Uniqueness Properties in the Theory of Differential Operators of Elliptic Type

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Introduction. Let L be a linear elliptic operator of order s with coefficients defined in a bounded domain B of E^N , where E^N denotes the space of N real variables $x = (x_1, \dots, x_N)$ with norm $|x| = (\sum x_i^2)^{\frac{1}{2}}$. For simplicity we assume $0 \in B$. In this paper we prove (Theorem 1) that the non-negative functions $u(x)$ which satisfy $Lu \leq 0$ possess the following uniqueness property: *If $u(x)$ vanishes at the origin of infinite order, that is, faster than any power of $|x|$, then $u(x)$ vanishes identically in B .* In particular, we get this uniqueness property for solutions of $Lu = 0$ which do not change sign in B . For second order elliptic equations this uniqueness property has been proved for arbitrary solutions. Special cases were proved by T. CARLEMAN [2], C. MÜLLER [8], E. HEINZ [5], P. HARTMAN & A. WINTNER [4] and P. LAX [7]; the general case was proved by N. ARONSZAJN [1] and H. CORDES [3]. For higher order elliptic equations it is still an open question whether this uniqueness property holds for arbitrary solutions.

HEINZ and ARONSZAJN used the ideas of CARLEMAN in [2] combined with an appropriate L^2 -estimate for $u(x) |x|^{-n}$ ($n = 1, 2, \dots$). Our method involves an appropriate L^1 -estimate of $u(x) |x|^{-n}$.

Later on, we give some converses of Theorem 1. It is shown that some uniqueness properties characterize the type of partial differential operator.

1. Theorem 1. *Let the operator*

$$(1) \quad L = \sum_{i=0}^s \sum_{i_1 + \dots + i_N = i} f_{i_1 \dots i_N}(x) \frac{\partial^i}{\partial x_1^{i_1} \dots \partial x_N^{i_N}}$$

be defined and elliptic in a domain $B \subset E^N$, the coefficients $f_{i_1 \dots i_N}(x)$ possessing $j = j_1 + \dots + j_N$ continuous derivatives in B . If $u(x)$ has s continuous derivatives in B and satisfies the conditions

(a) *for $x \in B$,*

$$(2) \quad Lu(x) \leq 0, \quad u(x) \geq 0,$$

(b) for every positive integer n ,

$$(3) \quad \sum_{j=0}^{s-1} \sum_{i_1+\dots+i_N=j} \left| \frac{\partial^j u(x)}{\partial x_1^{i_1} \dots \partial x_N^{i_N}} \right| = O(|x|^n) \quad (|x| \rightarrow 0),$$

then $u(x) \equiv 0$ in B .

For convenience, we give the proof in the case $N = 3$. In the sequel A_i will be used to denote appropriate positive constants. The function $u_n(x) = u(x) |x|^{-n}$ is defined in every sphere V ($|x| \leq R$) contained in B . We shall take $R < 1$. One can easily verify that

$$(4) \quad \frac{\partial^j |x|^n}{\partial x_1^{i_1} \partial x_2^{i_2} \partial x_3^{i_3}} = n(n-2) \dots (n-2j+2) |x|^{n-2j} x_1^{i_1} x_2^{i_2} x_3^{i_3} + R_j(x, n) \quad (j \leq s),$$

where

$$(5) \quad |R_j(x, n)| \leq A_1 n^{j-1} |x|^{n-j}.$$

Denoting by $D^j v(x)$ any j^{th} partial derivative of $v(x)$, and using (4), we have

$$(6) \quad \frac{\partial^s}{\partial x_1^{i_1} \partial x_2^{i_2} \partial x_3^{i_3}} (|x|^n u_n(x)) = n(n-2) \dots (n-2s+2) |x|^{n-2s} x_1^{i_1} x_2^{i_2} x_3^{i_3} u_n(x) + R_s(x, n) u_n(x) + \sum_{i=0}^{s-1} c_i (D^i |x|^n) (D^{s-i} u_n(x)),$$

where c_i are positive constants, $c_i \leq A_2$. Multiplying (6) by $f_{i_1, i_2, i_3}(x)$, $\sum j_i = s$, and summing up, we easily see that the inequality $|x|^{-n} L(|x|^n u_n(x)) = |x|^{-n} L u(x) \leq 0$ can be written in the form

$$(7) \quad \begin{aligned} & n(n-2) \dots (n-2s+2) \frac{1}{|x|^{2s}} \sum_{i_1+i_2+i_3=s} f_{i_1, i_2, i_3}(x) x_1^{i_1} x_2^{i_2} x_3^{i_3} u_n(x) \\ & + \frac{c(x)}{|x|^n} R_s(x, n) u_n(x) + \frac{1}{|x|^n} \sum_{i_1+i_2+i_3=s} f_{i_1, i_2, i_3}(x) \sum_{i=0}^{s-1} c_i (D^i |x|^n) (D^{s-i} u_n(x)) \\ & + \frac{1}{|x|^n} \sum_{i=0}^{s-1} \sum_{i_1+i_2+i_3=j} f_{i_1, i_2, i_3}(x) \sum_{i=0}^j c_{ij} (D^i |x|^n) (D^{j-i} u_n(x)) \leq 0, \end{aligned}$$

where c_{ij} are positive constants, $c_{ij} \leq A_3$ and $|c(x)| \leq A_3$.

We now integrate the inequality (7). Since this inequality involves s^{th} partial derivatives of $u_n(x)$, we integrate it over the domain $V_\epsilon : \epsilon \leq |x| \leq R$, where $\epsilon > 0$. Let us first carry out the integration of the terms

$$(8) \quad g_1(x) = f_{i_1, i_2, i_3}(x) \frac{D^i |x|^n}{|x|^n} D^{s-i} u_n(x) \quad (\sum j_i = s, 0 \leq i \leq s-1),$$

$$(9) \quad g_2(x) = f_{i_1, i_2, i_3}(x) \frac{D^i |x|^n}{|x|^n} D^{j-i} u_n(x) \quad (\sum j_i = j, 0 \leq i \leq j \leq s-1).$$

Denote, for brevity, $f_{i,i,i,i}(x) = f(x)$ and let

$$I_i(\epsilon) = \int_{V_\epsilon} g_i(x) dV_x \quad (i = 1, 2).$$

Integrating by parts, in $I_1(\epsilon)$, $s - i$ times, we get

$$(10) \quad \begin{aligned} I_1(\epsilon) = & \sum_{k=1}^{s-i} (-1)^{k-1} \int_{S_\epsilon} D^{k-1} \left(f(x) \frac{D^i |x|^n}{|x|^n} \right) D^{s-i-k} u_n(x) \cos(\nu, x_k) dS_x \\ & + (-1)^{s-i} \int_{V_\epsilon} D^{s-i} \left(f(x) \frac{D^i |x|^n}{|x|^n} \right) u_n(x) dV_x, \end{aligned}$$

where S_ϵ is the boundary of V_ϵ , ν is the outwardly directed normal to S_ϵ , and the integration by parts at the k^{th} time is carried out with respect to the variable x_k . To estimate $I_1(\epsilon)$, note that

$$\begin{aligned} \left| D^i \left(f(x) \frac{D^i |x|^n}{|x|^n} \right) \right| & \leq A_4 \frac{n^i}{|x|^{i+i}} \quad (0 \leq i \leq s; 0 \leq l \leq s - i), \\ |D^l u_n(x)| & = \left| D^l \frac{u(x)}{|x|^n} \right| \leq A_5 \frac{n^l}{|x|^{n+l}} \quad (0 \leq l \leq s - 1). \end{aligned}$$

Using these estimates and (10), we get

$$\begin{aligned} |I_1(\epsilon)| & \leq \sum_{k=1}^{s-i} \int_{|x|=\epsilon} \left| D^{k-1} \left(f(x) \frac{D^i |x|^n}{|x|^n} \right) D^{s-i-k} u_n(x) \right| dS_x \\ & \quad + A_6 R^2 \frac{n^{s-1}}{R^{n+s-1}} + A_7 n^i \int_{V_\epsilon} \frac{u_n(x)}{|x|^s} dV_x. \end{aligned}$$

Taking $\epsilon \rightarrow 0$ and noting (see (3)) that $D^l u_n(x)$ ($0 \leq l \leq s - 1$) vanishes of infinite order at the origin, we obtain

$$(11) \quad |I_1(0)| \leq A_6 \frac{n^{s-1}}{R^{n+s-1}} + A_7 n^i \int_V \frac{u(x)}{|x|^{n+s}} dV_x \quad (0 \leq i \leq s - 1).$$

Similarly, one can prove that

$$(12) \quad |I_2(0)| \leq A_8 \frac{n^{j-1}}{R^{n+j-1}} + A_9 n^i \int_V \frac{u(x)}{|x|^{n+i}} dV_x \quad (0 \leq i \leq j \leq s - 1).$$

The terms $g_1(x)$ and $g_2(x)$ in (8) and (9) are typical terms of the last two sums on the left side of (7). To estimate the integral of $|x|^{-n} R_s(x, n) u_n(x)$, we make use of (5). We obtain

$$(13) \quad \left| \int_V \frac{R_s(x, n)}{|x|^n} u_n(x) dV_x \right| \leq A_{10} n^{s-1} \int_V \frac{u_n(x)}{|x|^s} dV_x.$$

Combining (11), (12) and (13), we derive from (7) the inequality

$$(14) \quad \begin{aligned} n(n-2) \cdots (n-2s+2) \int_V \left(\sum_{i_1+i_2+i_3=s} f_{i_1,i_2,i_3}(x) x_1^{i_1} x_2^{i_2} x_3^{i_3} \right) \frac{u_n(x)}{|x|^{2s}} dV_x \\ \leq A_{10} \frac{n^{s-1}}{R^{n+s-1}} + A_{11} n^{s-1} \int_V \frac{u(x)}{|x|^{n+s}} dV_x. \end{aligned}$$

Since L is elliptic, we have

$$(15) \quad \sum_{i_1+i_2+i_3=s} f_{i_1 i_2 i_3}(x) x_1^{i_1} x_2^{i_2} x_3^{i_3} \geq \alpha |x|^s \quad (\alpha > 0).$$

Substituting (15) into (14) and taking $n \geq n_0$, where n_0 is sufficiently large, we easily conclude that

$$(16) \quad \int_V \frac{u(x)}{|x|^{n+s}} dV_x \leq \frac{A_{12} R^3}{n} \frac{1}{R^{n+s}}.$$

(16) is the desired L^1 -inequality for $u(x)$. The proof can now be completed by a simple argument due to CARLEMAN [2] which was used also in [1] and [5].

If $u(x) \equiv 0$ in V , then there exists a point $x^0 \in V$ such that

$$|x^0| < R, \quad u(x^0) \neq 0.$$

From (16) we deduce that

$$\frac{A_{13}}{|x^0|^{n+s}} \leq \frac{A_{12} R^3}{n} \frac{1}{R^{n+s}},$$

which is impossible for large n .

Having proved that $u(x)$ vanishes in V , we can, by a classical procedure of continuation, prove the vanishing of $u(x)$ in the whole domain B .

Remarks. (a) It is clear that Theorem 1 remains true if the assumption $Lu \leq 0$ is replaced by $Lu \leq Au(x) |x|^{-s}$ ($A > 0$).

(b) From the proof of Theorem 1, it is easy to see that Theorem 1 holds for the more general operators

$$(17) \quad L_1 = \sum_{i=0}^n \sum_{i_1+\dots+i_N=i} |x|^i f_{i_1 \dots i_N}(x) \frac{\partial^i}{\partial x_1^{i_1} \dots \partial x_N^{i_N}},$$

provided L satisfies the regularity and ellipticity assumptions of Theorem 1. Theorem 1, with L_1 instead of L , is sharp in the following sense: It is not sufficient to assume that (3) holds for all $n \leq n_0$, where n_0 is an arbitrary positive integer. This is shown by the counterexample of

$$N = 1, \quad u(x) = x^{n_0+2} \quad \text{and} \quad Lu = \frac{d^2}{dx^2} u - \frac{(n_0+2)(n_0+1)}{x^2} u.$$

(c) That Theorem 1 is not true for general elliptic operators which are parabolic at the origin is shown by the counterexample of

$$N = 2, \quad u(x_1, x_2) = e^{-x_1^{-2}} \quad \text{and} \quad Lu = x_1^{2m} \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - Au \quad (m \geq 4),$$

where A is a sufficiently large positive constant.

(d) If, in the assumptions of Theorem 1, $Lu(x) \leq 0$ is replaced by $Lu(x) = 0$, does there exist an integer $h = h(s, N)$, depending only on s and N , such that if $u(x)$ has a zero of order h at the origin, then $u(x) \equiv 0$ in B ? For $s = 2$, $h(2, N)$

exists and equals 1. Indeed, writing $Lu(x) = 0$ in the form

$$Lu \equiv \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = 0,$$

and using a slightly modified form of the minimum principle (obtained by slightly modifying the proof of the minimum principle in [6]), we conclude that if $c(x) \geq 0$ then the assumptions $u(x) \geq 0$ in B and $u(0) = 0$ imply $u(x) \equiv 0$ in B . For general $c(x)$ we can use the following simple device due to S. AGMON: Write $u(x) = \zeta(x) v(x)$. If we can find $\zeta(x)$ such that $\zeta(x) \geq \epsilon > 0$ and $L\zeta(x) \geq 0$, then the present case can be reduced to the previous one. Clearly we can take $\zeta(x) = \exp \{ \sum \alpha_i x_i \}$, where $|\alpha|$ is sufficiently large.

The example of

$$Lu = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}, \quad u(x) = x_1^{s-2} \cdots x_N^{s-2},$$

shows that $h(s, N) \geq N(s - 2)$. We leave open the question of the existence of a finite number $h(s, N)$.

2. Let L be the operator defined by (1), but without any assumptions with regard to its type. Denote by $A(L)$ the class of all functions $u(x)$ possessing first s continuous derivatives in B and satisfying in a neighborhood B_u (depending on u) of the origin the following properties:

(a) $Lu(x) \leq 0$, $u(x) \geq 0$;

(b) $\sum_{j=0}^{s-1} \sum_{i_1+\dots+i_N=j} \left| \frac{\partial^j u(x)}{\partial x_1^{i_1} \cdots \partial x_N^{i_N}} \right| = O(|x|^n) \quad (n = 1, 2, \dots; |x| \rightarrow 0).$

We have proved in Section 1 that if L is elliptic in B or elliptic in $B - \{O\}$ and parabolic at $x = 0$ of the form (17), then the set $A(L)$ consists only of the function zero. We denote this by $A(L) = [0]$. We now prove

Theorem 2. *If $A(L) = [0]$ then, at the origin, L defines a non-negative form, that is,*

$$(18) \quad \sum_{i_1+\dots+i_N=s} f_{i_1 \dots i_N}(0) \xi_1^{i_1} \cdots \xi_N^{i_N} \geq 0 \quad \text{for all real } \xi.$$

To prove the theorem, suppose (18) is not true. Then there exists a point $y = (y_1, \dots, y_N) \neq 0$ such that

$$\sum_{i_1+\dots+i_N=s} f_{i_1 \dots i_N}(0) y_1^{i_1} \cdots y_N^{i_N} < 0.$$

Denote $x \cdot y = \sum x_i y_i$; then the function

$$u(x) = \begin{cases} e^{-\epsilon(x \cdot y)^{-s}} & \text{if } x \cdot y \neq 0 \\ 0 & \text{if } x \cdot y = 0, \end{cases}$$

for $\epsilon > 0$ sufficiently small, belongs to $A(L)$, which is a contradiction.

From Remark (b) to Theorem 1 it is clear that the assumptions of Theorem 2 cannot imply that L is elliptic at the origin. The following theorem assumes some additional conditions on L which enable us to conclude that L is elliptic at the origin.

Theorem 3. *Let L be the operator defined by*

$$(19) \quad Lu = \sum_{i_1 + \dots + i_N = s} f_{i_1 \dots i_N} \frac{\partial^s}{\partial x_1^{i_1} \dots \partial x_N^{i_N}} u + f(x)u,$$

where $f_{i_1 \dots i_N}$ are constants and $f(x) \leq 0$. If $A(L) = [0]$, then L is elliptic.

Suppose the assertion is not true; then, by Theorem 2, L is necessarily parabolic. There exists a point $y = (y_1, \dots, y_N) \neq 0$ such that

$$\sum_{i_1 + \dots + i_N = s} f_{i_1 \dots i_N} y_1^{i_1} \dots y_N^{i_N} = 0.$$

The function

$$(20) \quad u(x) = \begin{cases} e^{-(x \cdot y)^{-1}} & \text{if } x \cdot y \neq 0 \\ 0 & \text{if } x \cdot y = 0 \end{cases}$$

belongs to $A(L)$, since $Lu(x) = f(x)u(x) \leq 0$. Since $u(x) \equiv 0$, we arrive at a contradiction.

Remark. One can easily see, by using the same function (20), that Theorem 3 remains true if the operator L in (19) is replaced by the more general operator

$$L + \sum f_i(x) \frac{\partial u}{\partial x_i},$$

where $f_i(x) = x_i c(x)$ and $c(x) \leq 0$.

REFERENCES

- [1] N. ARONSZAJN, *Sur l'unicité du prolongement des solutions des équations aux dérivées partielles elliptiques due second ordre*, C. R. Acad. Sci. Paris, 242 (1956), pp. 723–725.
- [2] T. CARLEMAN, *Sur les systèmes linéaires aux dérivées partielles du premier ordre à deux variables*, C. R. Acad. Sci. Paris, 197 (1933), pp. 471–474.
- [3] H. CORDES, *Über die eindeutige Bestimmtheit der Lösungen elliptischer Differentialgleichungen durch Anfangsvorgaben*, Nachr. Akad. Wiss. Göttingen, No. 11 (1956), pp. 239–258.
- [4] P. HARTMAN & A. WINTNER, *On the local behavior of solutions of non-parabolic partial differential equations III. Approximation by spherical harmonics*, Amer. J. Math., 77 (1955), pp. 453–483.
- [5] E. HEINZ, *Über die Eindeutigkeit beim Cauchyschen Anfangswertproblem einer elliptischen Differentialgleichung zweiter Ordnung*, Nachr. Akad. Wiss. Göttingen, No. 1 (1955), pp. 1–12.
- [6] E. HOPF, *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitzungsberichte Preuss. Akad. Wiss., 19 (1927), pp. 147–152.

- [7] P. D. LAX, *A stability theorem for solutions of abstract differential equations, and its applications to the study of the local behavior of solutions of elliptic equations*, Comm. Pure Appl. Math., **9** (1956), pp. 747–766.
- [8] C. MÜLLER, *On the behavior of the solutions of the differential equation $\Delta u = F(x, u)$ in the neighborhood of a point*, Comm. Pure Appl. Math., **7** (1954), pp. 505–515.

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