Dynamic Programming and the Variation of Green's Functions

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1. Introduction. In an earlier paper [1], the functional equation technique of dynamic programming was applied to obtain a variational equation for a Green's function corresponding to a second order ordinary differential equation. In the present paper, this method is extended to apply to elliptic partial differential operators, and a first consequence is the classical Hadamard variational formula. Further results require a more detailed argument which we shall present subsequently.

The technique presented here utilizes the principle of optimality (see [2]) in the following fashion. Given a one-parameter family of regions, monotone under inclusion, one takes the minimum value of a certain integral on any given region R , subject to certain restrictions, to be a functional of those restrictions and the region R . Then if $R^* \subset R$ the functional on R can be approximated by means of a related functional on R^* satisfying slightly different restrictions. This leads to a Gâteaux difference equation from which one easily derives the Hadamard relation.

The method is initially presented for the Laplace operator on R , and appropriate generalizations are indicated in §6 and §7.

2. Preliminaries. Let R be a bounded connected region of n -dimensional real Euclidean space, whose boundary ∂R is of class C_2 . For convenience we shall not explicitly write out the differentials of volume and surface area in integrals over R and ∂R . Given any twice-differentiable function u on R , let Δu and u_p represent the Laplacian of u and the restriction of u to its limiting values on ∂R , respectively. Then, if v and w are suitable functions on R and ∂R , the boundary value problem

$$(1) \quad \Delta u = v, \quad u_p = w$$

possesses the unique solution

$$(2) \quad u(p) = \int_{q \in R} g(p, q)v(q) + \int_{q \in \partial R} g_n(p, q)w(q),$$

where g is the Green's function for R normalized by the condition that

$$(3) \quad \int_{q \in S} \Delta g(p, q) = \int_{q \in \partial S} g_n(p, q) = 1.$$

Here S is any sphere with center p which lies inside R , and g_n is the exterior normal derivative of g on ∂S . The two integrals in (2) represent the solutions $u^{(1)}$ and $u^{(2)}$ of the boundary value problems

$$(4) \quad \Delta u = v, \quad u_\rho = 0$$

and

$$(5) \quad \Delta u = 0, \quad u_\rho = w,$$

respectively, and they are orthogonal in the sense that

$$(6) \quad \int_R \nabla u^{(1)} \cdot \nabla u^{(2)} = \int_{\partial R} u_\rho^{(1)} u_n^{(2)} - \int_R u^{(1)} \Delta u^{(2)} = 0,$$

where $\nabla u^{(i)}$ is the gradient of $u^{(i)}$, $i = 1, 2$.

3. A minimum problem. Among those functions u such that $u_\rho = 0$, the function $u^{(1)}$ maximizes the integral

$$\int_{p \in R} \int_{q \in R} g(p, q) (\Delta u - v)(p) (\Delta u - v)(q).$$

Hence, since the maximum value is zero, and since

$$(7) \quad \int_{q \in R} g(p, q) \Delta u(q) = u(p)$$

for any function u such that $u_\rho = 0$, one obtains an extremal condition

$$(8) \quad \begin{aligned} \min_{u|u_\rho=0} \int_R 2uw + |\nabla u|^2 &= \min_{u|u_\rho=0} \int_R (2v - \Delta u)u \\ &= \int_{p \in R} \int_{q \in R} g(p, q) v(p) v(q), \end{aligned}$$

with the minimizing function $u^{(1)}$. Define

$$(9) \quad f(v, w) = \min_{u|u_\rho=w} \int_R [2uw + |\nabla u|^2]$$

so that

$$(10) \quad f(v, 0) = \int_{p \in R} \int_{q \in R} g(p, q) v(p) v(q).$$

It should be noted that the first equality in (8) fails for those u such that $u_\rho = w \neq 0$.

Suppose that $u^{(2)}$ is given as the solution of (5). Then (9) may be rewritten, by means of (6), as

$$\begin{aligned}
 f(v, w) &= \min_{u|u_p=0} \int_R [2(u + u^{(2)})v + |\nabla u + \nabla u^{(2)}|^2] \\
 (11) \quad &= \min_{u|u_p=0} \int_R [2uv + |\nabla u|^2] + \int_R [2u^{(2)}v + |\nabla u^{(2)}|^2] \\
 &= f(v, 0) + 2 \int_R u^{(2)}v + \int_R |\nabla u^{(2)}|^2.
 \end{aligned}$$

In particular, writing ϵw in place of w ,

$$(12) \quad f(v, \epsilon w) = f(v, 0) + 2\epsilon \int_R u^{(2)}v + \epsilon^2 \int_R |\nabla u^{(2)}|^2.$$

Since $u^{(2)}$ is known explicitly in terms of w , this enables us to compute the Gâteaux difference

$$(13) \quad f(v, \epsilon w) - f(v, 0) = 2\epsilon \iint_{\substack{p \in R \\ q \in \partial R}} g_n(p, q)v(p)w(q) + o(\epsilon).$$

4. A functional equation. Let φ be a non-negative function of class C_2 on ∂R , and let ∂R^* be the surface obtained from ∂R by a displacement δn along the interior normal, where $\delta n = \epsilon\varphi$. If u is any differentiable function on R such that $u_p = 0$, then the restriction u_p^* of u to ∂R^* is $-u_n\delta n + o(\epsilon)$. We extend the definition of f to the class of regions R^* with boundaries ∂R^* by setting

$$(14) \quad f(\epsilon, v, w) = \min_{u|u_p^*=-w} \int_{R^*} [2uv + |\nabla u|^2].$$

If $u_p = 0$ then $|\nabla u|^2 = u_n^2$ on ∂R , so that the n -dimensional analog of the principle of optimality implies

$$(15) \quad f(0, v, 0) = \min_{u_n} [f(\epsilon, v, -u_n\delta n) + \int_{\partial R} \delta n u_n^2 + o(\epsilon)].$$

Set $\delta f(v, w) = f(\epsilon, v, w) - f(0, v, 0)$ and note that

$$(16) \quad \delta f(v, -u_n\delta n) = \delta f(v, 0) + o(\epsilon).$$

In this notation one may apply (13) and (15) to obtain

$$(17) \quad \min_{u_n} \left[\delta f(v, 0) - 2 \int_{p \in R} \int_{q \in \partial R} g_n(p, q)v(p)\delta n(q)u_n(q) + \int_{\partial R} \delta n u_n^2 \right] = o(\epsilon).$$

The Euler variational equation of (17) is

$$(18) \quad u_n(q) = \int_{p \in R} g_n(p, q)v(p),$$

and it follows that

$$(19) \quad \delta f(v, 0) = \int_{s \in \partial R} \int_{p \in R} \int_{q \in R} \delta n(s) g_n(p, s) g_n(q, s) v(p) v(q) + o(\epsilon).$$

5. The Hadamard variation. Let $g(\epsilon, p, q)$ represent the Green's function of the region R^* , and let $\delta g(p, q) = g(\epsilon, p, q) - g(0, p, q)$. We wish to derive the Hadamard relation between δg and δn . For the region R^* (10) becomes

$$(20) \quad f(\epsilon, v, 0) = \int_{p \in R^*} \int_{q \in R^*} g(\epsilon, p, q) v(p) v(q),$$

and since g vanishes and possesses a bounded normal derivative on ∂R it follows that

$$(21) \quad \delta f(v, 0) = \int_{p \in R} \int_{q \in R} \delta g(p, q) v(p) v(q) + o(\epsilon).$$

Since v is arbitrary, (19) and (21) together imply

$$(22) \quad \delta g(p, q) = \int_{s \in \partial R} \delta n(s) g_n(p, s) g_n(q, s) + o(\epsilon).$$

which is Hadamard's relation.

The preceding derivation is valid only when $R^* \subset R$. To prove (22) in general it suffices to consider R and R^* as regions both interior to a third region $\tilde{R}$, and to consider the difference of the variation of $\tilde{R}$ to R^* and the variation of $\tilde{R}$ to R . This device is due to Hadamard and is also applied in the standard derivation of (22).

6. Laplace-Beltrami operator. The Hadamard relation remains valid if Δ is replaced by any other self-adjoint second order differential operator which possesses a Green's function g . Thus, for example, Δ may be replaced by an arbitrary Laplace-Beltrami operator merely by furnishing R with an appropriate Riemannian metric. In this case there is no change in the preceding derivation.

7. Inhomogeneous operator. Alternatively, we may add a multiplication to obtain the operation $u \rightarrow \Delta u + \alpha(p)u$. Assuming that $\alpha(p)$ is sufficiently small, we may again consider the functional f defined by

$$(23) \quad f(v, w) = \min_{u|u_p=w} \int_R [2uw + |\nabla u|^2 - \alpha u^2].$$

The appropriate orthogonality relation is now

$$(24) \quad \int_R [\nabla u^{(1)} \cdot \nabla u^{(2)} - \alpha u^{(1)} u^{(2)}] = \int_{\partial R} u_p^{(1)} u_n^{(2)} - \int_R u^{(1)} [\Delta u^{(2)} + \alpha u^{(2)}] = 0.$$

The remainder of the argument proceeds as before.

Since the variational formula is independent of α , one may conclude that it is valid whenever the Green's function exists.

BIBLIOGRAPHY

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