On the Regularity of the Solutions of Non-linear Elliptic and Parabolic Systems of Partial Differential Equations

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Introduction. Consider the nonlinear system of partial differential equations

$$(1) \quad \begin{aligned} \Phi_i(x_1, \dots, x_r, u_1, \dots, u_N, \nabla u_1, \dots, \nabla u_N, \dots, \\ \nabla^{s_i+t_k} u_k, \dots) = 0 \quad (j = 1, \dots, N) \end{aligned}$$

where $\nabla^i v$ denotes the tensor $\partial^i v / \partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r}$ and Φ_i involves derivatives of u_k of order not greater than $s_i + t_k$. The system (1) is *elliptic* with respect to the functions $u_1, \dots, u_N$ if and only if the equations of variations

$$(2) \quad \begin{aligned} \sum_{k=1}^N L_{jk}(x, \nabla) v_k \equiv \left[\frac{d}{d\lambda} \Phi_i(x_1, \dots, x_r, u_1 + \lambda v_1, \dots, \right. \\ \left. \nabla^{s_i+t_k}(u_k + \lambda v_k), \dots) \right]_{\lambda=0} = 0 \end{aligned}$$

form a linear elliptic system in the sense of [3], namely, if $L_{jk}^0(x, \nabla)$ denotes the terms of $L_{jk}(x, \nabla)$ of exact order $s_j + t_k$ ($\equiv 0$ if $s_j + t_k < 0$), then $\det(L_{jk}^0(x, \xi)) \neq 0$ for all real $\xi \neq 0$. Unless the contrary is stated, Φ_i and u_k will be complex-valued functions. If $s_i = t_i = m$ ($i = 1, \dots, N$), (1) takes the simpler form

$$(3) \quad \begin{aligned} \Phi_i(x_1, \dots, x_r, u_1, \dots, u_N, \nabla u_1, \dots, \nabla^{2m} u_N) = 0 \\ (j = 1, \dots, N). \end{aligned}$$

For the elliptic system (1) it was proved by DOUGLIS & NIRENBERG [3] that if the Φ_i are C^∞ in their arguments and if the u_k belong to $C^{s+t+\alpha}$ in the interior, where $s = \max \{s_i\}$ and $0 < \alpha < 1$, then the u_k belong to C^∞ . Recently, MORREY [17] has proved that if the Φ_i are analytic, then also the u_k are analytic. This

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theorem, for the system (3) with $N = 1$, $2m = 2$, $\nu = 2$ was first proved by S. BERNSTEIN [1]. Other proofs were given by RADÓ [22], LEWY [13], GEVREY [19] and by BERNSTEIN himself [2]. The case $N = 1$, $2m = 2$, ν arbitrary was proved by GIRAUD [10] and by HOPF [11]. The proof for the general system (3) was first given by PETROWSKI [21]. Both proofs of PETROWSKI and MORREY involve continuation into the complex domain.

In this paper we prove general regularity theorems for the system (1), which include, as a special case, the theorem on the analyticity of the solutions, both in the interior and on the boundary. Our method consists of some nonlinear estimates combined with known methods for linear systems.

Theorem 1 is a general regularity theorem for the system (3). The proof makes use of a method of proving the regularity of the solutions of linear systems developed by the author in [6], [7], [8], and it is also related to GEVREY's paper [9] mentioned above.

Theorems 2 and 3 consider the regularity of the solutions of the nonlinear system

$$(4) \quad \frac{\partial u_i}{\partial t} = \sum_{i=1}^N \sum_{\Sigma k_i=2m} a_{ii}^{k_1 \dots k_\nu}(t) \frac{\partial^{2m} u_i}{\partial x_1^{k_1} \dots \partial x_\nu^{k_\nu}} + \Phi_i(x_1, \dots, x_\nu, t, u_1, \dots, \nabla_x^{2m-1} u_N) \quad (i = 1, \dots, N),$$

which is parabolic in the sense of PETROWSKI (see, for instance, [5; p. 59]), namely, for every real ξ , $|\xi| = 1$, all the characteristic roots of the matrix

$$\left(\sum_{\Sigma k_i=2m} (-1)^m a_{ii}^{k_1 \dots k_\nu}(t) \xi_1^{k_1} \dots \xi_\nu^{k_\nu} \right)$$

have negative real roots. The proofs of Theorems 2 and 3 are based on [8] and on the nonlinear estimates obtained in the proof of Theorem 1.

Theorems 4 and 5 are regularity theorems for the system (1) and in their proof we use some results obtained by MORREY & NIRENBERG for linear systems [19], together with our nonlinear estimates. Theorem 4, like Theorems 1–3, is a regularity theorem in the interior, while Theorem 5 is a regularity theorem on the boundary. As a special case of Theorem 5, we have the following result: If the system (1) is strongly elliptic (to be defined in §1), if the Φ_i are analytic, if the Dirichlet data are analytic and if the boundary surface is analytic, then all the solutions u_k , which are assumed to belong to $C^{s+\alpha}$ ($s = \max \{s_i\}$, $0 < \alpha < 1$) up to the boundary, can be continued analytically across the boundary. Another proof of this theorem was very recently found by MORREY [18].

In §1 we state the results of this paper. In §§2 and 3 the proof of Theorem 1 is given. In §2 we find some nonlinear estimates and state the main result in Lemma 1. The procedure in §3 is similar to that of the linear case described in [7]. In §4 and §5 we sketch the proofs of Theorems 4 and 5 respectively.

1. Let $\{M_n\}$ and $\{\bar{M}_n\}$ be two sequences of positive numbers. A function $F(x, v)$, defined for $x = (x_1, \dots, x_\nu)$ in a ν -dimensional domain D and for $v =$

$(v_1, \dots, v_\mu)$ in a μ -dimensional open set E , is said to belong to the class $C\{M_{n-a}; D \mid \bar{M}_{n-a}; E\}$ (a an integer) if and only if $F(x, v)$ is infinitely differentiable (i.d.) and to every pair of closed subsets $D_0 \subset D$ and $E_0 \subset E$ there correspond constants H_0 and H such that for $x \in D_0$ and $v \in E_0$

$$(5) \quad \left| \frac{\partial^{i+k} F(x, v)}{\partial x_1^{i_1} \dots \partial x_\nu^{i_\nu} \partial v_1^{k_1} \dots \partial v_\mu^{k_\mu}} \right| \leq H_0 H^{i+k} M_{i-a} \bar{M}_{k-a} \\ (\sum j_i = j, \sum k_i = k; j, k = 0, 1, 2, \dots).$$

We use the notation $M_{-i} = 1$ ($i = 1, 2, \dots$). If $F(x, v) \equiv f(x)$, we simply write $f(x) \in C\{M_{n-a}; D\}$ (see [16]). Note that $C\{n!; D\}$ is the class of all functions analytic in D .

A function $F(x, t, v)$ (t real, $(x, t) \in D'$, $v \in E$) is said to belong to the class $C\{M_{n-a}; D' \mid \bar{M}_{n-a}; E\}$ if and only if it is i.d. in (x, t, v) and the inequalities (5) hold for $(x, t) \in D'_0$ and $v \in E_0$, where D'_0 is any closed subset of D' . Note that no assumptions are made on the derivatives of F in which t -derivatives appear.

Let $u(x) = (u_1(x), \dots, u_N(x))$ be a solution of the system (3) in a domain D . Denote by E_1 the $(\mu$ -dimensional) set

$$(u_1(x), \dots, u_N(x), \nabla u_1(x), \dots, \nabla u_N(x), \dots, \nabla^{2m} u_N(x)), \quad x \in D.$$

We assume that the Φ_i are real-valued functions defined in a $(\nu + \mu)$ -dimensional set (D, E) , where E is an open set which contains the closure $\bar{E}_1$ of E_1 .

Theorem 1. *Let $u(x)$ be a real solution of the system (3), which is assumed to be elliptic. Assume that the Φ_i belong to $C\{M_{n-3}; D \mid M_{n-3}; E\}$ and that, for some $A > 0$, the M_n satisfy the monotonicity conditions*

$$(6) \quad \binom{n}{i}_{M_i M_{n-i}} \leq A M_n \quad (i = 1, 2, \dots; n = 1, 2, \dots).$$

If $u(x)$ belongs to $C^{2m+\alpha}$ where $0 < \alpha < 1$, then $u(x)$ belongs to $C\{M_{n-2m-2}; D\}$.

Remark 1. If $2m = 2$ and $N = 1$ then it is sufficient to assume that $u \in C^2$ since then it follows from [20] that $u \in C^\infty$. In the general case, using a theorem of DOUGLIS & NIRENBERG [3; p. 532], we conclude from $u \in C^{2m+\alpha}$ that $u \in C^\infty$. We may, therefore, assume in what follows that $u \in C^\infty$.

Remark 2. Later on we shall make use of the inequalities

$$(6') \quad \frac{1}{i} \binom{n}{i-1}_{M_{i-1} M_{n-1-i}} \leq C_0 C^i M_{n-2} \\ (i = 1, 2, \dots, n-1; n = 1, 2, \dots),$$

where C_0 and C are appropriate constants.

The proof of (6') follows by considering separately the cases $1 \leq i \leq \frac{1}{2}n$ and $\frac{1}{2}n < i \leq n-1$ and using (6).

Corollary. *Let the M_n be such that $C\{M_{n-3}; D\}$ is a quasi-analytic class (see [16]). Then $u_1, \dots, u_N$ belong to this quasi-analytic class and, therefore, they possess the following uniqueness property: If $u_k(x)$ vanishes at a point $x^0 \in D$ together with its derivatives of all orders, then it vanishes identically in D .*

Analogous corollaries hold for Theorems 2, 4 and 5.

The parabolic case can be treated in a similar way, if we combine the results of [8] with the results of §2. We shall state one theorem in this direction. Denote by (4') the linear system obtained from (4) by deleting the Φ_i .

Theorem 2. *Let $u(x, t) = (u_1(x, t), \dots, u_N(x, t))$ be a solution in a domain D' of the system (4) which is assumed to be of parabolic type. Assume that the system (4') satisfies the condition (Λ) of EIDELMAN [4; p. 367] and that the $a_{ij}^{k_1 \dots k_r}(t)$ are i.d. Assume also that the Φ_i and the $\partial \Phi_i / \partial t$ belong to $C\{M_{n-3}; D' \mid M_{n-3}; E\}$ and that the M_n satisfy the inequalities (6). If $u(x, t)$ is i.d. in D' , then $u(x, t)$ belongs to $C\{M_{n-2m-3}; D'\}$.*

We now introduce classes $C\{M_{n-a}; D' \mid 2m \mid \bar{M}_{n-a}; E\}$ ($a \geq 0$) of i.d. functions $F(x, t, v)$ which, instead of satisfying (5), satisfy the inequalities

$$(5') \quad \left| \frac{\partial^{i+i+k} F(x, t, v)}{\partial x_1^{i_1} \dots \partial x_r^{i_r} \partial t^i \partial v^{k_1} \dots \partial v_{\mu}^{k_{\mu}}} \right| \leq H_0 H^{i+i+k} M_{j+2mi-a} \bar{M}_{k-a}.$$

Similarly we define $C\{M_{n-a}; D' \mid 2m\}$.

The discussion of [8] §6, together with Theorem 2 and the method of majorization used in §2, can be used to prove

Theorem 3. *If to the assumptions of Theorem 2 we add the assumptions that the $a_{ij}^{k_1 \dots k_r}(t)$ belong to $C\{M_{n-3}; D' \mid 2m\}$ and that the Φ_i belong to $C\{M_{n-3}; D' \mid 2m \mid M_{n-3}; E\}$, then all the solutions $u(x, t)$ belong to $C\{M_{n-2m-3}; D' \mid 2m\}$.*

The proofs of Theorems 2 and 3 will be omitted.

Classes $C\{M_{n-a}; D \mid \bar{M}_{n-a}; E\}$ can be defined also with respect to the Φ_i appearing in (1). We then have

Theorem 4. *Let $u(x)$ be a solution of (1), which is assumed to be elliptic. Assume that the Φ_i belong to $C\{M_{n-3}; D \mid M_{n-3}; E\}$ and that (6) holds. If the $u_k(x)$ belong to $C^{s+\epsilon k+\alpha}$ where $s = \max \{s_j\}$ and $0 < \alpha < 1$, then the $u_k(x)$ belong to $C\{M_{n-s-\epsilon k-1+\lfloor \frac{1}{2}\nu \rfloor}; D\}$.*

Here $\lfloor \frac{1}{2}\nu \rfloor$ denotes the largest integer $\leq \frac{1}{2}\nu$.

Let S be a portion of the boundary of D , and let $D_s = D \cup S$. We introduce classes $C\{M_{n-a}; D_s \mid \bar{M}_{n-a}; E\}$, whose definition follows word by word the definition of $C\{M_{n-a}; D \mid \bar{M}_{n-a}; E\}$. We shall assume that S is a surface which can be represented locally in the form $g(x_1, \dots, x_r) = 0$ with $\sum (\partial g / \partial x_i)^2 \neq 0$. For simplicity we may assume that S can be represented globally in the form $x_r = f(x_1, \dots, x_{r-1})$, where $(x_1, \dots, x_{r-1})$ varies in a domain F .

We shall assume that the system (1) is strongly elliptic, namely, the system

(2) is strongly elliptic in the sense of [3]. That means that $t_i = s_i \geq 0$ and that for every x

$$\operatorname{Re} \left\{ \sum_{i,k} (-1)^{s_i} L_{ik}^0(x, \xi) v^i \bar{v}^k \right\} \neq 0$$

when $\xi \neq 0$, $v \neq 0$ (ξ real, v complex).

We consider a solution u_k which takes on S the Dirichlet boundary values

$$(7) \quad \left(\frac{\partial}{\partial x_r} \right)^i u_k = \varphi_{ki} \quad (j = 0, 1, \dots, t_k - 1; k = 1, \dots, N),$$

where $\varphi_{ki} = \varphi_{ki}(x_1, \dots, x_{r-1})$ are defined in the domain F .

Theorem 5. Let $u(x)$ be a solution in D_S of (1) which is assumed to be strongly elliptic and let $u(x)$ satisfy the boundary conditions (7). Assume that the Φ_i belong to $C\{M_{n-3}; D_S \mid M_{n-3}; E_\mu\}$, where E_μ is the μ -dimensional Euclidean space, that the φ_{ki} belong to $C\{M_{n-s-t_k-2}; F\}$, that f belongs to $C\{M_{n-2s-2}; F\}$, and that (6) holds. If the u^k belong to $C^\infty(D_S)$ then the u^k belong to $C\{M_{n-s-t_k-1+\lfloor \frac{1}{2}v \rfloor}; D_S\}$.

Remark. It was communicated to the author by L. NIRENBERG that S. AGMON, A. DOUGLIS and L. NIRENBERG have proved in a paper, which is now in the process of being written, that if the u_k belong to $C^{s+t_k+\alpha}(D_S)$ where α is any positive number < 1 , then the u_k belong also to $C^\infty(D_S)$. Consequently, it is sufficient to assume in Theorem 5 that the u_k belong to $C^{s+t_k+\alpha}(D_S)$.

The proof of Theorem 1 will be given in the next two sections. For simplicity we shall give the proof in the case $N = 1$; the proof of the general case is similar. Instead of considering the system (3), we shall have to consider the equation

$$(8) \quad \Phi(x_1, \dots, x_r, u, \nabla u, \dots, \nabla^{2m} u) = 0.$$

We denote by $u^{(j)}$ any j^{th} partial derivative of u . We shall also use the notation

$$(9) \quad a_{i_1, \dots, i_k} = \frac{\partial \Phi}{\partial u_{i_1, \dots, i_k}}.$$

2. In §3 we shall need upper bounds for p^{th} partial derivatives of functions of the form $X = X(x_1, \dots, x_r, u_1, \dots, u_\mu)$. If we substitute $u_i = u_i(x)$ in X , then we denote the resulting function by $X(x)$. $(X(x))^{(p)} = \partial^p X(x) / \partial x_{i_1} \dots \partial x_{i_p}$ is a linear combination with positive coefficients of terms of the form

$$(10) \quad \frac{\partial^k X}{\partial x_{i_1} \dots \partial x_{i_k}} u_{i_1}^{(s_1)} \dots u_{i_\lambda}^{(s_\lambda)} \quad (1 \leq k \leq p, 1 \leq s_i \leq p, 0 \leq \lambda \leq \mu).$$

(By $\lambda = 0$ we simply mean that no derivatives of the u_i appear in (10).)

Let $\Psi = \Psi(x_1 + \dots + x_r, v_1, \dots, v_\mu)$ be a function all of whose k^{th} partial derivatives ($1 \leq k \leq p$) at the point $x_i = 0, v_i = 0, 1 \leq i \leq r, 1 \leq j \leq \mu$ are greater than the absolute values of the corresponding k^{th} partial derivatives of X at any point $P^0 = (x_1^0, \dots, x_r^0, u_1^0, \dots, u_\mu^0)$, where $u_i^0 = u_i(x^0)$ ($j = 1,$

$\dots, \mu)$. Let $v_j(\xi)$ ($1 \leq j \leq \mu$) be a function of a real variable ξ , whose k^{th} derivative ($1 \leq k \leq p$) at the point $\xi = 0$ is greater than each of the absolute values of the k^{th} partial derivatives of $u_j(x)$ at the point x^0 . Suppose also that $v_j(0) = 0$.

By comparing terms of the form (10) with the corresponding terms in $d^p \Psi(\xi)/d\xi^p$ ($\Psi(\xi) = \Psi(\xi, v_1(\xi), \dots, v_\mu(\xi))$) it follows that

$$(11) \quad |(\mathbf{X}(x))^{(p)}|_{x=x^0} \leq \left. \frac{d^p \Psi(\xi)}{d\xi^p} \right|_{\xi=0}.$$

In the sequel B_i will be used to denote appropriate constants independent of p . If $\mathbf{X} \in C\{M_{n-2}; D | M_{n-2}; E\}$, then

$$\left| \frac{\partial^{q+r} \mathbf{X}}{\partial x_1^{q_1} \dots \partial x_r^{q_r} \partial u_1^{r_1} \dots \partial u_\mu^{r_\mu}} \right| \leq K_0 K^q M_{q-2} K^r M_{r-2} \quad (\sum q_i = q, \sum r_i = r),$$

where K_0 and K are constants. We can then take for $\Psi(\xi, v_1, \dots, v_\mu)$ the function

$$K_0 \sum_{q=0}^p \frac{K^q M_{q-2} \xi^q}{q!} \sum_{r_1, \dots, r_\mu=0}^p \frac{K^{r_1} v_1^{r_1} \dots K^{r_\mu} v_\mu^{r_\mu} M_{r-2}}{r_1! \dots r_\mu!}.$$

If, in particular, $v_1 = \dots = v_\mu = v$, we can take for Ψ the simpler function

$$(12) \quad \Psi(\xi, v) = B_0 \sum_{i=0}^p \frac{K^i M_{i-2}}{i!} \xi^i \sum_{i=0}^p \frac{B_1 M_{i-2}}{i!} v^i.$$

We write

$$(13) \quad \Psi_1(v) = B_0 \sum_{i=0}^p \frac{B_1 M_{i-2}}{i!} v^i, \quad \Psi_2(\xi) = \sum_{i=0}^p \frac{K^i M_{i-2}}{i!} \xi^i.$$

Suppose now that the function $u(x)$ is i.d. in D and that it satisfies, in a subdomain G of D ($\bar{G} \subset D$),

$$(14) \quad |u^{(q)}(x)| \leq H_0 H_1^{q-2-2m} M_{q-2-2m}$$

for all $2m+2 \leq q \leq n+1$, where n is an arbitrary integer $\geq 2m+1$, and suppose also that

$$(15) \quad |u^{(q)}(x)| \leq H_0$$

for all $0 \leq q \leq 2m+1$. Assume that $H_1 \geq 1$. We can always assume that the $M_i \geq 1$.

If we take $u_\mu(x) = u^{(2m)}(x)$, then it follows that

$$\begin{aligned} |u_\mu^{(i)}(x)| &= |u^{(2m+i)}(x)| \leq H_0 H_1^{i-2} M_{i-2} \quad (2 \leq i \leq n+1-2m) \\ |u_\mu^{(1)}(x)| &= |u^{(2m+1)}(x)| \leq H_0. \end{aligned}$$

Defining $p = n+1-2m$, we can take

$$(16) \quad v_\mu(\xi) = H_0 \left(\xi + \frac{M_0}{2!} \xi^2 + \dots + H_1^{i-2} \frac{M_{i-2}}{i!} \xi^i + \dots + H_1^{p-2} \frac{M_{p-2}}{p!} \xi^p \right).$$

We introduce the following notation: $v(\xi) \ll h(\xi)$ if and only if $v^{(i)}(0) \leq h^{(i)}(0)$ for $1 \leq i \leq p$. From (16) it follows that

$$(v_\mu(\xi))^2 = H_0^2 \left(\xi^2 + \dots + \left(\sum_{\lambda=2}^{i-2} \frac{M_{\lambda-2}}{\lambda!} H_1^{\lambda-2} \frac{M_{i-\lambda-2}}{(i-\lambda)!} H_1^{i-\lambda-2} + \frac{2M_{i-3}}{(i-1)!} H_1^{i-3} \right) \xi^i + \dots \right).$$

To estimate the coefficient of ξ^i in $(v_\mu(\xi))^2$, note first that if we take $i = 1$ in (6), we get

$$(6'') \quad qM_q \leq A_1 M_{q+1} \quad \left(A_1 = \frac{A}{M_1} \right).$$

Note also that

$$(17) \quad \sum_{\lambda=2}^{i-2} \frac{1}{\lambda^2} \frac{1}{(i-\lambda)^2} = \sum_{\lambda=2}^{i-2} \frac{1}{i^2} \left(\frac{1}{\lambda} + \frac{1}{i-\lambda} \right)^2 \leq \frac{c_1}{i^2} \quad (c_1 > 0).$$

Using (6), (6'') and (17), we conclude that the coefficient of ξ^i is smaller than

$$\begin{aligned} H_0^2 \frac{H_1^{i-2}}{H_1} \left(\frac{B_2}{i^2} \max_{\lambda} \frac{M_{\lambda-2}}{(\lambda-2)!} \frac{M_{i-\lambda-2}}{(i-\lambda-2)!} + \frac{2M_{i-3}}{(i-1)!} \right) \\ \leq H_0^2 \frac{H_1^{i-2}}{H_1} \left(\frac{B_2 A M_{i-4}}{i^2 (i-4)!} + \frac{2M_{i-3}}{(i-1)!} \right) \leq B_3 H_0^2 \frac{H_1^{i-2} i M_{i-3}}{H_1 i!}. \end{aligned}$$

(Note that in the last inequality B_3 is necessarily greater than 1.) Consequently, we get

$$(v_\mu(\xi))^2 \ll \frac{H_0^2 B_3}{H_1} \left(H_1 \xi^2 + H_1 \frac{3M_0}{3!} \xi^3 + \dots + H_1^{p-2} \frac{pM_{p-3}}{p!} \xi^p \right).$$

Proceeding by induction, we obtain

$$\begin{aligned} (18) \quad (v_\mu(\xi))^i &\ll \frac{H_0^i B_3^{i-1}}{H_1^{i-1}} \left(H_1^{i-1} \xi^i + \frac{H_1^{i-1} (i+1) i \dots 3M_0}{(i+1)!} \xi^{i+1} \right. \\ &\quad \left. + \dots + \frac{H_1^{p-2} p(p-1) \dots (p-i+2) M_{p-i-1}}{p!} \xi^p \right) \quad (1 \leq i < p), \\ (v_\mu(\xi))^p &\ll H_0^p B_3^{p-1} \xi^p. \end{aligned}$$

Suppose now that each of the u_i (in X) is a function of the form $u^{(k)}(x)$, $0 \leq k \leq 2m$. From (14) and (15) it follows that $u^{(k)}(x)$ is also majorized by $v_\mu(\xi)$. Using (18), (13) (with $v = v_\mu(\xi)$) and (6'), we get

$$\begin{aligned} (19) \quad \left. \frac{d^i \Psi_1}{d\xi^i} \right|_{\xi=0} &\leq B_0 \sum_{i=1}^{j-1} \frac{B_1^i M_{i-2}}{i!} \frac{H_0^i B_3^{i-1}}{H_1^{i-1}} H_1^{i-2} j(j-1) \dots (j-i+2) M_{j-i-1} \\ &\quad + B_0 B_1^i H_0^i B_3^{i-1} M_{j-2} \leq B_4 H_0 H_1^{i-2} M_{j-2} \quad (2 \leq j \leq p), \end{aligned}$$

provided that $H_1 > 2B_1H_0B_3$. One can verify directly that

$$(20) \quad \left. \frac{d\Psi_1}{d\xi} \right|_{\xi=0} \leq B_5H_0.$$

The function $\Psi_2(\xi)$, defined in (13), satisfies

$$(21) \quad \left. \frac{d^i\Psi_2}{d\xi^i} \right|_{\xi=0} = K^i M_{i-2}.$$

From (12), (19) and (21) we get, on taking $H_1 > K$ and using (6), (6'') and (17),

$$(22) \quad \begin{aligned} \left. \frac{d^p\Psi(\xi, v)}{d\xi^p} \right|_{\xi=0} &\leq \sum_{i=2}^p \binom{p}{i} B_4H_0H_1^{i-2}M_{i-2}K^{p-i}M_{p-i-2} \\ &+ B_5pH_0K^{p-1}M_{p-3} + B_6K^pM_{p-2} \leq B_7H_0H_1^{p-2}M_{p-2}. \end{aligned}$$

In proving (22) we assumed that (14) holds for $2m+2 \leq q \leq n+1$. However, the $(n+1)^{\text{th}}$ partial derivatives of u appear only in terms of the form $(\partial X/\partial u^{(2m)}) u^{(2m+p)} = (\partial X/\partial u^{(2m)}) u^{(n+1)}$; consequently, their coefficients are bounded uniformly with respect to n . From this remark and from (22) and (11) we conclude:

Lemma 1. *Let $u(x)$ satisfy (15) and (14) with $2m+1 \leq q \leq n$. (If $n = 2m+1$, only (15) is assumed.) Then for all $x \in G$*

$$(23) \quad \begin{aligned} \left| \frac{\partial^{n+1-2m}}{\partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r}} X(x_1, \dots, x_r, u(x), \nabla u(x), \dots, \nabla^{2m}u(x)) \right| \\ \leq B_8 |u^{(n+1)}(x)|^* + B_7H_0H_1^{n-1-2m}M_{n-1-2m}, \end{aligned}$$

where $|u^{(n+1)}(x)|^*$ denotes $\max \{|u^{(n+1)}(x)|\}$, the maximum being taken over all the $(n+1)^{\text{th}}$ partial derivatives of u .

It is important to note that B_7 and B_8 depend on X , u , G and H_0 but are independent of n and H_1 . Note also that H_1 was assumed to be larger than some constant depending on X , u , G and H_0 but independent of n .

3. To prove Theorem 1, it is sufficient to show that for every point $x^0 \in D$ there exists a neighborhood where

$$|u^{(n)}(x)| \leq C_1C_2M_{n-2-2m} \quad (C_1 > 0, \quad C_2 > 0);$$

an application of the Heine-Borel Theorem will then complete the proof.

Given x^0 , let V be a cube with center x^0 and diagonal R , whose edges are parallel to the coordinate axes. Let H_0 be a positive constant for which the inequalities

$$(24) \quad |u^{(q)}(x)| \leq H_0 \quad (0 \leq q \leq 4m)$$

hold in V . From now on H_0 will be fixed.

For every function $w(x)$ defined in V , we denote by $|w|_s$ the maximum of $|w(x)|$ in the cube which is homothetic with V and such that the distance

between its boundary and the boundary S of V is δ . We shall prove by induction that

$$(25) \quad |u^{(a)}|_{\delta} \leq H_0 \left(\frac{H}{\delta} \right)^{a-2-2m} M_{q-2-2m} \quad (q = 2m + 2, 2m + 3, \dots).$$

From the remark made at the beginning of this section it is clear that the proof of the theorem is thereby completed.

Assuming the truth of (25) for $q \leq n$, we shall prove its truth for $q = n + 1$. In this proof, H and the diagonal R of V will be determined. However, to avoid unnecessary complications, we take from the beginning $H > 1$, $R < 1$. Then (24) implies (25) for $2m + 2 \leq q \leq 4m$ and, therefore, we may assume in what follows that $n \geq 4m$.

Differentiating (8) with respect to x_μ and using (9), we get

$$(26) \quad \begin{aligned} \sum a_{i_1 \dots i_{2m}} \frac{\partial^{2m}}{\partial x_{i_1} \dots \partial x_{i_{2m}}} \frac{\partial u}{\partial x_\mu} \\ = - \sum_{k=0}^{2m-1} \sum a_{i_1 \dots i_k} \frac{\partial^k}{\partial x_{i_1} \dots \partial x_{i_k}} \frac{\partial u}{\partial x_\mu} - \frac{\partial \Phi}{\partial x_\mu} \equiv \Phi_\mu. \end{aligned}$$

For simplicity we agree to write $\partial^\lambda a / \partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r} = \partial^\lambda a / \partial x_1^{\beta_1} \dots \partial x_r^{\beta_r} = a^{(\lambda)}$ even when $(\alpha_1, \dots, \alpha_r) \neq (\beta_1, \dots, \beta_r)$. The bounds on $a^{(\lambda)}$ will depend on λ but not on the special sequence $(\alpha_1, \dots, \alpha_r)$. A similar notation will be used for u .

Differentiating (26) $n - 2m + 1$ times, we obtain

$$(27) \quad \begin{aligned} \sum a_{i_1 \dots i_{2m}} \frac{\partial^{2m}}{\partial x_{i_1} \dots \partial x_{i_{2m}}} u^{(n-2m+2)} \\ = - \sum \sum_{\lambda=1}^{n-2m+1} \binom{n-2m+1}{\lambda} a_{i_1 \dots i_{2m}}^{(\lambda)} u^{(n+2-\lambda)} + \Phi_\mu^{(n-2m+1)} \equiv G(x). \end{aligned}$$

Let y belong to V and denote by δ' the distance between y and the boundary S of V . Define

$$(28) \quad \delta = \delta' \left(1 - \frac{1}{n} \right).$$

Denote by V_δ the cube with center y and edges parallel to the coordinate axes of length $2(\delta' - \delta)$. Let S_δ be the boundary of V_δ .

Writing (27) in the form $Lu^{(n-2m+2)} = G(x)$, where L denotes the linear elliptic operator on the left side of (27), and using a representation by a fundamental solution ([12], [14], [15]) in the domain V_δ , we get

$$(29) \quad \begin{aligned} u^{(n-2m+2)}(y) &= \int_{S_\delta} \sum \cos(\tau x_i) B^i(\varphi(x, y), u^{(n-2m+2)}(x)) dS_x \\ &\quad + \int_{V_\delta} \varphi(x, y) G(x) dV_x = \bar{J}_1 + \bar{J}_2, \end{aligned}$$

where $\bar{J}_1$ is the surface integral, τ is the outwardly directed normal to S_s , $\varphi(x, y)$ is a fundamental solution of the adjoint L^* of L , and

$$\begin{aligned} B^i(w, v) = & \sum_{k=1}^{2m} \sum_{i_1, \dots, i_{k-1}}^r \left\{ \left[(a_{i_1, \dots, i_k}^* v) \frac{\partial^{k-1} w}{\partial x_{i_1} \dots \partial x_{i_k}} \right]_{i_1=j} \right. \\ & + \left[(-1) \frac{\partial}{\partial x_{i_1}} (a_{i_1, \dots, i_k}^* v) \frac{\partial^{k-2} w}{\partial x_{i_1} \dots \partial x_{i_k}} \right]_{i_1=j} + \dots \\ & \left. + \left[(-1)^{k-1} \left| \frac{\partial^{k-1}}{\partial x_{i_1} \dots \partial x_{i_{k-1}}} (a_{i_1, \dots, i_k}^* v) \right| w \right]_{i_k=j} \right\} \end{aligned}$$

where $a_{i_1, \dots, i_k}^*$ is the coefficient of $\partial^k / \partial x_{i_1} \dots \partial x_{i_k}$ in L^* .

In deriving (29), one makes use of Green's identity

$$vL^*w - wLv = \sum (\partial / \partial x_i) B^i(w, v).$$

$\varphi(x, y)$ is i.d. in $x, y \in V$, $x \neq y$, and it satisfies the following property: The functions

$$(30) \quad \psi(x, y) = |x - y|^{\nu - 2m + \sum j_i + k_i} \frac{\partial^{i+k}}{\partial x_1^{i_1} \dots \partial y_r^{k_r}} \varphi(x, y)$$

($\nu - 2m + \sum j_i + k_i > 0$)

are bounded in V .

Differentiating (29) $2m - 1$ times with respect to y , we obtain

$$(31) \quad u^{(n+1)}(y) = \bar{J}_1^{(2m-1)} + \bar{J}_2^{(2m-1)} = J_1 + J_2,$$

where $\bar{J}_i^{(2m-1)} = J_i$ ($i = 1, 2$). We proceed to estimate the J_i . In the sequel, A_i will be used to denote appropriate constants independent of n .

To estimate J_1 , observe that it is sufficient to estimate $I^{(2m-1)}$, where

$$I = \int_{S_s} (u^{(n-2m+2)}(x) a(x))^{(i)} (\varphi(x, y))^{(2m-1-i)} \cos(\tau x_i) dS_x \quad (0 \leq i \leq 2m - 1).$$

Here $a(x)$ is any of the coefficients $a_{i_1, \dots, i_m}(x)$ of L . Consider first the case where $0 \leq i \leq 2m - 2$. Then

$$|I^{(2m-1)}| \leq A_1 \sum_{\lambda=0}^i |u^{(n-2m+2+\lambda)}|_s \int_{S_s} |(\varphi(x, y))^{(4m-2-i)}| dS_x.$$

Using the inductive assumption (note that $n - 2m + 2 + \lambda \geq 2m + 2$) and the boundedness of the $\psi(x, y)$ in (30), we conclude that

$$|I^{(2m-1)}| \leq A_2 H_0 \left(\frac{H}{\delta} \right)^{n-4m+i} M_{n-4m+i} \left(\frac{1}{\delta' - \delta} \right)^{2m-1-i}.$$

Using (28) and (6''), we finally get

$$(32) \quad |I^{(2m-1)}| \leq A_3 \frac{H_0 H^{n-2-2m}}{\delta^{n-1-2m}} M_{n-1-2m}.$$

The case $i = 2m - 1$ will be estimated in a different way. We shall assume that $u^{(n+1)}$ in (31) is not of the form $\partial^{n+1}u/\partial x_i^{n+1}$. Suppose, for instance, that $u^{(n+1)} = \partial u^{(n)}/\partial x_1 = \partial u^{(n)}/\partial x_2$. Since $n - 2m + 1 \geq 2$, we may then assume that $u^{(n-2m+1)} = \partial u^{(n-2m)}/\partial x_1 = \partial u^{(n-2m)}/\partial x_2$. We can now integrate by parts in $I^{(2m-1)}$. It will be sufficient to consider the function

$$K = \int_{S'_i} a(x) u^{(n+1)}(x) (\varphi(x, y))^{(2m-1)} dS_x,$$

where S'_i is a face of S_i on which $x_i = \text{const}$. Integrating by parts, we obtain

$$K = \int_{\sigma'_i} u^{(n)}(x) a(x) (\varphi(x, y))^{(2m-1)} d\sigma_x - \int_{S'_i} u^{(n)}(x) (a(x) (\varphi(x, y))^{(2m-1)})^{(1)} dS_x,$$

where σ'_i denotes appropriate $(\nu - 2)$ -dimensional faces of the boundary of S'_i . Using the inductive assumption, the boundedness of the $\psi(x, y)$ in (30), (6'') and (28), one can easily see that $|K|$ is majorized by the right side of (32). Thus we have proved that

$$(33) \quad |J_1| \leq A_4 H_0 \frac{H^{n-2-2m}}{\delta^{n-1-2m}} M_{n-1-2m},$$

except for the cases where $u^{(n+1)} = \partial^{n+1}u/\partial x_i^{n+1}$, $1 \leq i \leq \nu$. These cases will be treated later on.

To estimate J_2 , we first note that

$$(34) \quad |J_2| \leq |G|_i \int_{V_i} |(\varphi(x, y))^{(2m-1)}| dV_x \leq A_5 \left(\frac{\delta}{n}\right) |G|_i,$$

so that it remains to estimate $|G|_i$. It is sufficient to estimate the typical term

$$I = \sum_{\lambda=1}^{n-2m+1} \binom{n-2m+1}{\lambda} a^{(\lambda)} u^{(n+2-\lambda)},$$

where $a(x)$ is any of the functions $a_{i_1, \dots, i_m}(x)$; estimates for the other terms which appear in G can be obtained in exactly the same way. Writing

$$I = (n - 2m + 1) a^{(1)} u^{(n+1)} + \sum_{\lambda=2}^{n-2m+1} \dots$$

and applying Lemma 1 (with $H_1 = H/\delta$, $X = a$ and $n + 1 - 2m = \lambda$), we obtain (if H is taken to be larger than some constant depending on Φ , u , V and

H_0 but independent of n),

$$\begin{aligned}
|I| &\leq A_6 n |u^{(n+1)}|_\delta^* \\
&\quad + \sum_{\lambda=2}^{n-2m} \binom{n-2m+1}{\lambda} A_7 H_0 \left(\frac{H}{\delta}\right)^{\lambda-2} M_{\lambda-2} H_0 \left(\frac{H}{\delta}\right)^{n-2m-\lambda} M_{n-2m-\lambda} \\
&\quad + A_8 H_0 \left(\frac{H}{\delta}\right)^{n-2m-1} M_{n-2m-1} \\
&\leq A_6 n |u^{(n+1)}|_\delta^* + A_9 H_0 \left(\frac{H}{\delta}\right)^{n-1-2m} n M_{n-1-2m},
\end{aligned}$$

where $|u^{(n+1)}|_\delta^*$ denotes $\max \{|u^{(n+1)}|_\delta\}$, the maximum being taken over all the $(n+1)^{\text{th}}$ partial derivatives of u . Here we made use of (6), (6'') and of the inequality

$$\sum_{\lambda=1}^{n-1} \frac{1}{\lambda^2} \frac{1}{(n-\lambda)} \leq \frac{A_{10}}{n}.$$

We conclude that

$$(35) \quad |G|_\delta \leq A_{11} n |u^{(n+1)}|_\delta^* + A_{12} H_0 \left(\frac{H}{\delta}\right)^{n-1-2m} n M_{n-1-2m},$$

and, together with (34) and (33), we get

$$(36) \quad |u^{(n+1)}|_{\delta'} \leq A_{13} R |u^{(n+1)}|_\delta^* + H_0 \left(\frac{H}{\delta}\right)^{n-1-2m} M_{n-1-2m} \left(A_{14} R + \frac{A_4}{H}\right).$$

It remains to estimate the functions $\partial^{n+1} u / \partial x_i^{n+1}$. To do that, we take $\mu = i$ and differentiate (26) $n - 2m$ times with respect to x_i . We obtain

$$\begin{aligned}
(37) \quad -a_{i_1 \dots i_{n-2m}} \partial^{n+1} u / \partial x_i^{n+1} &= \sum_{\lambda=1}^{n-2m} \binom{n-2m}{\lambda} a_{i_1 \dots i_{n-2m}}^{(\lambda)} u^{(n+1-\lambda)} \\
&\quad + \sum' a_{i_1 \dots i_{n-2m}} u^{(n+1)} + \Phi_i^{(n-2m)},
\end{aligned}$$

where in $\sum'$ only mixed $(n+1)^{\text{th}}$ derivatives of u appear. Using (36) to estimate each of the terms which appear in $\sum'$ and estimating the remaining terms in the way which we used to estimate $|G|_\delta$, we conclude, after noting that $a_{i_1 \dots i_{n-2m}} \neq 0$, that $\partial^{n+1} u / \partial x_i^{n+1}$ is also majorized by the right side of (36). Noting that $\delta^{n-1-2m} \geq A_{15} \delta'^{n-1-2m}$, we get

$$(38) \quad |u^{(n+1)}|_\delta^* \leq A_{16} R |u^{(n+1)}|_\delta^* + H_0 \left(\frac{H}{\delta'}\right)^{n-1-2m} M_{n-1-2m} \left(A_{17} R + \frac{A_{18}}{H}\right).$$

We shall need the following lemma ([7] [8]):

Lemma 2. *Let $g(t)$ be a positive monotone decreasing function, defined in the interval $0 \leq t \leq 1$ and satisfying*

$$(39) \quad g(t) \leq \frac{1}{10} g\left(t\left(1 - \frac{1}{n}\right)\right) + \frac{C}{t^{n-1-2m}} \quad (n \geq 4m, \quad C > 0),$$

then $g(t) < CA_{19}/t^{n-1-2m}$.

The proof of the lemma follows easily by repeatedly applying (39).

Applying the lemma to the function $g(t) = |u^{(n+1)}|_t^*$ and using (38), we conclude that if $R < 1/10A_{19}$ then

$$|u^{(n+1)}|_\delta^* \leq H_0 \left(\frac{H}{\delta}\right)^{n-1-2m} M_{n-1-2m} A_{19} \left(A_{17}R + \frac{A_{18}}{H}\right).$$

Taking R to be such that also $RA_{17}A_{19} < \frac{1}{2}$ and taking $H > 2A_{18}A_{19}$, we arrive at (25) with $q = n + 1$ and the proof of Theorem 1 is completed.

4. In this section we shall sketch a proof of Theorem 4. Given an arbitrary point $x^0 \in D$, denote by B_r the sphere $|x - x_0| < r$ and let B_R be contained in D for some $R < 1$. To prove Theorem 4 it is sufficient to prove the existence of constants H_0 and H such that

$$(40) \quad d_p(u, B_r) = \left[\int_{B_r} \sum_{i=1}^N |\nabla^{s+t_i+p} u_i|^2 dx \right]^{\frac{1}{2}} \leq \frac{H_0 H^{p-2}}{(R-r)^{p-2}} M_{p-2} \left(2 \leq p < \infty, \quad \frac{R}{2} \leq r < R \right),$$

where $|\nabla^i v|^2$ denotes the sum of the squares of the absolute values of the components of the tensor $\nabla^i v$; for, using Sobolev's lemma in the form used in [19], we can easily conclude from (40) that if $|x - x_0| \leq r$, $\frac{1}{2}R \leq r \leq R$, then

$$(41) \quad |\nabla^p u_i(x)| \leq \frac{H_0 H^p}{(R-r)^p} M_{p-s-t_i-1+\lfloor \frac{1}{2}p \rfloor}.$$

Denoting

$$(42) \quad e_p(f, B_r) = \left[\int_{B_r} \sum_{i=1}^N |\nabla^{s-s_i+p} u_i|^2 dx \right]^{\frac{1}{2}},$$

MORREY & NIRENBERG [19] proved that if $\sum L_{jk}^0 u_k = f_i$ is a linear elliptic system with constant coefficients such that only the derivatives of order $s_i + t_k$ occur in L_{jk}^0 , then

$$(43) \quad d_0^2(u, B_r) \leq K_1 \left(e_0^2(f, B_{r+\delta}) + \sum_{q=1}^{s+t} d_{-q}^2(u, B_{r+\delta}) \delta^{-2q} \right) \quad (\delta < r < r + \delta < R),$$

where K_1 is a constant depending only on $p, N, s_1, \dots, t_N$ and the coefficients of L_{jk}^0 and $t = \max \{t_i\}$. We note that in the definition of $d_p(u, B_r)$ in (40) $-(s+t) \leq p < \infty$ we take $\nabla^a u_i = 0$ whenever $a < 0$. To avoid complica-

tions, we transform (43) into the form

$$(44) \quad \delta_0^2(u, B_r) \leq K_2 \left(\epsilon_0^2(f, B_{r+\delta}) + \sum_{q=1}^{s+t} \delta_{-q}^2(u, B_{r+\delta}) \delta^{-2q} \right) \\ (\delta < r < r + \delta < R),$$

where

$$(45) \quad \delta_p(u, B_r) = \max \left[\int_{B_r} |u_i^{(s+t_i+p)}|^2 dx \right]^{\frac{1}{2}}$$

$$(46) \quad \epsilon_p(f, B_r) = \max \left[\int_{B_r} |f_i^{(s-s_i+p)}|^2 dx \right]^{\frac{1}{2}},$$

the maximum being taken over the index j and over all the partial derivatives of order $s + t_i + p$ in (45), and of order $s - s_i + p$ in (46).

Suppose that $X_j(x_1, \dots, x_n, u_1, \dots, \nabla^{s_j+t_j} u_k, \dots)$ ($j = 1, \dots, N$) involves derivatives of u_k of order not greater than $s_j + t_k$. Following the method of §2, we can prove

Lemma 1'. *Let the $u_k(x)$ satisfy the inequalities*

$$(47) \quad \delta_q(u, B_r) \leq H_0 H_1^{q-2} M_{q-2} \quad (2 \leq q \leq n, \quad H_1 \geq 1),$$

and assume that the X_i belong to $C\{M_{n-2}; D \mid M_{n-2}; E\}$. Then

$$(48) \quad \epsilon_n(X, B_r) \leq A H_0 H_1^{n-2} M_{n-2},$$

where A is a constant independent of n and H_1 , and H_1 is assumed to be larger than some constant depending on H_0 , B_R and the functions u_k and X_i , but independent of n .

With the aid of Lemma 1' we can now follow the method of [19]. We shall however mention a somewhat simplified modification of that method. It is sufficient to prove that

$$(49) \quad \delta_p(u, B_r) \leq \frac{K_0 K^{p-2}}{(R-r)^{p-2}} M_{p-2} \quad (2 \leq p < \infty; \quad \frac{1}{2}R \leq r < R),$$

since (40) follows very easily from (49). We shall prove (49) by induction on p : Assuming its truth for $2 \leq p \leq n$, we shall prove its truth for $p = n + 1$.

Differentiating (1) once, we obtain a linear system for $u_k^{(1)}$ of the form $\sum L_{ik} u_k^{(1)} = f_i$. Writing it in the form

$$(50) \quad \sum_{k=1}^N L_{ik}^0 u_k^{(1)} = f_i + \sum_{k=1}^N (L_{ik}^0 - L_{ik}) u_k^{(1)} \equiv F_i$$

where L_{ik}^0 denotes the terms of L_{ik} of exact order $s_i + t_k$ ($\equiv 0$ if $s_i + t_k < 0$) taken at the point $x = x^0$, and differentiating (50) n times, we obtain

$$(51) \quad \sum_{k=1}^N L_{ik}^0 u_k^{(n+1)} = F_i^{(n)}.$$

To the system (51) we apply the inequality (44) with $\delta = (R - r)/n$, and in estimating the right hand we make use of Lemma 1'. After some calculations similar to those which we made in estimating J_2 in §3 and after using Lemma 2, we obtain, on taking K to be larger than some constant independent of n , the inequality (49) with $p = n + 1$, and the proof of Theorem 4 is completed.

5. The proof of Theorem 5 proceeds in a way similar to the proof of Theorem 4; it is also based on a method for linear systems developed in [19] and on Lemma 1' (with d_p , e_p , δ_p and ϵ_p defined in a slightly different manner). However, the method of [19] applies only to the special case in which $\varphi_{ki} = 0$ and $f = 0$, and, therefore, we have first to reduce the general case to that special case. The reader will note that the transformations (52) and (53) which make the necessary reduction do not affect the strong ellipticity of the system.

The transformation

$$(52) \quad x_k = y_k \quad (k = 1, \dots, \nu - 1), \quad x_\nu = y_\nu + f(y_1, \dots, y_{\nu-1})$$

maps the surface S into a portion of the hyperplane $y_\nu = 0$ and the derivatives $(\partial/\partial x_\nu)^i$ are transformed into the normal derivatives $(\partial/\partial y_\nu)^i$. Note that the transformation (52) is one-to-one.

Let us first consider the affect of this transformation on the validity of Lemma 1, by considering the typical example of $\Phi(y, u^{(2m)}) \equiv \Psi(h(y), u^{(2m)}g(y))$, where $\Psi(x, u) \in C\{M_{n-2}; D \mid M_{n-2}; E_\mu\}$ and $h(y), g(y) \in C\{M_{n-2}; D^*\}$ (D^* is such that $h(y) \in D$ whenever $y \in D^*$). Now, if u satisfies (14) and (15), and if H_1 is sufficiently large, then the functions $v_\mu(\xi)$ in (16), with H_0 replaced by AH_0 , still majorize (in the sense of $\ll$) the function $u^{(2m)}g$; A is a proper constant independent of n and H_1 . Consequently, (19) is true with $u_\mu = u^{(2m)}g$. By the method of §2 one can easily show that if $\Psi_2(x) \in C\{M_{n-2}; D\}$, then $\Psi_2(h(y))$ belongs to $C\{M_{n-2}; D^*\}$, and, together with the previous remark, the inequality (23) of Lemma 1 with $X = \Phi(y, u^{(2m)}(y))$ follows.

In complete analogy with the above considerations, one can show that Lemma 1' remains true after making the transformation (52) in X_j . (Note that $u_k^{(i)}$, in X_j , is replaced by $u_k^{(i)}(f_1^{(1)})^i + \dots + u_k^{(1)}f_1^{(i)}$ if we make a transformation $x = f_1(y)$. Note also that $(f_1^{(a)})^b \in C\{M_{n-2s+a-2}; F\}$ if $f_1 \in C\{M_{n-2s-2}; F\}$; in particular, this holds for the function $x = f_1(y)$ defined by (52).)

We can make a transformation of the given boundary values into the zero boundary values by defining new dependent functions

$$(53) \quad v_k = u_k - \sum_{i=0}^{t_k-1} \frac{(x_\nu)^i}{j!} \varphi_{ki}.$$

It is easy to see, on using the inequalities

$$|\varphi_{ki}^{(n+s+t_k)}| \leq A_0 A^{n-2} M_{n-2} \quad (A_0, A > 0),$$

that Lemma 1' remains true for the dependent functions v_k .

Combining both transformations (52) and (53), we conclude that for the

functions $v_k(f_1(y))$, under the assumption that the X_i of the original system belong to $C\{M_{n-2}; D \mid M_{n-2}; E_\mu\}$, Lemma 1' remains true. Here $x = f_1(y)$ denotes the transformation (52). As already remarked at the beginning of this section, the proof of Theorem 5 for the transformed system, with $w_k(y) = v_k(f_1(y))$ as the dependent functions, proceeds by analogy with the proof of Theorem 4, if we use Lemma 1' and the results of [19] (and especially formula (3.11) which is the analogue of (43)). We find that the k^{th} component of the solution belongs to $C\{M_{n-2-\epsilon k-1-\lfloor \frac{1}{2}v \rfloor}; D^*\}$ (for an appropriate set D^*). We now make transformations inverse to (52) and (53), and, after some calculations similar to those of §2, we finally get the desired result.

Remark. From the proof of Theorem 5 it is clear that the domain E_μ which appears in the assumptions of Theorem 5 can be replaced by a smaller domain depending on the functions φ_{k_i} and f .

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