$, of complex X, Z, Z^* space, where X, Z, Z^* are complex variables and G^* is the conjugate domain to G .

[‡]If we have two series $A \equiv \sum A_m Z^m$, $B \equiv \sum B_m Z^m$, and $|A_m| \leq B_m$, we shall say that A is majorized by B and we shall write $A \ll B$. If $A \ll B$, then $dA/dZ \ll dB/dZ$ and $\int_0^Z A dZ \ll \int_0^Z B dZ$. Further $B \ll B/(1-Z/r)$.

We shall now show (by induction) that in D_r

$$(11) \quad q_{Z^*}^{(n,n,p)}(Z, Z^*) \ll \frac{M(1+\epsilon)^p p!}{(1 - Z/r)^p (1 - Z^*/r)^2 r^{p+2}}.$$

For $p = 1$, (11) follows from (3) and (10).

Suppose now that for some p , $p \geq 1$, $q^{(n,n,p)}$ satisfies the inequality

$$(12) \quad q_{Z^*}^{(n,n,p)}(Z, Z^*) \ll \frac{N_p(1+\epsilon)^p p!}{(1 - Z/r)^p (1 - Z^*/r)^2 r^{p+2}}, \quad (Z, Z^*) \in D_r.$$

We shall show that the quantity N_p (which at first is assumed to depend on p) can be determined so that it *is independent of* p . From (3.0) it follows that

$$(13) \quad \begin{aligned} |q_{Z^*}^{(n,n,p+1)}(|Z|, |Z^*|)| &\leq |q_{ZZ^*}^{(n,n,p)}(|Z|, |Z^*|)| + \frac{\tilde{F} |q^{(n,n,p)}(|Z|, |Z^*|)|}{(1 - |Z|/r)(1 - |Z^*|/r)} \\ &\leq \frac{N_p(1+\epsilon)^p p! p}{r^{p+3}(1 - |Z|/r)^{p+1}(1 - |Z^*|/r)^2} + \frac{\tilde{F} N_p(1+\epsilon)^p p!}{r^{p+1}(1 - |Z|/r)^{p+1}(1 - |Z^*|/r)^2} \\ &= \frac{N_p(1+\epsilon)^p (p+1)! \left[\frac{p}{p+1} + \tilde{F} \frac{r^2}{p+1} \right]}{r^{p+3}(1 - |Z|/r)^{p+1}(1 - |Z^*|/r)^2}. \end{aligned}$$

If p_0 is sufficiently large, $r^2 \tilde{F}/(p+1) < \epsilon$ for all $p \geq p_0$. Hence, choosing $M = \max(N_1, \dots, N_{p_0})$,

$$(14) \quad q_{Z^*}^{(n,n,p+1)}(Z, Z^*) \ll \frac{M(1+\epsilon)^{p+1}(p+1)!}{r^{p+3}(1 - Z/r)^{p+1}(1 - Z^*/r)^2}$$

and the inequality (11) holds for all p .

The convergence of (2a) can be shown as follows: In any compact subdomain of G we have $|g(Z)| < C$, and

$$|L_p(g)| \leq \frac{C|Z|^p}{p!}.$$

Hence

$$|\psi^{(n,n)}(Z, Z^*; g)| \leq C + \sum_{p=1}^{\infty} \frac{M(1+\epsilon)^p C |Z|^p}{r^{p+1}(1 - |Z|/r)^p (1 - |Z^*|/r)}$$

which converges if

$$\frac{|Z|(1+\epsilon)}{r(1 - |Z|/r)} < 1,$$

i.e. for

$$(1+\epsilon) \frac{|Z|}{r} < \left(1 - \frac{|Z|}{r}\right) \quad \text{or} \quad \frac{|Z|}{r} < \frac{1}{2+\epsilon}.$$

Considering the equation (8.1) and substituting the right hand side of (2b) for $\psi^{(n,n-2)}(Z, Z^*)$ and (2a) for $\psi^{(n,n)}$, one sees that (8.1) is satisfied if the $q^{(n,n-2,p)}$ are connected by the relations (3.1).

We proceed now to the derivation of an inequality of the form (11) for $q_{Z^*}^{(n,n-2,p)}(Z, Z^*)$.

From (11) follows

$$(15) \quad |q^{(n,n,s)}(|Z|, |Z^*|)| \leq \frac{s!M_1(1+\epsilon)^s}{(1-|Z|/r)^s(1-|Z^*|/r)^{p+1}},$$

where $M_1 < \infty$ is a conveniently chosen constant.

The desired inequality is again proved by induction. Suppose that for $s \leq p$ and $|Z| \leq \frac{1}{3}r$, $|Z^*| \leq \frac{1}{3}r$

$$(16) \quad q_{Z^*}^{(n,n-2,s)}(Z, Z^*) \ll \frac{s!k^{(s)}(1+\epsilon)^s}{(1-|Z|/r)^s(1-|Z^*|/r)^{2p+3}},$$

$k^{(3)} > 1$, holds. According to (15) and (3.1) one obtains

$$(17) \quad \begin{aligned} & |q_{Z^*}^{(n,n-2,p+1)}(|Z|, |Z^*|)| \\ & \leq \frac{p!\tilde{M}_1(1+\epsilon)^p}{(1-|Z|/r)^p(1-|Z^*|/r)^{p+1}} + \frac{p!k^{(p)}p(1+\epsilon)^p}{(1-|Z|/r)^{p+1}(1-|Z^*|/r)^{2p+3}} \\ & \quad + \frac{k^{(p)}\tilde{F}p!(1+\epsilon)^p}{(1-|Z|/r)^{p+1}(1-|Z^*|/r)^{2p+1}} \\ & \leq \frac{(p+1)!k^{(p)}(1+\epsilon)^p}{(1-|Z|/r)^{p+1}(1-|Z^*|/r)^{2p+3}} \left[\frac{\tilde{M}_1r^2}{k^{(p)}(p+1)} + \frac{p}{(p+1)} + \frac{\tilde{F}r^2}{p+1} \right]. \end{aligned}$$

Since $\tilde{M}_1$, r and $\tilde{F}$ are fixed, a p_0 can be determined such that for $p \geq p_0$

$$\frac{\tilde{F}r^2}{p+1} \leq \frac{\epsilon}{2} \quad \text{and} \quad \frac{\tilde{M}_1r^2}{k^{(p)}(p+1)} \leq \frac{\epsilon}{2}.$$

Consequently for sufficiently large p , say $p \geq p_0$,

$$(18) \quad \frac{\tilde{M}_1r^2}{k^{(p)}(p+1)} + \frac{p}{(p+1)} + \frac{\tilde{F}r^2}{p+1} \leq \frac{\epsilon}{2} + 1 + \frac{\epsilon}{2} = 1 + \epsilon.$$

Therefore a constant M_2 (which is independent of s) can be determined so that (16) holds with $k^{(s)}$ replaced by M_2 .

Proceeding in a similar way one can obtain analogous bounds for $|q^{(m,m-2s,p)}(|Z|, |Z^*|)|$, $s = 2, 3, \dots, [\frac{1}{2}m]$. Hence the convergence of the series (2b) follows exactly as for (2a). Since every $\phi_n(X, Z, Z^*; g)$ is a sum of finitely many functions $X^{n-2\nu}\psi^{(n,n-2\nu)}(Z, Z^*; g)$, see (1), we have proved its existence in $[|X| < \infty] \times G$.

II. In I the function $q_{Z^*}^{(n,n-2\nu,p)}$ is determined and one can choose the integration constant so that the $q^{(n,n-2\nu,p)}(Z, 0)$ satisfy the conditions (4. ν). Then $\phi_n(X, Z, 0; g)$ will satisfy the condition (6a).

Further, the $L_\nu(g)$, see (2c), vanish for $Z = 0$, $\psi^{(n,n)}(0, Z^*; g) = g(0)$ and $\psi^{(n,n-2\nu)}(0, Z^*; g) = 0$ for $\nu = 1, 2, \dots [\frac{1}{2}n]$.

Consequently $\phi_n(X, 0, Z^*; g)$ satisfies the condition (6b).

III. In the following it will be shown that the p -fold integral, see (2c), can be replaced by an ordinary integral, see (2d). The proof will be given by induction. By integrating by parts, one obtains

$$\begin{aligned}
 \int_0^Z Z_1^s \int_0^{Z_1} Z_2^p g(Z_2) dZ_2 dZ_1 &\equiv \int_0^Z \left[\int_0^{Z_1} Z_2^p g(Z_2) dZ_2 \right] d\left(\frac{Z_1^{s+1}}{s+1}\right) \\
 (19) \qquad &= \frac{1}{s+1} \left[Z^{s+1} \int_0^Z Z_1^p g(Z_1) dZ_1 - \int_0^Z Z_1^{s+p+1} g(Z_1) dZ_1 \right] \\
 &= \frac{1}{s+1} \int_0^Z [Z^{s+1} Z_1^p - Z_1^{s+p+1}] g(Z_1) dZ_1.
 \end{aligned}$$

For $p > 2$

$$\begin{aligned}
 L_p(g) &= \int_0^Z \left[\int_0^{Z_1} \dots \int_0^{Z_{p-1}} g(Z_p) dZ_p \dots dZ_2 \right] dZ_1 \\
 &= \int_0^Z L_{p-1}(g) dZ_1 = \int_0^Z \int_0^{Z_1} \sum_{\nu=0}^{p-2} a_\nu^{(p-1)} Z_1^{p-\nu-2} Z_2^\nu g(Z_2) dZ_2 dZ_1 \\
 (20) \qquad &\equiv \int_0^Z \sum_{\nu=0}^{p-2} a_\nu^{(p-1)} Z_1^{p-\nu-2} \left[\int_0^{Z_1} Z_2^\nu g(Z_2) dZ_2 \right] dZ_1 \\
 &= \int_0^Z \left[\int_0^{Z_1} Z_2^\nu g(Z_2) dZ_2 \right] d\left[\sum_{\nu=0}^{p-2} \frac{a_\nu^{(p-1)}}{p-\nu-1} Z_1^{p-\nu-1} \right] \\
 &= \int_0^Z \left[\sum_{\nu=0}^{p-2} \frac{a_\nu^{(p-1)}}{p-\nu-1} Z^{p-\nu-1} Z_1 - \left(\sum_{\nu=0}^{p-2} \frac{a_\nu^{(p-1)}}{p-\nu-1} \right) Z_1^{p-1} \right] g(Z_1) dZ_1.
 \end{aligned}$$

This yields the relations (5a), (5b).

3. An integral operator generating real solutions of the equation (1.2). An integral operator ϕ_n (see (2.1)) generating solutions of the equation (1.2) has been constructed in §2.

Using ϕ_n in the present section an operator τ_n generating real solutions of the equation (1.2) will be determined. (τ_n yields solutions which for real values of $x_1 = X$, $x_2 = -i(Z - Z^*)$, $x_3 = Z + Z^*$ are real.) τ_n is defined by the relation

$$\begin{aligned}
 \tau_n(X, Z, Z^*; g(Z)) &= \frac{1}{2} [\phi_n(X, Z, Z^*; g(Z)) + \bar{\phi}_n(X, Z^*, Z; \bar{g}(Z^*))] \\
 &\equiv \frac{1}{2} \sum_{\nu=0}^{[\frac{1}{2}n]} X^{n-2\nu} [\psi^{(n,n-2\nu)}(Z, Z^*; g(Z)) + \bar{\psi}^{(n,n-2\nu)}(Z^*, Z; \bar{g}(Z^*))] \\
 (1) \qquad &\equiv \frac{1}{2} \left\{ X^n \left[g(Z) + q^{(n,n,1)}(Z, Z^*) \int_0^Z g(Z_1) dZ_1 + \dots \right. \right.
 \end{aligned}$$

$$\begin{aligned}
(1) \quad & + \bar{g}(Z^*) + \bar{q}^{(n,n,1)}(Z^*, Z) \int_0^{Z^*} \bar{g}(Z_1^*) dZ_1^* + \dots \Big] \\
& + X^{n-2} \left[q^{(n,n-2,1)}(Z, Z^*) \int_0^Z g(Z_1) dZ_1 + \dots \right. \\
& \left. + \bar{q}^{(n,n-2,1)}(Z^*, Z) \int_0^{Z^*} \bar{g}(Z_1^*) dZ_1^* + \dots \right] + \dots \Big\},
\end{aligned}$$

see (2.1), (2.2a), (2.2b) and (2.2c).

The operator $\tau_n(X, Z, Z^*; g(Z))$ has the property that

$$(2a) \quad \tau_n(X, Z, 0; g(Z)) = \frac{1}{2}[g(Z) + \bar{g}(0)]X^n,$$

$$(2b) \quad \tau_n(X, 0, Z^*; g(Z)) = \frac{1}{2}[g(0) + \bar{g}(Z^*)]X^n.$$

(2a) and (2b) follow from the fact that the $q^{(n,n-2\nu,p)}$ satisfy the relation (2.4 ν).

Notation. The function $g(Z)$ will be denoted as the associate of the integral operator (1).

In the following it will be assumed that the functions $g_n(Z)$ are normalized by the requirement that $g_n(0)$ be real.

Corollary. *Let*

$$(3) \quad \tau(X, Z, Z^*) = \sum_{n=0}^N \tau_n(X, Z, Z^*; g_n(Z)),$$

where every associate $g_n(Z)$ is regular in the domain D_n which contains the origin O .

Let D be a simply connected domain such that $O \in D$ and such that $D \subset D_1 \cap D_2 \dots \cap D_n$. Then

$$(4a) \quad \tau(X, Z, 0) = \sum_{n=0}^N \tau_n(X, Z, 0; g_n(Z)) = \frac{1}{2} \sum_{n=0}^N X^n [g_n(Z) + \bar{g}_n(0)],$$

$$(4b) \quad \tau(X, 0, Z^*) = \sum_{n=0}^N \tau_n(X, 0, Z^*; g_n(Z)) = \frac{1}{2} \sum_{n=0}^N X^n [g_n(0) + \bar{g}_n(Z^*)].$$

In connection with our further considerations we shall show that the solutions

$$(5) \quad \Phi(X, Z, Z^*) = \sum_{\nu=0}^{\lfloor \frac{1}{2}n \rfloor} X^{n-2\nu} \phi_{n-2\nu}(Z, \bar{Z}), \quad Z = \frac{1}{2}(ix_2 + x_3),$$

which are regular for $Z, \bar{Z}$ belonging to a bounded star domain[†] B^2 can be continued to a certain domain $H^4(B^2)$ (described below) of the four-dimensional space Z, Z^* . One obtains $H^4(B^2)$, the hull of B^2 , as follows. Let[‡] $\tilde{B}^2$ be the domain of the real plane which one obtains by reflecting every point (x_2, x_3) along the

[†]With respect to the origin.

[‡]It is clear that if x_2, x_3 vary in the domain $2B^2$ of the real plane, $Z = \frac{1}{2}(ix_2 + x_3)$ varies in the domain B^2 , while $\bar{Z} = \frac{1}{2}(-ix_2 + x_3)$ varies in $\tilde{B}^2$, and if $g(Z)$ is regular in B^2 , $\bar{g}(\bar{Z})$ is regular in $\tilde{B}^2$.

x_2 axis of the real plane. From every point $\{Z^{(0)}\}$ of the boundary C^1 of B^2 , we draw in the four dimensional Z, Z^* space the plane $\{Z = Z^{(0)}\}$, and from every point of the boundary $\tilde{C}^1$ of $\tilde{B}^2$ we draw the plane $\{Z^* = \tilde{Z}^{(0)}\}$. The totality of planes which one obtains in this way form two hypersurfaces in the four dimensional Z, Z^* space, which divide this space into parts. The part which includes the domain B^2 of the real plane is denoted by $H^4(B^2)$.

Lemma: Let ψ be a real solution of the equation

$$(6) \quad \mathbf{L}(\psi) \equiv \psi_{ZZ^*} - F\psi - T = 0$$

defined in the domain B^2 of the real plane. Suppose further that $F(Z, Z^*)$ is an entire function and $T(Z, Z^*)$ is a function of two complex variables which is regular in the (closed) domain $H^4(\tilde{B}^2)$ of the four dimensional space. Then $\psi(Z, Z^*)$ can be continued to $H^4(\tilde{B}^2)$.

Proof: We consider at first a special solution of (6).

$$(7) \quad \begin{aligned} \Psi(Z, Z^*) = & \int_0^Z \int_0^{Z^*} T(Z_1, Z_1^*) \delta_1 \\ & + \int_0^Z \int_0^{Z^*} F(Z_1, Z_1^*) \int_0^{Z_1} \int_0^{Z_1^{*}} T(Z_2, Z_2^*) \delta_2 \\ & + \int_0^Z \int_0^{Z^*} F(Z_1, Z_1^*) \int_0^{Z_1} \int_0^{Z_1^{*}} F(Z_2, Z_2^*) \int_0^{Z_2} \int_0^{Z_2^{*}} T(Z_3, Z_3^*) \delta_3 + \dots, \\ & \delta_n = \prod_{\nu=1}^n dZ_\nu dZ_\nu^*, \end{aligned}$$

is a solution of (6) defined in $H^4(B^2)$.

Since we assume that $T(Z_1, Z_1^*)$ is regular for $Z_1 \in B^2$ and $\dagger Z_1^* \in \tilde{B}^2$ and since the $\int_0^{Z_1} \int_0^{Z_1^{*}} T(Z_1, Z_1^*) dZ_1 dZ_1^*$ is $\lim \sum T(Z_1^{(\nu)}, Z_1^{*(\mu)}) \Delta_{\nu\mu}$, this integral exists for all values $Z \in B^2, Z^* \in \tilde{B}^2$ and is an analytic function of two complex variables Z, Z^* . If $\tilde{B}_1^2$ is a closed subdomain of B^2 there exists a constant T_0 such that

$$(8) \quad \int_0^Z \int_0^{Z^*} T(Z_1, Z_1^*) dZ_1 dZ_1^* < T_0$$

for $Z \in \tilde{B}_1^2, Z^* \in \tilde{\tilde{B}}_1^2$. Since $|F(Z, Z^*)| \leq F_0$, for $Z \in B_1^2, Z^* \in \tilde{B}_1^2$, where F_0 is a conveniently chosen constant,

$$(9) \quad |\Psi(Z, Z^*)| \leq T_0 + \frac{T_0 F_0}{(1!)^2} + \frac{T_0 F_0^2}{(2!)^2} + \frac{T_0 F_0^3}{(3!)^2} + \dots < \infty.$$

We proceed now to the proof of our statement for any solution ψ of (6) satisfying the hypotheses of our lemma. The difference $\psi^{(0)} = \Psi - \psi$ satisfies the equation

$$(10) \quad \psi_{ZZ^*}^{(0)} + F\psi^{(0)} = 0$$

$\dagger \tilde{B}^2$ is the domain which one obtains by reflecting B^2 along the real axis.

and according to well known results every solution of (10) which is regular in B^2 for $Z^* = \bar{Z}$ can be continued analytically to $H^4(B^2)$. Thus $\psi = \Psi - \psi^{(0)}$ can be continued to $H^4(B^2)$.

We can now apply our lemma when constructing the solutions $\phi_n(X, Z, Z^*; g)$.

According to the theorem mentioned before, the solutions $\psi^{(n,n)}(Z, Z^*)$ are regular for $Z \in B^2$, $Z^* \in \bar{B}^2$, where B^2 is a simply connected domain (which contains the origin) in which $g(Z)$ is regular. Writing $T = n(n-1)\psi^{(n,n)}$ and applying our lemma it follows that $\psi^{(n,n-2)}(Z, Z^*)$ is regular for $Z \in B$, $Z^* \in \bar{B}$. In this way we can show successively that all $\psi^{(n,n-2\nu)}$, $\nu = 1, 2, \dots, [\frac{1}{2}n]$, are regular for $Z \in B$, $Z^* \in \bar{B}$. Since $\phi_n(X, Z, Z^*; g)$ is a combination of finitely many $\psi^{(n,m)}$, $\frac{1}{2}[\phi_n(X, Z, Z^*; g) + \phi_n(X, Z^*, Z; \bar{g})]$ is regular for $Z \in B$, $Z^* \in \bar{B}$.

4. Conditions for singularities on a cylinder $x_2^2 + x_3^2 = r^2$ of a solution ψ .
A known theorem in the theory of functions states: Let

$$(1) \quad g(Z) = \sum_{\nu=0}^{\infty} A_{\nu} Z^{\nu}$$

be an analytic function of a complex variable which is regular at the origin. The necessary and sufficient condition in order that the point $re^{i\alpha}$ on the circle of convergence

$$(2) \quad |Z| = r, \quad r = \limsup_{\nu \rightarrow \infty} (|A_{\nu}|)^{-1/\nu},$$

be a singular point of $g(z)$ is that for the expressions

$$(3) \quad B_{\mu} = (-1)^{\mu} \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} A_{\lambda} r^{\lambda} e^{i\lambda\alpha}$$

the relation

$$(4) \quad \limsup_{\mu \rightarrow \infty} (|B_{\mu}|)^{1/\mu} = 2$$

hold. See [6] p. 36. (The necessary and sufficient condition in order that $re^{i\alpha}$ be a regular point of $g(Z)$ is that the left hand side of (4) be smaller than 2.)

In the following a generalization of this result to the theory of differential equation (1.2) will be discussed.

Theorem 4.1. *Let*

$$(5) \quad \begin{aligned} S(X, Z, Z^*) &= \sum_{n=0}^N \tau_n(X, Z, Z^*; g_n(Z)) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} A_{nmp} X^n Z^m Z^{*p}, \quad A_{nmp} = \bar{A}_{n\bar{p}m}, \end{aligned}$$

be a real solution of the differential equation (1.2), and let the $g_n(0)$, $n = 0, 1, 2, \dots, N$, be real. Suppose further that the sequences $\{A_{nm0}\}$, $m = 0, 1, 2, \dots$, have the

property that

$$(6) \quad \limsup_{m \rightarrow \infty} (|A_{nm0}|)^{1/m} = r = \text{const.} \quad \text{for } n = 0, 1, \dots, N.$$

A sufficient condition in order that the (straight) lines $Z = re^{i\alpha_n}$ be singular lines of (5) is that

$$(7) \quad \limsup_{\mu \rightarrow \infty} (|B_{n\mu 0}|)^{1/\mu} = 2, \quad n = 0, 1, \dots, N,$$

where the $B_{n\mu 0}$ are connected with the A_{nm0} by the relation

$$(3a) \quad B_{n\mu 0} = (-1)^\mu \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} A_{n\lambda 0} r^\lambda e^{i\lambda\alpha_n},$$

and the α_n are distinct.

Proof: As has been shown in §2 and §3 the singularities of

$$(8) \quad \tau_n(X, Z, Z^*; g_n(Z))$$

coincide with those of

$$(9) \quad g_n(Z) = \sum_{m=0}^{\infty} A_{nm0} Z^m.$$

According to the previously mentioned result the relations (6) and (7) are necessary and sufficient in order that the point $Z = re^{i\alpha_n}$ be a singularity of $g_n(Z)$ and that $g_n(Z)$ be regular in $|Z| < r$.

Consequently (6) and (7) are a sufficient condition in order that $\tau_n(X, Z, Z^*; g_n(Z))$ be regular in the cylinder

$$(10) \quad x_2^2 + x_3^2 < r^2, \quad |x_1| < \infty$$

and singular along the line

$$(11) \quad x_2 = r \cos \alpha_n, \quad x_3 = r \sin \alpha_n, \quad |x_1| < \infty.$$

Since $S(X, Z, Z^*)$ is the sum of finitely many $\tau_n(X, Z, Z^*; g_n(Z))$, $S(X, Z, Z^*)$ is regular in (10), and the lines (11) are singularities. Here $n = 0, 1, \dots, N$.

5. Conditions in order that ψ have pole-like singularities along $Z = Z_v^{(n)} = \text{const.}$ In the present section the well known Hadamard relations for the coefficients of the series development of a meromorphic function will be used to obtain relations in the theory of the differential equation (1.2).

If one assumes that $g(Z)$ has a pole at $Z = Z_0$ the solution $\tau_n(X, Z, Z^*; g)$ will have for $Z = Z_0$ a singularity which will be denoted as the *pole-like singularity of type τ_n* . We should note that a pole of order n at $Z = Z_0$ will lead to a logarithmic singularity of $L_p(g)$ for $p = n$. Hence in order to define $\tau_n(X, Z, Z^*; g)$ we must make cuts in the domain G from each pole to the boundary, so that the remaining part is simply connected.

Theorem 5.1. *Let the real solution $\tau_n(X, Z, Z^*; g(Z))$ of (1.2) have the power series development*

$$(1) \quad \begin{aligned} \tau_n(X, Z, Z^*; g(Z)) &= \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} A_{nmp} X^n Z^m Z^{*p} \\ &+ \sum_{\nu=1}^{[\frac{1}{2}n]} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \bar{A}_{nmp}^{(n-2\nu)} X^{n-2\nu} Z^m Z^{*p}, \quad A_{nmp} = \bar{A}_{npm}. \end{aligned}$$

Further, let

$$(2) \quad s_{np} = \limsup_{m \rightarrow \infty} |Q_{nm}^{[p]}|^{1/m}$$

where

$$(3) \quad Q_{nm}^{[p]} = \begin{vmatrix} A_{n,m,0} & A_{n,m+1,0} & \cdots & A_{n,m+p,0} \\ A_{n,m+1,0} & A_{n,m+2,0} & \cdots & A_{n,m+p+1,0} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,m+p,0} & A_{n,m+p+1,0} & \cdots & A_{n,m+2p,0} \end{vmatrix}.$$

I. If from some p on, say from $p = p_0$, all s_{np} vanish, then the associate g of τ_n (see (3.1) and (2.2)) equals

$$(4) \quad g(Z) = \frac{p(Z)}{q(Z)}$$

where $p(Z)$ is an entire function and $q(Z)$ a polynomial. The solution $\tau_n(X, Z, Z^*; g(Z))$ has finitely many singularity lines

$$(5) \quad Z = Z_*^{(n)},$$

where $Z_*^{(n)}$ are the zeros of $q(Z)$. $\tau_n(X, Z, Z^*; g(Z))$ admits the representation (3.1) with $g(Z) = p(Z)/q(Z)$.

II. If the ratio $s_{np}/s_{n,p-1}$ goes to 0 when $p \rightarrow \infty$ then the function $\tau_n(X, Z, Z^*; g(Z))$ has in every cylinder of finite radius finitely many pole-like singularities of type τ_n . The singularity lines are given by (5).

III. If the ratio $s_{np}/s_{n,p-1}$ remains constant from some value p on then $g(Z) = p(Z)/q(Z)$ where $q(Z)$ is a polynomial and $p(Z)$ is regular in some circle $|Z| < R$. Hence $\tau_n(X, Z, Z^*; g(Z))$ has pole-like singularities at $Z = Z_\mu$ where Z_μ are the zeros of $q(Z)$.

IV. If the ratio $s_{np}/s_{n,p-1}$ converges to a limit $1/R$ different from 0 then the function $\tau_n(X, Z, Z^*; g(Z))$ has in every cylinder

$$(6) \quad |Z| \leq r < R$$

finitely many pole-like singularities and admits infinitely many pole-like singularities in the neighbourhood of the cylinder $|Z| = R$.

Proof: According to (1) and (3.2a)

$$(7) \quad \frac{1}{2}[g(Z) + \bar{g}(0)]X^n = \sum_{m=0}^{\infty} A_{nm0} X^n Z^m,$$

so that

$$(8) \quad \frac{1}{2}[g(Z) + \bar{g}(0)] = \sum_{m=0}^{\infty} A_{nm0} Z^m,$$

and therefore

$$(9) \quad g(Z) = 2 \sum_{m=0}^{\infty} A_{nm0} Z^m - \bar{g}(0).$$

When considering the function $g(Z)$ of one complex variable, we apply the Hadamard criteria. (See [7], p. 131 ff.). According to these results $g(Z)$ has finitely many poles and admits the representation (4) in the case I. Here $q(Z)$ is a polynomial. If $Z_p^{(n)}$ are zeros of $q(Z)$, then along every singularity line (5) $\tau_n(X, Z, Z^*; g(Z))$ has a pole-like singularity of type τ_n . Using Hadamard's results in the cases II, III, IV, we can make analogous conclusions† about the behavior of $g(Z)$ at the singular points (5) which then can be interpreted as corresponding results about $\tau_n(X, Z, Z^*; g(Z))$.

Remark: In the case II the moduli $|Z_p^{(n)}|$ of the succeeding poles are the ratios $s_{n,p-1}/s_{n,p}$. If the genus of the function $g(Z) + \bar{g}(0)$ is finite it will be a quantity γ such that $\sum_p (s_{n,p}/s_{n,p-1})^\gamma$ converges.

Corollary. Suppose that the series development

$$(10) \quad T(X, Z, Z^*) = \sum_{n=0}^N \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} A_{nmp} X^n Z^m Z^{*p}, \quad A_{nmp} = \bar{A}_{npm},$$

is a real solution of (1.2). Further suppose that each of the $s_{n,p}$, $n = 0, 1, 2, \dots, N$, satisfies one of the conditions I—IV of Theorem 5.1. Then

$$(11) \quad T(X, Z, Z^*) = \sum_{n=0}^N \tau_n(X, Z, Z^*; g_n(Z))$$

where every $\tau_n(X, Z, Z^*; g_n(Z))$ has the properties indicated in Theorem 5.1.

The corollary follows immediately from the fact that according to (2.4 ν) we have $\psi^{(n, n-2\nu)}(Z, 0; g_n(Z)) = 0$ for $\nu = 1, 2, \dots, [\frac{1}{2}n]$. Therefore according to (3.1)

$$(12) \quad \begin{aligned} T(X, Z, 0) &= \sum_{n=0}^N \frac{1}{2}[\phi_n(X, Z, 0; g_n(Z) + \bar{\phi}_n(X, 0, Z; \bar{g}_n(0))] \\ &= \sum_{n=0}^N \frac{1}{2}\{X^n(g_n(Z) + \bar{g}_n(0))\} = \sum_{n=0}^N \left\{X^n \left(\sum_{m=0}^{\infty} A_{nm0} Z^m\right)\right\}. \end{aligned}$$

Consequently

$$(13) \quad g_n(Z) = \sum_{m=0}^{\infty} A_{nm0} Z^m - \bar{g}_n(0)$$

†Clearly various other results in the theory of functions of a complex variable lead to results about $\tau_n(X, Z, Z^*; g(Z))$. We do not carry out these considerations, since they represent an immediate generalization of the theorems about functions of one complex variable.

and about every $\tau_n(X, Z, Z^*; g_n(Z))$ one can make the conclusions which were made before about $\tau_n(X, Z, Z^*; g(Z))$, see Theorem 5.1.

6. A multiplication theorem. As a last application of our procedure a theorem will be discussed which represents an analogue of the Hadamard multiplication theorem in the theory of differential equations (1.2). See [6], pp. 20 and 105 ff.

Theorem 6.1. *Suppose that each of the functions ψ_k , with power series development*

$$(1) \quad \psi_k(X, Z, Z^*) = \sum_{n=0}^N \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} A_{nmp}^{(k)} X^n Z^m Z^{*p}, \quad k = 1, 2, 3,$$

is a solution of an equation of the form

$$(2) \quad \mathbf{L}_k(\psi) \equiv \psi_{XX} + \psi_{ZZ^*} + F_k(Z, Z^*)\psi = 0, \quad k = 1, 2, 3,$$

where the functions $F_k(Z, Z^)$ are entire functions. Here the coefficients of the function elements*

$$(3) \quad \sum_{n=0}^N \left[(A_{n00}^{(k)} + \bar{A}_{n00}^{(k)}) + \sum_{m=1}^{\infty} A_{nm0}^{(k)} Z^m \right] X^n, \quad k = 1, 2, 3,$$

of $\psi_k(X, Z, 0)$ are connected by the relations

$$(4a) \quad (A_{n00}^{(1)} + \bar{A}_{n00}^{(1)})(A_{n00}^{(2)} + \bar{A}_{n00}^{(2)}) = A_{n00}^{(3)} + \bar{A}_{n00}^{(3)}, \quad n = 0, 1, \dots, N,$$

$$(4b) \quad A_{nm0}^{(1)} A_{nm0}^{(2)} = A_{nm0}^{(3)}, \quad n = 0, 1, \dots, N, \quad m = 1, 2, \dots.$$

Let $S_n^{(k)}$, $k = 1, 2$, $n = 0, 1, 2, \dots, N$, be the principal star domains[†] with centers at $Z = 0$ in which the functions $\sum_{m=0}^{\infty} A_{nm0}^{(k)} Z^m$ are regular, and let

$$(5) \quad S_n^{(3)} = S_n^{(1)} \odot S_n^{(2)} + S_n^{(1)} + S_n^{(2)}.$$

Then the function $\psi_3(X, Z, Z^)$ is regular in the domain[‡]*

$$(6) \quad [S_0^{(3)} \cap S_1^{(3)} \cap \dots \cap S_N^{(3)}] \times [|X| < \infty].$$

Proof: The statement of the Theorem 6.1, follows almost immediately from the results of §3.

Let $S_n^{(k)}$, $k = 1, 2$, be the principal star domain in which $\sum_{m=0}^{\infty} A_{nm0}^{(k)} Z^m$ is regular. (The regularity domains of $\sum_{m=0}^{\infty} A_{nm0}^{(k)} Z^m$ and $\sum_{m=0}^{\infty} A_{nm0}^{(k)} Z^m + \bar{A}_{n00}^{(k)}$ are naturally identical.) According to the Hadamard multiplication theorem

[†]A domain which possess the property that its intersection with every ray from the origin O is a connected interval is denoted as a star domain with the center at O . The largest domain of this kind in which the function $f(Z)$ is regular is denoted as the principal star domain of $f(Z)$ with respect to the origin.

[‡]See [6] p. 20. It should be noted that some other similar results, see e.g. [6, 9], can be generalized in a similar way to the case of differential equation (2).

the function whose function element is

$$\begin{aligned}
 (7) \quad & (A_{n00}^{(3)} + \overline{A_{n00}^{(3)}}) + \sum_{m=1}^{\infty} A_{nm0}^{(3)} Z^m \\
 & = \sum_{m=0}^{\infty} A_{nm0}^{(1)} A_{nm0}^{(2)} Z^m + \overline{A_{n00}^{(2)}} \sum_{m=0}^{\infty} A_{nm0}^{(1)} Z^m + \overline{A_{n00}^{(1)}} \sum_{m=0}^{\infty} A_{nm0}^{(2)} Z^m + \overline{A_{n00}^{(1)}} A_{n00}^{(2)}
 \end{aligned}$$

is regular in the domain (5). By the results of §3 the solution $\psi_3(X, Z, Z^*)$ such that

$$\psi_3(X, Z, 0) = \sum_{n=0}^N \left[(A_{n00}^{(3)} + \overline{A_{n00}^{(3)}}) + \sum_{m=1}^{\infty} A_{nm0}^{(3)} Z^m \right] X^n$$

is regular in the domain (6).

BIBLIOGRAPHY

- [1] STEFAN BERGMAN, Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen, *Récueil Mathématique, N.S.* 2 (1937) pp. 1169–1198.
- [2] ———, Classes of solutions of linear partial differential equations in three variables, *Duke Math. J.* 13 (1946) pp. 419–458.
- [3] ———, On solutions with algebraic character of linear partial differential equations, *Trans. Amer. Math. Soc.* 68 (1950) pp. 461–507.
- [4] ———, Operatorenmethoden in der Gasdynamik, *Zeitschrift für angewandte Mathematik und Mechanik* 32 (1952) pp. 35–45.
- [5] ———, The coefficient problem in the theory of linear partial differential equations, *Trans. Amer. Math. Soc.* 73 (1952) pp. 1–34.
- [6] LUDWIG BIEBERBACH, *Analytische Fortsetzung*, Ergebnisse der Mathematik und ihrer Grenzgebiete, No. 3 (New series), Berlin (1955).
- [7] J. HADAMARD, Essai sur l'étude des fonctions données par leur développement de Taylor, *Journal de mathém. pures et app., IVth series* 8 (1892) pp. 101–186.
- [8] ERWIN KREYSZIG, On certain partial differential equations and their singularities, *Journal of Rational Mech. and Anal.* 5 (1956) pp. 805–820.
- [9] GEORGE PÓLYA, Untersuchungen über Lücken und Singularitäten von Potenzreihen, Zweite Mitteilung, *Math. Anal.* (2) 34 (1933) pp. 731–777.

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