On Some Approximation Methods for Solutions of the Dirichlet Problem for Linear Elliptic Equations of Arbitrary Order

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An important tool in the theory of the Dirichlet problem for a single second-order elliptic partial differential equation is the maximum principle, no analogue of which has yet been found in the more general case of higher order equations or of systems of equations. The portions of the theory of a second-order equation which depend upon the maximum principle in an essential way, such as the PERRON-REMAK theory of sub-harmonic functions or WIENER's generalized solution of the Dirichlet problem, seem therefore to have no significant generalization to the higher order case. There are certain portions of the classical second-order theory and of the more special theory of harmonic functions, however, whose development is usually given using the maximum principle but for which the monotonicity properties of the differential operator are not fundamentally relevant and for which a generalization can be given to higher order equations and to systems of equations using the Hilbert space theory of elliptic differential operators. We shall consider several such topics in this paper, the Bergman kernel function, the Schwarz alternating method, and the Poincaré "sweeping out" process. Our discussion, which is restricted for the sake of notational simplicity to a single elliptic equation, is valid for the more general class of strongly elliptic systems of differential equations.

In Section 1 we present a summary of the definitions and principal results of the Hilbert space approach to the Dirichlet problem for the general linear elliptic partial differential equation. In Section 2 we establish the existence of kernel functions for such problems as well as their approximability in terms of complete orthonormal sequences of solutions of the homogeneous equation. Section 3 is devoted to the Schwarz alternating method which is considered as a special case of a more general process of iterative solution of the Dirichlet problem generalizing the Poincaré *balayage* process.

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Section 1: Let G be an open set in Euclidean n -space with points $x = (x_1, \dots, x_n)$, $D_i = \partial/\partial x_i$, $C_c^\infty(G)$ the family of all complex-valued infinitely differentiable functions with compact support in G . For a multi-index $\alpha = (\alpha_1, \dots, \alpha_r)$, $1 \leq \alpha_i \leq n$, we set $|\alpha| = r$ while $D^\alpha = D_{\alpha_1} \cdots D_{\alpha_r}$. ($D\emptyset =$ the identity if $\emptyset$ is the empty set, $|\emptyset| = 0$.)

$L^2(G)$ is the Hilbert space of equivalence classes of complex-valued square-integrable functions on G with the usual inner product, $(u, v) = \int_G u(x) \overline{v(x)} dx$ ($\bar{v}$ the complex conjugate of v , integration with respect to Lebesgue n -measure). $C^j(G)$ is the family of j -times continuously differentiable functions in G , $C^j(\bar{G})$ the subfamily of $C^j(G)$ of those f such that $D^\alpha f$ is continuous on the closure of G for all α with $|\alpha| \leq j$.

Definition 1: $W^{m,2}(G) = \{u \mid u \in L^2(G); \text{ the distribution derivative } D^\alpha u \in L^2(G) \text{ for every } \alpha \text{ with } |\alpha| \leq m\}$.

For every j with $0 \leq j \leq m$ we may define a form $(u, v)_j$ for u and v in $W^{m,2}(G)$, which is linear in u and conjugate linear in v , by

$$(1) \quad (u, v)_j = \sum_{|\beta|=j} (D^\beta u, D^\beta v).$$

If we define an inner product on $W^{m,2}(G)$ by

$$(2) \quad [u, v] = \sum_{i=0}^m (u, v)_i,$$

it is easy to see that $W^{m,2}(G)$ is a Hilbert space with respect to this inner product with norm $\|u\| = [u, u]^{\frac{1}{2}}$. For each j with $0 \leq j \leq m$ we may define the j -semi-norm on $W^{m,2}(G)$ by $\|u\|_j = (u, u)_j^{\frac{1}{2}}$.

Definition 2: $\dot{W}^{m,2}(G)$ is the closure of $C_c^\infty(G)$ in $W^{m,2}(G)$.

Suppose a function u lies in $W^{m,2}(G) \cap C^{m-1}(G)$ and has null Dirichlet data of order m on the boundary of G , i.e. for every $\epsilon > 0$ there exists a compact set C_ϵ contained in G such that $|D^\beta u| < \epsilon$ on $G - C_\epsilon$ for all $|\beta| \leq m - 1$. Then u lies in $\dot{W}^{m,2}(G)$. $\dot{W}^{m,2}(G)$ will be described as the family of functions in $W^{m,2}(G)$ whose Dirichlet data of order m vanishes on the boundary of G in the variational sense. (A general definition of Dirichlet data for functions in $W^{m,2}(G)$ will be described in Section 3 below, from which this terminology will derive a more literal descriptiveness.)

Let A be a partial differential operator of order $2m$ ($m \geq 1$) with coefficients defined on G . For the sake of simplicity we shall assume throughout that A has infinitely differentiable coefficients in G , although the results depend only upon a very limited amount of differentiability of the coefficients. A may be written (in general, not uniquely) in the form:

$$(3) \quad Au = \sum_{|\alpha|, |\beta| \leq m} D^\beta (a_{\alpha\beta}(x) D^\alpha u),$$

where the $a_{\alpha\beta}$ are infinitely differentiable real-valued functions on G . The characteristic form $a(x, \xi)$ of the differential operator A is defined by

$$a(x, \xi) = \sum_{|\alpha|, |\beta|=m} a_{\alpha\beta}(x) \xi^\alpha \xi^\beta,$$

where ξ is a real n -vector $(\xi_1, \dots, \xi_n)$, $\xi^\alpha = \xi_{\alpha_1} \dots \xi_{\alpha_m}$, and x is a point of G . A is said to be elliptic at x if there exists a constant $c(x)$ such that $a(x, \xi) \geq c(x) \sum_{|\beta|=m} (\xi^\beta)^2$, elliptic in G if it is elliptic for every point of G , uniformly elliptic in G if in addition there exists $c_0 > 0$ such that $c(x) \geq c_0$ for all x in G .

If f and g are two given smooth functions on G , the Dirichlet problem in the strict sense asks for a solution u in $C^{2m}(G)$ of the equation $Au = f$ in G such that $u - g$ has null Dirichlet data in the strict sense on the boundary of G . We may transform this problem into another problem concerning the solution of an operator equation in the Hilbert space $\dot{W}^{m,2}(G)$ (BROWDER [2], [3]; GÄRDING [8]; VISIK [16]).

Corresponding to the formula (3) for A , we define the Dirichlet form $a(u, v)$ for u, v in $\dot{W}^{m,2}(G)$:

$$(4) \quad a(u, v) = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\beta|} (a_{\alpha\beta} D^\alpha u, D^\beta v).$$

Suppose we are given a function $u \in C^{2m}(G) \cap \dot{W}^{m,2}(G)$ for which $Au = f$ with $u - g$ having null Dirichlet data on the boundary of G , $f \in L^2(G)$, $g \in \dot{W}^{m,2}(G)$. If we form the equation $(Au, \phi) = (f, \phi)$ for $\phi \in C_c^\infty(G)$ and integrate by parts, we obtain

$$(5) \quad a(u, \phi) = (f, \phi), \quad u - g \in \dot{W}^{m,2}(G).$$

Definition 3: A function u in $\dot{W}^{m,2}(G)$ is said to be a solution of the Dirichlet problem in the variational sense with inhomogeneous part f and boundary function g provided it satisfies condition (5).

Let A^+ be the formal adjoint differential operator to A :

$$A^+ u = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|+|\beta|} D^\alpha (a_{\alpha\beta}(x) D^\beta u).$$

Every solution of (5) satisfies the equation $(u, A^+ \phi) = (f, \phi)$ for ϕ in $C_c^\infty(G)$, i.e. is a distribution solution of $Au = f$. Let $r_n = [\frac{1}{2}n]$. It has been shown for A elliptic that if $f \in \dot{W}^{r,2}(G)$ for $r \geq r_n + 1$, then $u \in C^{2m}(G)$ and $Au = f$ is satisfied in the ordinary sense (BROWDER [3], FRIEDRICHS [7]). If we suppose moreover that A is uniformly elliptic on the bounded domain G with the boundary of G composed of surfaces of class C^{m+r_n} , $a_{\alpha\beta}$ lie in $C^{r_n}(\bar{G})$, $f \in \dot{W}^{r_n+1}(G)$, $g \in \dot{W}^{r_n+m}(G)$, then every solution u of (5) will lie in $C^{m-1}(\bar{G})$ and have the same Dirichlet data in the strict sense as g on the boundary of G (BROWDER [4], GUSEVA [9], NIRENBERG [12]).

We shall henceforward consider only the Dirichlet problem in the variational sense for an elliptic operator A , bearing in mind the results of the last paragraph connecting this problem with the Dirichlet problem in the strict sense. We shall show that (5) is equivalent to an operator equation in $\dot{W}^{m,2}(G)$.

By the Schwarz inequality there exists $c_1 > 0$ such that $|a(u, v)| \leq c_1 \|u\| \|v\|$ for all u, v in $\dot{W}^{m,2}(G)$. Furthermore $|(f, v)| \leq \|f\|_0 \|v\|_0 \leq \|f\|_0 \|v\|$ for v in $\dot{W}^{m,2}(G)$. For fixed u in $\dot{W}^{m,2}(G)$, $a(u, v)$ is a bounded conjugate-linear form in v for v ranging over the Hilbert space $\dot{W}^{m,2}(G)$. Thus there exists a unique element $Tu \in \dot{W}^{m,2}(G)$ such that $[Tu, v] = a(u, v)$ for $v \in \dot{W}^{m,2}(G)$. T is a bounded linear mapping of $\dot{W}^{m,2}(G)$ into $\dot{W}^{m,2}(G)$ and $\|T\| \leq c_1$. Similarly there exists an element Cf of $\dot{W}^{m,2}(G)$ such that $[Cf, v] = (f, v)$ for all v in $\dot{W}^{m,2}(G)$ and C is a bounded linear transformation of $L^2(G)$ into $\dot{W}^{m,2}(G)$. (Indeed, C is a bounded linear transformation of $W^{m,2}(G)$ into $\dot{W}^{m,2}(G)$. Cf. [5].) Since $\dot{W}^{m,2}(G)$ is a linear subset of $L^2(G)$, we may also consider C as a bounded linear transformation of $\dot{W}^{m,2}(G)$.

Let $v = u - g$. In terms of T, C , and v , condition (5) becomes

$$(6) \quad [Tv, \phi] + [Tg, \phi] = [Cf, \phi] \quad \text{for } \phi \in C_0^\infty(G); v \in \dot{W}^{m,2}(G).$$

Since $C_0^\infty(G)$ is dense in $\dot{W}^{m,2}(G)$, this is simply equivalent to $Tv = Cf - Tg$, $v \in \dot{W}^{m,2}(G)$. The question as to the solutions of (5), the Dirichlet problem in the variational sense, consists therefore of inverting the operator T on $\dot{W}^{m,2}(G)$. A particularly simple situation occurs when G satisfies the condition:

$$(g_0) \quad G \text{ is bounded.}$$

If A is uniformly elliptic on G and if $a_{\alpha\beta} \in C^0(\bar{G})$, it follows from a Fourier transform argument (GÄRDING [8], BROWDER [26]) that there exist $c_2 > 0$, $k_0 \geq 0$ such that:

$$(7) \quad (-1)^m \operatorname{Re} a(u, u) \geq c_2 [u, u] - k_0(u, u), \quad u \in \dot{W}^{m,2}(G).$$

Let $T_1 = (-1)^m T + k_0 C$, T and C being restricted to $\dot{W}^{m,2}(G)$. Then (7) is equivalent to $\operatorname{Re} [T_1 u, u] \geq c_2 \|u\|^2$ for $u \in \dot{W}^{m,2}(G)$. In particular, $\|T_1 u\| \geq [T_1 u, u]/\|u\| \geq c_2 \|u\|$ and T_1 has a bounded inverse T_1^{-1} defined on its range, which is a closed subspace of $\dot{W}^{m,2}(G)$. $R(T_1)$, the range of T_1 , must equal the whole of $\dot{W}^{m,2}(G)$ since if $v \in R(T_1)$, $0 = \operatorname{Re} [T_1 v, v] \geq c_2 \|v\|^2$ implies $v = 0$. Thus T_1^{-1} is a bounded linear transformation of $\dot{W}^{m,2}(G)$. Since $T = (-1)^m T_1 + (-1)^{m+1} k_0 C$, the element v from $\dot{W}^{m,2}(G)$ is a solution of $Tv = Cf - Tg$ if and only if

$$(8) \quad T_1 v - k_0 C v = (-1)^m C f + (-1)^{m+1} T g.$$

If (g_0) holds, it follows from a well known lemma of RELICH ([3], page 32) that C is a compact linear transformation of $\dot{W}^{m,2}(G)$. Hence so is $C_1 = -k_0 T_1^{-1} C$. Applying T_1^{-1} to both sides of (8), we see that (8) is equivalent to

$$(9) \quad v + C_1 v = (-1)^m T_1^{-1} C f + (-1)^{m+1} T_1^{-1} T g.$$

We shall restrict ourselves in the remainder of this paper to elliptic differential operators A for which the inequality (7) holds with $k_0 = 0$, *i.e.* satisfying the condition:

$$(p_0) \quad \operatorname{Re} a(u, u) \geq c_2 \|u\|^2, \quad u \in \dot{W}^{m,2}(G).$$

This class includes in particular all homogeneous $2m^{\text{th}}$ order elliptic operators with constant coefficients as well as second-order elliptic operators with non-positive zeroth order term. Since (g_0) was used only to ensure the compactness of C , and since (p_0) implies that C may be taken as zero, the Dirichlet problem in the variational sense may be solved under the condition (p_0) even on unbounded domains. In any case, $C = 0$ and the solution u of (5) in this case is given by

$$(10) \quad u = g + (-1)^{m+1} T_1^{-1} T g + (-1)^m T_1^{-1} C f.$$

If we set $P = I + (-1)^{m+1} T_1^{-1} T$, we see

Definition 4: The function $u = Pg$ is the solution of the equation $Au = 0$ having the same Dirichlet data in the variational sense on the boundary of G as the prescribed function g in $W^{m,2}(G)$.

Definition 5: H is the subspace of $W^{m,2}(G)$ consisting of all solutions u of the equation $Au = 0$.

P is a (in general, non-orthogonal) projection of $W^{m,2}(G)$ onto H .

The structure of the subspace H is very special because of the ellipticity of A . In particular, all elements of H lie in $C^\infty(G)$ and we have the following:

(h₁) For each multi-index α there is a continuous function $c_\alpha(x)$ defined on G such that, for all u in H ,

$$|D^\alpha u(x)| \leq c_\alpha(x) \|u\|_0$$

([2c], [7]).

Section 2: Let $\langle u, v \rangle$ be an inner product defined on a subset F of $W^{m,2}(G)$ with $\|u\|_F = \langle u, u \rangle^{1/2}$. We suppose that F contains $C_c^\infty(G)$ and satisfies the following conditions:

(f₁) F is a Hilbert space with respect to $\|\cdot\|_F$.

(f₂) There exists $c_4 > 0$ such that, for all u in F , $\|u\| \leq c_4 \|u\|_F$.

(f₃) If $\langle u, \phi \rangle = 0$ for all $\phi \in C_c^\infty(G)$ for a given u in F , then $u \in C^\infty(G)$.

Let us remark that any inner product on $W^{m,2}(G)$ yielding a norm equivalent to $\|\cdot\|$ obviously satisfies conditions (f₁) and (f₂). If $\langle u, v \rangle = a(u, v)$ where $a(u, v)$ is a Dirichlet form corresponding to an elliptic differential operator with infinitely differentiable coefficients, then (f₃) is satisfied.

By the property (h₁) of the subspace H we have for each x in G

$$(11) \quad |Pu(x)| \leq c(x) \|Pu\|_0 \leq c(x) \|Pu\| \leq c_3 c_4 c(x) \|u\| \leq c_3 c_4 c(x) \|u\|_F.$$

Hence $(Pu)(x)$ is a bounded linear functional of u in F and uniquely representable in the form:

$$(12) \quad (Pu)(x) = \langle u, k_x \rangle, \quad u \in F,$$

where k_x is an element of F depending upon the point x in G . By the same argument, for each multi-index α there exists a unique element k_x^α in F such that

$$(13) \quad D^\alpha (Pu)(x) = \langle u, k_x^\alpha \rangle, \quad u \in F.$$

Consider the measurable functions $k(x, y)$, $k^\alpha(x, y)$ defined up to a set of measure zero in $G \times G$ by the prescription $k(x, y) = k_x(y)$, $k^\alpha(x, y) = k_x^\alpha(y)$. Let D_x^α , D_y^β denote the partial derivatives in the distribution sense with respect to the x and y variables respectively.

Lemma 1: $D_x^\alpha k(x, y) = k^\alpha(x, y)$.

Proof of Lemma 1: It suffices, first, to show that $D_x^\alpha k_x$ is an element of F and that, for all u in F , $D_x^\alpha(Pu)(x) = \langle u, D_x^\alpha k_x \rangle$, since k_x^α is the unique element of F satisfying this last relation. It suffices to show, moreover, that if g is a function from G to F such that $\langle u, g \rangle = s(x)$ where $s(x)$ is a differentiable function in G , then g is a weakly differentiable function from G to F and the distribution derivative $D_i g$ coincides with the weak derivative in F . The weak differentiability of the function g is equivalent to the differentiability of the function $s(x)$. Since the difference quotients converge weakly in F and, since by (f_2) the injection mapping of F into $W^{m,2}(G)$ is bounded and hence weakly continuous, they converge to the same limit weakly in $W^{m,2}(G)$ and hence as distributions. Since the distribution derivative is the limit in the distribution topology of the difference quotients, the two derivatives must be the same.

We know that for $\phi \in C_c^\infty(G)$, $P\phi = 0$. Therefore $\langle \phi, k_x^\alpha \rangle = 0$ for all α and, by (f_3) , $D_y^\beta k^\alpha(x, y)$ is continuous in G for every x in G and every α and β . It follows, from the fact that $D_x^\alpha D_y^\beta k(x, y)$ is a locally summable function for every α and β , that $k(x, y)$ lies (after change on a set of measure zero) in $C^\infty(G \times G)$ (SCHWARTZ [13], p. 47). Thus we have

Theorem 1: For each F contained in $W^{m,2}(G)$ and satisfying conditions (f_1) , (f_2) , and (f_3) above, there exists a kernel function $k(x, y) \in C^\infty(G \times G)$ such that if $k_x(y) = k(x, y)$, $k_x \in F$ for every x in G and $\langle u, k_x \rangle = Pu(x)$.

We now specialize the space F and the operator A under consideration. We shall assume that A is formally self-adjoint and has a Dirichlet form satisfying the following conditions:

$$(p_1) \quad a(u, v) = a(u, v); \quad a(u, u) \geq c_s \|u\|^2 \text{ for } u, v \in W^{m,2}(G)$$

$$(c_s > 0, \text{ independent of } u).$$

The prototype of such operators A is the case of the second-order operator $\sum_{i,j} D_i(a_{ij}(x) D_j u) + c(x)u$, $c(x) < -c_s$. The properties of the kernel function for operators satisfying condition (p_2) defined in Section 3 will be discussed in that section and includes the case of $c(x) \equiv 0$.

If A satisfies (p_1) , we choose F to be $W^{m,2}(G)$ and take $\langle u, v \rangle = a(u, v)$. It is easy to see that conditions (f_1) , (f_2) , and (f_3) are satisfied. Since $W^{m,2}(G)$ is separable, F and its closed subspace H are separable Hilbert spaces.

Theorem 2: Suppose that A satisfies (p_1) , F is the whole of $W^{m,2}(G)$ with the inner product $\langle u, v \rangle = a(u, v)$, and $k(x, y)$ is the kernel function described in Theo-

rem 1. Let $\{\psi_i(x)\}$ be a complete orthonormal sequence in H with respect to the inner product $\langle u, v \rangle$ and $s_r(x, y) = \sum_{i=1}^r \overline{\psi_i(x)} \psi_i(y)$. Then $s_r(x, y)$ converges uniformly to $k(x, y)$ on compact subsets of $G \times G$. Similarly for every α and β , $D_x^\alpha D_y^\beta k(x, y)$ is the uniform limit of $D_x^\alpha D_y^\beta s_r(x, y)$ on every compact subset of $G \times G$.

Proof of Theorem 2: Since $P\phi = 0$ for all $\phi \in C_0^\infty(G)$, (12) implies that $a(k_x, \phi) = 0$ for all such ϕ . Therefore $k_x \in H$ for each x in G . Similarly $D_x^\alpha k_x \in H$ for every x in G and every multi-index α . On the other hand, since $\psi_i \in H$, $P\psi_i = \psi_i$. Hence $P\psi_i(x) = \psi_i(x) = \langle \psi_i, k_x \rangle$. Since $k_x \in H$ for every x in G , we have $k_x = \sum_{i=1}^\infty \overline{\psi_i(x)} \psi_i$, the infinite series converging in H . Let $s_{r,x}$ be the element of H defined by $s_{r,x} = \sum_{i=1}^r \overline{\psi_i(x)} \psi_i$. By property (h_1) of H , since the sequence $s_{r,x}$ for fixed x converges strongly to k_x in H as $r \rightarrow \infty$, it converges uniformly on every compact subdomain together with all the derivatives $D_y^\beta s_{r,x}$. Since $s_r(x, y) = s_r(y, x)$ and, for each x , $s_r(x, y)$ converges to $k(x, y)$ uniformly in y on compact subsets of G , $k(x, y) = k(y, x)$. It follows immediately that, for fixed y in G , $s_r(x, y)$ converges to $k(x, y)$ uniformly in x on every compact subset of G . Similarly $D_x^\alpha s_r(x, y)$ converges to $D_x^\alpha k(x, y)$ as $r \rightarrow \infty$ uniformly in x on every compact subset of G . Since $D_x^\alpha (P\psi_i)(x) = D_x^\alpha \psi_i(x) = \langle \psi_i, D_x^\alpha k_x \rangle$, we have $D_x^\alpha D_y^\beta s_r(x, y)$ converging to $D_x^\alpha D_y^\beta k(x, y)$ uniformly for a fixed x in G and y in a compact subset of G or for a fixed y in G and x in a compact subset of G .

To complete the proof of the sharper result stated in Theorem 3 we consider a compact subset K of G . We have already observed that $D_x^\alpha k_x \in H$ for every multi-index α and that $D_x^\alpha \psi_i(x) = \langle \psi_i, D_x^\alpha k_x \rangle$. By the Parseval equality $\|D_x^\alpha k_x\|_F^2 = \sum_{i=1}^\infty |D_x^\alpha \psi_i(x)|^2$. Since $|\langle u, D_x^\alpha k_x \rangle| = |D_x^\alpha (Pu)(x)| \leq c'_\alpha(x) \|u\|_F$ for all u in $W^{m,2}(G)$, we have $\|D_x^\alpha k_x\|_F \leq c'_\alpha(x)$, where $c'_\alpha(x)$, which is a constant multiple of the $c_\alpha(x)$ of (h_1) , is continuous on G . Thus $\sum_{i=1}^\infty |D_x^\alpha \psi_i(x)|^2 \leq c_{\alpha,6}^2(K)$ for all x in K . In particular,

$$(14) \quad |D_x^\alpha D_y^\beta s_r(x, y)| \leq \left\{ \sum_{i=1}^r |D_x^\alpha \psi_i(x)|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^r |D_y^\beta \psi_i(y)|^2 \right\}^{\frac{1}{2}} \\ \leq c_{\alpha,6}(K) \cdot c_{\beta,6}(K); \quad (x, y) \in K \times K.$$

It follows that

$$(15) \quad \iint_{K \times K} |D_x^\alpha D_y^\beta k(x, y) - D_x^\alpha D_y^\beta s_r(x, y)|^2 dx dy \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

since the integrand is uniformly bounded on a set of finite measure and converges to zero everywhere. But then, by a theorem of SOBOLEV and SCHWARTZ (SCHWARTZ [13], p. 47), $s_r(x, y)$ converges uniformly to $k(x, y)$ on $K_1 \times K_1$ where K_1 is a compact subset of the interior of K . The same is true for the derivatives $D_x^\alpha D_y^\beta s_r(x, y)$, and the theorem is proved.

Section 3: In the present section we shall assume our differential operator A to be formally self-adjoint and such that it can be written in the following

“homogeneous” form

$$(16) \quad Au = \sum_{|\alpha|, |\beta|=m} D^\beta(a_{\alpha\beta}(x)D^\alpha u) \quad (a_{\alpha\beta} = a_{\beta\alpha})$$

with the corresponding Dirichlet form $a(u, v)$ satisfying the following positivity condition:

$$(p_2) \quad \text{There exists a constant } c_7 > 0 \text{ such that } a(u, u) \geq c \|u\|_m^2 \\ \text{for all } u \text{ in } W^{m,2}(G).$$

Let F_0 be the finite-dimensional subspace of $W^{m,2}(G)$ spanned by the monomials with total degree less than m . If (p_2) is valid and if for an element u of $W^{m,2}(G)$ we have $a(u, u) = 0$, then $\|u\|_m = 0$ and $u \in F_0$. Conversely, if $u \in F_0$, $a(u, v) = 0$ for all v in $W^{m,2}(G)$.

We impose several restrictions upon the domain G . The first is

$$(g_1) \quad \text{If } S \text{ is a bounded set in } W^{m,2}(G), S \text{ is compact in } W^{m-1,2}(G).$$

It follows from theorems of SOBOLEV and KONDRASHOV that (g_1) is valid for any bounded domain G for which there exist $h > 0$ and $a_0 > 0$ such that at every point x of G there exists a right circular cone with vertex at x lying completely in G with height h and cross-sectional area a_0 (SOBOLEV [10], KONDRASHOV [15]). In particular, (g_1) will be satisfied by any domain for which we have

$$(g_2) \quad G \text{ is a bounded domain with boundary } B(G) \text{ composed of a finite} \\ \text{number of surfaces of class } C^m \text{ in } E^n.$$

It follows from (g_2) that $C^m(\bar{G})$ is dense in $W^{m,2}(G)$ (cf. LIONS [11], p. 208). Let B be the boundary of G and $V_0 = \sum_{|\beta| \leq m} \oplus (L_2(B))_\beta$, where for each multi-index β the corresponding summand is a replica of $L^2(B)$ formed with respect to integration in surface area. We define a mapping ζ of $C^m(\bar{G})$ into V_0 by setting $(\zeta u)_\beta = D^\beta u|_B$. It follows from (g_2) that ζ is a bounded linear mapping of the dense subset $C^m(\bar{G})$ of $W^{m,2}(G)$ into V_0 and hence extendable to a bounded linear mapping of $W^{m,2}(G)$ into V_0 which we shall also designate as ζ . Furthermore $\zeta u = 0$ if and only if $u \in \dot{W}^{m,2}(G)$ (SCHWARTZ [14]).

For the finite-dimensional subspace F_0 of $W^{m,2}(G)$ defined above, $\zeta(F_0)$ is a finite-dimensional and hence closed subspace of V_0 . If $V'_0 = V_0 \ominus \zeta(F_0)$, V'_0 is a closed linear subspace of V_0 . If $u \in W^{m,2}(G)$, the projection of ζu on V'_0 equals $\zeta u - \zeta f$, where f is some element of F_0 and hence lies in $\zeta(W^{m,2}(G))$. Thus $\zeta(W^{m,2}(G)) = \zeta(F_0) \dot{+} \zeta(W^{m,2}(G)) \cap V'_0$. We set $W_1 = \zeta^{-1}(V'_0)$. Since $(\zeta u, \zeta f)_v$ for each fixed f in F_0 is a continuous linear functional of u in $W^{m,2}(G)$, W_1 is a closed subspace of $W^{m,2}(G)$. Further, $W^{m,2}(G) = W_1 \dot{+} F_0$.

Lemma 2: *The inner product $a(u, v)$ on W_1 yields a norm equivalent to the $W^{m,2}$ -norm.*

Proof of Lemma 2: Since W_1 has no elements in common with F_0 , $a(u, u) > 0$ for $u \in W_1$. Suppose that Lemma 2 were false. Then there must exist a sequence u_r with $u_r \in W_1$ such that $\|u_r\| = 1$ while $a(u_r, u_r) \rightarrow 0$ as $r \rightarrow \infty$. By (p₂) it follows that $\|u_r\|_m \rightarrow 0$ as $r \rightarrow \infty$. Choosing a subsequence by (g₁), we may assume that $\{u_r\}$ is a Cauchy sequence in $W^{m-1,2}(G)$. But then

$$\|u_r - u_s\|^2 \leq \sum_{i < m} \|u_r - u_s\|_i^2 + (\|u_r\|_m + \|u_s\|_m)^2 \rightarrow 0$$

as $r, s \rightarrow \infty$, so that $\{u_r\}$ is a Cauchy sequence in $W^{m,2}(G)$. Hence u_r converges in $W^{m,2}(G)$ to some function v from $W^{m,2}(G)$ and $\|v\| = \lim_r \|u_r\| = 1$. Since W_1 is closed and the sequence lies in W_1 , $v \in W_1$. On the other hand, $\|u\|_m$ is a continuous function of u in $W^{m,2}(G)$, so that $\|v\|_m = \lim_r \|u_r\|_m = 0$ and $v \in F_0$. Since $W_1 \cap F_0 = \{0\}$, $v = 0$ which contradicts the fact that $\|v\| = 1$. Thus the lemma must be true.

A particular consequence of Lemma 2 is that W_1 is a Hilbert space with respect to the inner product $a(u, v)$. If we specialize the space F of Theorem 1 to be our Hilbert space W_1 with inner product $a(u, v)$ and repeat the argument of Theorem 2, we obtain

Theorem 3: *There is an unique function $k(x, y) \in C^\infty(G \times G)$ which is the kernel function for W_1 in the sense of Theorem 1. Moreover, if $\psi_i(x)$ are a complete orthonormal sequence in $W_1 \cap H$ with respect to the inner product $a(u, v)$, then*

$$(17) \quad k(x, y) = \sum_{i=1}^{\infty} \overline{\psi_i(x)} \psi_i(y),$$

the series converging uniformly on compact subsets of $G \times G$ together with all partial derivatives.

Let G_1 be an open subset of G and J_{G_1} the natural projection of $W^{m,2}(G)$ into $W^{m,2}(G_1)$ with $J_{G_1}u$ defined to be the restriction of u to G_1 . Let K_{G_1} be the norm-preserving mapping of $W^{m,2}(G_1)$ into $W^{m,2}(G)$ obtained by extending by continuity the identity mapping of $C_c^\infty(G_1)$ into $C_c^\infty(G)$. Thus $K_{G_1}u$ is the element of $W^{m,2}(G)$ obtained from u by setting u equal to zero outside G_1 . Finally, let Q_{G_1} be the mapping of $W^{m,2}(G_1)$ into itself defined by letting $Q_{G_1}u$ be the solution of $\Delta v = 0$ in G_1 having the same Dirichlet data on the boundary of G_1 as u . We remark that if Δ satisfies (p₂) on G , the Dirichlet problem is uniquely solvable on every bounded open subset G_1 of G . In particular, if G satisfies (g₂), it is bounded and the Dirichlet problem on G is uniquely solvable. Let Pu denote, as before, the solution of the equation $\Delta v = 0$ in G having the same Dirichlet data as the given function u in $W^{m,2}(G)$.

If $u \in W^{m,2}(G)$, $J_{G_1}u$ is its restriction to the subset G_1 and $Q_{G_1}(J_{G_1}u)$ is the solution of $\Delta v = 0$ on G_1 having the same Dirichlet data on the boundary of G_1 as $J_{G_1}u$. Then the function $J_{G_1}u - Q_{G_1}(J_{G_1}u)$ lies in $W^{m,2}(G_1)$. Its extension to G , setting it equal to zero outside G_1 , is $g = K_{G_1}(J_{G_1}u - Q_{G_1}(J_{G_1}u))$. Finally, $v = Pu = u - g$ is the function from $W^{m,2}(G)$ which is equal to u outside

G_1 and which inside G_1 is the solution of the equation $Av = 0$ having the same Dirichlet data as u on the boundary of G_1 . P_{G_1} for each open subset G_1 of G is a bounded idempotent linear mapping of $W^{m,2}(G)$ into itself.

Let us note that $P_{G_1}u = u - g$, where $g \in \dot{W}^{m,2}(G)$. As a consequence, $\zeta P_{G_1}u = \zeta u$ for all u in $W^{m,2}(G)$. In particular, if $u \in W_1$, then $P_{G_1}u \in W_1$ for any open subset G_1 of G .

We shall construct a new inner product on $W^{m,2}(G)$ which yields a Hilbert space structure on that set with norm equivalent to the $W^{m,2}$ -norm and in which P_{G_1} is an orthogonal projection for every open subset G_1 of G . By our previous remarks and Lemma 2, $W^{m,2}(G) = W_1 + F_0$, where W_1 is a closed subspace of $W^{m,2}(G)$ and the inner product $a(u, v)$ on W_1 yields a norm equivalent to $\|\cdot\|$.

Let $u = w_1 + f$, $v = w_2 + g$, where $w_1, w_2 \in W$; $f, g \in F_0$. We define the inner product $\langle u, v \rangle = a(w_1, w_2) + [f, g] = a(u, v) + [f, g]$. Note that $\langle u, u \rangle = a(w_1, w_1) + [f, f] \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$. Furthermore, $W^{m,2}(G)$ is complete with respect to the norm $\|u\| = \langle u, u \rangle^{1/2}$, for if $\{u_r\}$ ($u_r = w_r + f_r$, $w_r \in W_1$, $f_r \in F_0$) form a Cauchy sequence with respect to this norm, then $\{w_r\}$ and $\{f_r\}$ are Cauchy sequences with respect to the $W^{m,2}$ -norm and therefore converge in that norm to elements w_0 and f_0 of W_1 and F_0 , respectively, since W_1 and F_0 are closed subspaces of $W^{m,2}(G)$. It follows that u_r converges to $w_0 + f_0$ in the new norm.

We shall now verify that P_{G_1} is an orthogonal projection with respect to this new inner product. Let us remark first that P_{G_1} is the identity operator on F_0 since F_0 is contained in H . Note next that for any v in $W^{m,2}(G)$, $v = P_{G_1}v + h_1$ where h_1 is the limit in $W^{m,2}(G)$ of a sequence $\{\phi_r\}$ with $\phi_r \in C_0^\infty(G_1)$. In particular, if u is any element of $W^{m,2}(G)$, $a(P_{G_1}u, h_1) = \lim_r a(P_{G_1}u, \phi_r) = 0$, since $P_{G_1}u$ is a solution of $Av = 0$ in G_1 . Therefore $a(P_{G_1}u, v) = a(P_{G_1}u, P_{G_1}v)$. Finally, $\langle P_{G_1}u, v \rangle$, for $u = w_1 + f$, $v = w_2 + g$ as above, equals $a(P_{G_1}u, v) + [f, g]$. Thus $\langle P_{G_1}u, v \rangle = a(P_{G_1}u, P_{G_1}v) + [f, g] = \langle u, P_{G_1}v \rangle$.

Let G_j be a sequence (finite or infinite) of open subsets of G , $1 \leq j \leq J \leq \infty$. Suppose that $\{G_j\}$ forms a covering of G . Let π be an infinite sequence of positive integers $(n_1, n_2, \dots)$ with $1 \leq n_k \leq J$.

Definition 6: The sequence π is said to be a permissible sequence if for each j with $1 \leq j \leq J$ there exists an integer K_j such that every block of K_j successive indices n_k includes the integer j at least once.

It is easy to verify that for each J , finite or infinite, there are permissible sequences of J indices. A very important special case is that of $J = 2$ and the sequence π consisting of alternating 1's and 2's. Let $P_j = P_{G_j}$, $P = P_G$.

Theorem 4: Let u be a given function in $W^{m,2}(G)$ and let $R_s u = P_{n_s} P_{n_{s-1}} \dots P_{n_1} u$ for a given permissible sequence π . Then:

(a) $R_s u$ converges weakly to Pu in $W^{m,2}(G)$ as $s \rightarrow \infty$.

(b) $R_s u$ converges strongly to Pu in $W^{m-1,2}(G)$ as well as in $L^p(G)$ with $p = n/(n - 2m)$ for $2m < n$, all $L^p(G)$ for $n = 2m$, and in $C^*(\bar{G})$ if $m + s > \frac{1}{2}n$.

(c) Let $\bar{G}_i$ be the boundary of G_i , $G' = G - \bigcup_i \bar{G}_i$. Then $R_s u$ converges uniformly to Pu on every compact subset of G' together with all partial derivatives.

(d) If $J = 2$ and π is the alternating sequence, $R_s u$ converges strongly to Pu in $W^{m,2}(G)$.

Remark on Theorem 4: The sequence R_s for the alternating sequence when $J = 2$ is that of the Schwarz alternating process; the more general sequence R_s for an arbitrary permissible sequence yields a generalization of the Poincaré balayage process.

Proof of Theorem 4: (b) follows from (a), together with theorems of SOBOLEV and KONDRASHOV on the imbedding of $W^{m,2}(G)$ in L^p , $C^*(\bar{G})$ (SOBOLEV [15], KONDRASHOV [10]). Note that (g₁) shows that weak convergence in $W^{m,2}(G)$ implies strong convergence in $W^{m-1,2}(G)$. (c) follows directly from (a) and (b) since every compact subset K of G' can be decomposed into a finite number of subsets K_λ each contained in some G_i and for $s \geq s_0$, for some s_0 , all $R_s u$ will be solutions of the equation $\Delta v = 0$ on some fixed neighborhood of each K_λ .

Since P_1 and P_2 are orthogonal projections with respect to our new inner product, (d) is equivalent to a well known theorem of RELICH and VON NEUMANN. Note that $R_{2s+1} = (P_1 P_2 P_1)^s$ where $P_1 P_2 P_1$ is self-adjoint and $0 \leq P_1 P_2 P_1 \leq I$. It follows easily that $R_{2s+1} u$ is strongly convergent. Since $R_{2s} u = P_2 R_{2s-1} u$, $R_{2s} u$ is strongly convergent to a limit w . Let v be the strong limit of $R_{2s+1} u$. Then $P_2 v = w$, $P_1 w = v$ implies $\|w\| \leq \|v\| \leq \|w\|$ and the equality of norms implies $v = w$, $P_1 v = P_2 v = v$.

(a) follows from the following lemma whose proof is due to S. KAKUTANI. KAKUTANI's original proof considered only a finite number of distinct projections, and we have made a slight alteration to allow for an infinite number of different P_i .

Lemma 3: Let $\{P_i\}$ be a sequence of orthogonal projections in a Hilbert space W , π a permissible sequence of indices, $R_s = P_{\pi_s} P_{\pi_{s-1}} \cdots P_{\pi_1}$, M_i the range of P_i , M_0 the intersection of all M_i , and P_0 the projection on M_0 . Then $R_s u$ converges weakly in W to $P_0 u$ as $s \rightarrow \infty$.

Proof of Lemma 3: Since M_0 is contained in M_i for every j , $P_i M_0 = M_0$ and $P_i(W \ominus M_0) \subset W \ominus M_0$. Since R_s is the identity on M_0 , it suffices to show that $R_s u$ converges weakly to zero if $u \in W \ominus M_0$. Thus we may assume $M_0 = \{0\}$.

Let $u_s = R_s u$. Since $u_{s-1} - u_s = (I - P_{\pi_s})u_{s-1}$ is orthogonal to $u_s = P_{\pi_s} u_{s-1}$, $\|u_{s-1} - u_s\|^2 = \|u_{s-1}\|^2 - \|u_s\|^2$. Summing, we see that $\sum_s \|u_{s-1} - u_s\|^2 \leq \|u\|^2$ and as a consequence $\|u_{s-1} - u_s\| \rightarrow 0$ as $s \rightarrow \infty$. Iterating the argument, for every fixed integer r , $\|u_{s+r} - u_s\| \rightarrow 0$ as $s \rightarrow \infty$.

Since $\|u_s\| \leq \|u\|$, by the compactness of closed balls in Hilbert space, it suffices to show that if $\{u_{s_k}\}$ is a weakly convergent subsequence of the sequence

u_s with limit v , then $v = 0$. Let j be an arbitrary integer. If u_{s_k} converges weakly to v as $k \rightarrow \infty$, then, by the conclusion of the preceding paragraph, so does $u_{s_{k+r}}$ for $0 \leq r \leq K_j$. So does any subsequence obtained by interspersing this finite family of subsequences. For each k choose r_k , $0 \leq r_k \leq K_j$, so that $n_{s_{k+r_k}} = j$. Then $u_{s_{k+r_k}}$ converges weakly to v as $k \rightarrow \infty$. But $u_{s_{k+r_k}} \in M_j$, and since M_j is a closed subspace, it contains all weak limits of sequences from M_j . Thus $v \in M_j$. Since j was arbitrary, $v \in M_0$ and since by assumption $M_0 = \{0\}$, $v = 0$.

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