On Jordan Algebras with Two Generators

N. JACOBSON & L. J. PAIGE¹

1. Introduction. A (non-associative) algebra $\mathfrak{A}$ is called a Jordan algebra if its multiplication satisfies the identities

$$(1.1) \quad ab = ba, \quad (a^2b)a = a^2(ba).$$

We shall assume that the base ring Φ or $\mathfrak{A}$ is a field of characteristic not two. If $\mathfrak{A}$ is an associative algebra over a field of characteristic not two and multiplication composition $(\times)$, then $\mathfrak{A}$ defines a Jordan algebra $\mathfrak{A}^+$ relative to the Jordan multiplication $ab = \frac{1}{2}(a \times b + b \times a)$. A Jordan algebra is called special if it is isomorphic to a subalgebra of an algebra $\mathfrak{A}^+$, $\mathfrak{A}$ associative. It has been known for a long time that there exist exceptional (non-special) Jordan algebras and it has been shown by P. COHN [2] that there exist special Jordan algebras with homomorphic images which are not special.

Recently it was proved by A. I. SHIRSHOV [7] that the free Jordan algebra with two generators is special. The main purpose of this note is to give a comparatively simple alternative proof of this result which gives at the same time a simple basis for the algebra.² A sketch of the procedure is as follows. We begin by studying the free special Jordan algebra $\mathfrak{J}_0$ with two generators. This is the subalgebra relative to Jordan multiplication generated by two generators in a free associative algebra $\mathfrak{F}$. We obtain a basis for $\mathfrak{J}_0$ which is defined inductively using Jordan multiplication (§3). We then imitate this procedure in the free Jordan algebra $\mathfrak{J}$, thus defining a set S in $\mathfrak{J}$ in 1-1 correspondence with the basis for $\mathfrak{J}_0$. To prove that S is a basis it suffices to show that $[S]^2 \subset [S]$, where $[S]$ is the space spanned by S . The proof of this makes use of some three variable identities which are of independent interest (§2).

In considering $\mathfrak{J}_0$, it is possible to encompass the case of characteristic two by defining a special Jordan algebra as a vector space with a unary composition $a \rightarrow a^{[2]}$ and a binary composition $(a, b) \rightarrow \{aba\}$ such that there is a 1-1 linear

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²SHIRSHOV's result is valid also for rings containing no elements of additive order two. Our argument could be extended to that case but we shall not insist on this.

mapping β of the space into an associative algebra satisfying $(a^{121})^\beta = (a^\beta)^2$, $\{aba\}^\beta = a^\beta b^\beta a^\beta$. Roughly speaking, a special Jordan algebra (of any characteristic) is a subspace of an associative algebra closed under $a \rightarrow a^2$ and $(a, b) \rightarrow aba$. If the characteristic is not two this reduces to the requirement of closure relative to the Jordan multiplication.

A consequence of SHIRSHOV's theorem and an earlier one of COHN [2] is that any Jordan algebra (of characteristic not two) with two generators is special. The corresponding result fails for characteristic two even for one generator (§3).

2. Basic identities. The defining identities (1.1) for a Jordan algebra $\mathfrak{A}$ have a number of well-known consequences [1, 5] which we shall need. In these and throughout this note we write $a_1 a_2 \cdots a_n$ for $(\cdots ((a_1 a_2) a_3) \cdots a_n)$. Then we have

$$(2.1) \quad abdc + acdb + cbda = (ab)(cd) + (ac)(bd) + (ad)(bc);$$

$$(2.2) \quad abcd - acbd = adbc - adcb + cdba - bdca.$$

A Jordan algebra $\mathfrak{A}$ is power associative; that is, if $a^1 = a$ and $a^k = a^{k-1}a$ then $a^k a^r = a^{k+r}$. This is a consequence of

$$(2.3) \quad R_{a^k} R_{a^r} = R_{a^r} R_{a^k}, \quad k, r \geq 1,$$

where R_a is the mapping $x \rightarrow xa (= ax)$. Also we have

$$(2.4) \quad R_{a^{k+r}} = 2R_{a^k} R_{a^{k+r}} - (2R_{a^k}^2 - R_{a^{2k}}) R_{a^r}, \quad k, r \geq 1.$$

As in [5], we define the Jordan triple product $\{abc\}$ for a, b, c , in $\mathfrak{A}$ by

$$(2.5) \quad \{abc\} = abc + bca - cab.$$

If $\mathfrak{A}$ is an associative algebra with associative multiplication $a \times b$ and we take $ab = \frac{1}{2}(a \times b + b \times a)$, then

$$(2.6) \quad \{abc\} = \frac{1}{2}(a \times b \times c + c \times b \times a).$$

In any Jordan algebra $\{abc\}$ is linear in any one of the arguments if the other two are fixed. If $\mathfrak{A}$ has an identity 1, then

$$(2.7) \quad \{1ab\} = \{a1b\} = \{ab1\} = ab.$$

Moreover, $\{abc\} = \{cba\}$ and, by (2.3),

$$(2.8) \quad \{a^k a^r b\} = a^{k+r} b;$$

$$(2.9) \quad \{a^k b a^k\} a^r = \{a^k (b a^r) a^k\}.$$

We now list some additional identities for $\{abc\}$ which will play an important role in the sequel. Except for (2.10), which is equivalent to an identity in [5], these appear to be new three variable identities for Jordan algebras.

$$(2.10) \quad \{a^r \{a^k b a^k\} a^r\} = \{a^{k+r} b a^{k+r}\}, \quad k, r \geq 1.$$

$$(2.11) \quad \{aba\}c = 2\{a(ab)c\} - \{a^2bc\}.$$

$$(2.12) \quad 2\{abc\}a = \{a(bc)a\} + \{a^2bc\}.$$

$$(2.13) \quad 2\{a^rbc\}a = \{a\{a^{r-1}bc\}a\} + \{a^{r+1}bc\}, \quad r > 1.$$

$$(2.14) \quad \{a^r\{a^kba^k\}c\} = 2\{a^{r+k}(a^kb)c\} - \{a^{r+2k}bc\}, \quad r, k \geq 1.$$

Proofs. Equation (2.11). Expand the right hand side to obtain

$$(2.15) \quad \begin{aligned} 2\{a(ab)c\} - \{a^2bc\} &= 2abac + 2abca \\ &\quad - 2(ab)(ac) - a^2bc - a^2(bc) + a^2cb. \end{aligned}$$

From (2.1), we have

$$2abca = -a^2cb + 2(ab)(ac) + a^2(bc).$$

Substitution in (2.15) gives

$$2\{a(ab)c\} - \{a^2bc\} = 2abac - a^2bc = \{aba\}c.$$

Equations (2.12) and (2.13). Since every Jordan algebra can be embedded in a Jordan algebra with an identity we may assume that $1 \in \mathfrak{A}$. If we set $a^0 = 1$ and use (2.7) then (2.12) and (2.13) can be considered as the single relation (2.13) in which $r \geq 1$. We have

$$\begin{aligned} \{a\{a^{r-1}bc\}a\} &= 2\{a^{r-1}bc\}aa - \{a^{r-1}bc\}a^2 \\ &= 2(a^{r-1}bca - a^{r-1}cba)a + 2a^{r-1}(bc)aa \\ &\quad - a^{r-1}bca^2 - bca^{r-1}a^2 + a^{r-1}cba^2. \end{aligned}$$

If we take $a = a^{r-1}$, $b = b$, $c = c$, $d = a$ in (2.2), we obtain

$$a^{r-1}bca - a^{r-1}cba = a^rbc - a^rcb + acba^{r-1} - abca^{r-1}.$$

Substituting in the last equation, we obtain

$$(2.16) \quad \begin{aligned} \{a\{a^{r-1}bc\}a\} &= 2(a^rbc - a^rcb + acba^{r-1} - abca^{r-1})a + 2a^{r-1}(bc)aa \\ &\quad - a^{r-1}bca^2 - bca^{r-1}a^2 + a^{r-1}cba^2. \end{aligned}$$

Now consider

$$\{a^{r+1}bc\} = a^{r-1}a^2bc - a^{r-1}a^2cb + bca^{r+1}.$$

Applying (2.2) with $a = a^2$, $b = b$, $c = c$, $d = a^{r-1}$, we obtain

$$\{a^{r+1}bc\} = a^2bca^{r-1} - a^2cba^{r-1} + a^{r-1}bca^2 - a^{r-1}cba^2 + bca^{r+1}.$$

If we apply (2.2) with $a = a$, $b = b$, $c = c$, $d = a$ to the first two terms of the last equation, we obtain

$$(2.17) \quad \{a^{r+1}bc\} = 2(abca - acba)a^{r-1} + a^{r-1}bca^2 - a^{r-1}cba^2 + bca^{r+1}.$$

We now add (2.16) and (2.17), making liberal use of (2.3), to obtain

$$\begin{aligned}
 \{a\{a^{r-1}bc\}a\} + \{a^{r+1}bc\} &= 2(a^rbc - a^rcb)a + bc(2R_{a^{r-1}}R_a^2 - R_{a^{r-1}}R_{a^*} + R_{a^{r+1}}) \\
 &= 2(a^rbc - a^rcb)a + 2bc(R_aR_{a^*}) \\
 &= 2(a^rbc - a^rcb + bca^r)a \\
 &= 2\{a^rbc\}a,
 \end{aligned}$$

which is the required equation.

Equation (2.14). In contrast to the previous proofs, we begin with a decomposition of the left side of (2.14).

$$\{a^r\{a^kba^k\}c\} = \{a^kba^k\}a^rc + \{a^kba^k\}ca^r - \{a^kba^k\}(a^rc).$$

We consider the terms separately and write first

$$\{a^kba^k\}a^rc = (a^kba^ka^r - a^ka^kba^r)c + a^kba^ka^rc.$$

Using (2.2) with $a = a^k$, $b = b$, $c = a^k$, $d = a^r$, we obtain

$$a^kba^ka^r - a^ka^kba^r = a^{r+k}ba^k - a^{r+k}a^kb + a^{r+k}ba^k - a^rba^ka^k.$$

We substitute in the previous expression and obtain, using (2.3),

$$(2.18) \quad \{a^kba^k\}a^rc = 2a^kba^{r+k}c - a^{r+2k}bc.$$

Next consider,

$$\{a^kba^k\}ca^r = (a^kba^kc - a^ka^kbc)a^r + a^kba^kca^r.$$

If we use (2.2) with $a = a^k$, $b = b$, $c = a^k$, $d = c$, we obtain

$$(a^kba^kc - a^ka^kbc)a^r = (a^kcba^k - a^kca^kb)a^r + (a^kcba^k - bca^ka^k)a^r.$$

Applying (2.2) to the first two terms on the right with $a = a^kc$, $b = b$, $c = a^k$, $d = a^r$ and then substituting in the previous expressions yields

$$\begin{aligned}
 (2.19) \quad \{a^kba^k\}ca^r &= a^kca^rba^k - a^kca^ra^kb + a^{r+k}b(a^kc) - a^rba^k(a^kc) \\
 &\quad + (a^kcba^k - bca^ka^k)a^r + a^kba^kca^r.
 \end{aligned}$$

Now we consider,

$$\{a^kba^k\}(a^rc) = a^kba^k(a^rc) - a^ka^kb(a^rc) + a^kba^k(a^rc).$$

We apply (2.2) with $a = a^k$, $b = b$, $c = a^k$, $d = a^rc$ to the first two terms on the right hand side and obtain

$$(2.20) \quad \{a^kba^k\}(a^rc) = 2a^rca^kba^k - a^rca^ka^kb - a^rcba^ka^k + a^kba^k(a^rc).$$

Subtraction of (2.20) from (2.19) gives

$$\begin{aligned}
 (2.21) \quad \{a^kba^k\}ca^r - \{a^kba^k\}(a^rc) &= (a^rcba^k - a^rca^kb)a^k + a^{r+k}b(a^kc) \\
 &\quad - a^rba^k(a^kc) + (a^kcba^k - bca^ka^k)a^r + a^kba^kca^r - a^kba^k(a^rc).
 \end{aligned}$$

We apply (2.2) with $a = a^r$, $b = b$, $c = a^k$, $d = c$ to obtain

$$a^r c b a^k - a^r c a^k b = a^r b a^k c - a^{r+k} b c + b c a^k a^r - a^k c b a^r.$$

Substituting in (2.21), we have

$$(2.22) \quad \begin{aligned} \{a^k b a^k\} c a^r - \{a^k b a^k\} (a^r c) &= a^r b a^k c a^k - a^r b a^k (a^k c) + a^{r+k} b (a^k c) \\ &\quad - a^{r+k} b c a^k + a^k b a^k c a^r - a^k b a^k (a^r c). \end{aligned}$$

Next we take $a = a^k b$, $b = a^k$, $c = a^r$, $d = c$ in (2.1) and obtain

$$a^k b a^k c a^r = -a^{r+k} c (a^k b) - a^r (a^k b) c a^k + a^k b a^r (a^k c) + a^k b c a^{r+k} + a^r c (a^k b a^k).$$

Substituting in (2.22), the right hand side becomes

$$= a^{r+k} b (a^k c) - a^{r+k} b c a^k + a^k b c a^{r+k} - a^{r+k} c (a^k b).$$

Using (2.1) with $a = a^{r+k}$, $b = b$, $c = a^k$, $d = c$ we obtain

$$-a^{r+k} b c a^k = a^k b c a^{r+k} + a^{r+2k} c b - a^{r+2k} (b c) - a^k c (a^{r+k} b) - a^{r+k} c (a^k b).$$

Combining the last two expressions, we have

$$(2.23) \quad \begin{aligned} \{a^k b a^k\} c a^r - \{a^k b a^k\} (a^r c) &= 2a^k b c a^{r+k} \\ &\quad - 2a^{r+k} c (a^k b) + a^{r+2k} c b - a^{r+2k} (b c). \end{aligned}$$

If we add (2.23) and (2.18), we obtain the desired result,

$$\begin{aligned} \{a^r \{a^k b a^k\} c\} &= 2a^{r+k} (a^k b) c + 2a^k b c a^{r+k} - 2a^{r+k} c (a^k b) \\ &\quad - a^{r+2k} b c - a^{r+2k} (b c) + a^{r+2k} c b, \\ &= 2\{a^{r+k} (a^k b) c\} - \{a^{r+2k} b c\}. \end{aligned}$$

3. The free special Jordan algebra with two generators. In this section we allow the base field Φ to be of any characteristic. Let $\mathfrak{F}$ be the free associative algebra with two generators a, b over Φ . We denote the product in $\mathfrak{F}$ by xy . Then $\mathfrak{F}$ has a basis consisting of 1 and the monomials in a and b . These have one of the following forms

$$(3.1) \quad \begin{aligned} (i) \quad &a^{r_1} b^{s_1} \cdots a^{r_k} b^{s_k}, & (ii) \quad &b^{s_1} a^{r_1} \cdots b^{s_k} a^{r_k}, \\ (iii) \quad &a^{r_1} b^{s_1} \cdots a^{r_k} b^{s_k} a^{r_{k+1}}, & (iv) \quad &b^{s_1} a^{r_1} \cdots b^{s_k} a^{r_k} b^{s_{k+1}}, \end{aligned}$$

where the r_i and s_i are > 0 . We call (i), (ii), (iii) and (iv) ab -elements, ba -elements, a -elements and b -elements respectively. The a -degrees and b -degrees are defined as usual. We shall need also a notion of *height* defined as follows: 1 is of height 0, the elements of (i) and (ii) in (3.1) are of height $2k$, the elements of (iii) and (iv) in (3.1) are of height $2k + 1$. The notion of height for arbitrary elements of $\mathfrak{F}$ is defined in the obvious way.

Let J be the involution in $\mathfrak{F}$ sending $a^{u_1} b^{v_1} \cdots a^{u_k} b^{v_k}$, $u_i \geq 0, v_i \geq 0$ into

$b^{v_k}a^{u_k} \dots b^{v_1}a^{u_1}$ and let $\mathfrak{S}$ denote the space of J -symmetric elements of $\mathfrak{F}$. A basis for $\mathfrak{S}$ consists of 1 and the elements of even height

$$(3.2) \quad a^{r_1}b^{s_1} \dots a^{r_k}b^{s_k} + b^{s_k}a^{r_k} \dots b^{s_1}a^{r_1}, \quad r_i, s_i > 0,$$

the a -elements of odd height,

$$(3.3) \quad \begin{aligned} & \text{(i)} \quad a^{r_1}b^{s_1} \dots a^{r_k}b^{s_k}a^{r_{k+1}}, \quad r_i = r_{k+2-i} > 0, \\ & \quad \quad \quad s_i = s_{k+1-i} > 0, \quad (j = 1, \dots, k), \\ & \text{(ii)} \quad a^{r_1}b^{s_1} \dots a^{r_k}b^{s_k}a^{r_{k+1}} + a^{r_{k+1}}b^{s_k}a^{r_k} \dots b^{s_1}a^{r_1}, \quad r_i, s_i > 0, \end{aligned}$$

where one of the equalities in (i) does not hold,

and the b -elements of odd height

$$(3.4) \quad \text{as in (3.3) with } a, b \text{ interchanged.}$$

Let $\mathfrak{S}_0$ be the smallest subspace of $\mathfrak{F}$ containing 1, a , b and closed relative to the compositions $x \rightarrow x^2$ and $(x, y) \rightarrow xyz$. Since $\mathfrak{S}$ is a subspace closed relative to these compositions, $\mathfrak{S} \supset \mathfrak{S}_0$. We shall call $\mathfrak{S}_0$ the *free special Jordan algebra* generated by a and b . It is easy to see that $\mathfrak{S}_0$ is closed relative to $x \rightarrow x^n$, $n = 1, 2, 3, \dots$, to $(x, y) \rightarrow \{xy\} \equiv xy + yx$ and $(x, y, z) \rightarrow \{xyz\} \equiv xyz + zyx$. If the characteristic is not two then $\mathfrak{S}_0$ can be characterized as the smallest subspace closed under the Jordan multiplication $\{xy\}$ (or $\frac{1}{2}(xy + yx)$).

Corresponding to the basis defined above for $\mathfrak{S}$, we define inductively a subset S_0 of $\mathfrak{S}_0$, $\mathfrak{S}$ -heights, a -elements, b -elements and ab -elements of S_0 as follows:

- (i) $1 \in S_0$ and this is the only element of $\mathfrak{S}$ -height 0 in S_0 .
- (ii) a^r, b^s for $r, s > 0$ are in S_0 and these are the only elements of $\mathfrak{S}$ -height 1 in S_0 . a^r is an a -element and b^s is a b -element.
- (iii) $\{a^r b^s\}$, $r, s > 0$ are ab -elements of $\mathfrak{S}$ -height 2 in S_0 .

Now suppose that the elements of S_0 of $\mathfrak{S}$ -height $< m$ have been specified. If m is even we define the elements of S_0 of $\mathfrak{S}$ -height m to be

$$(3.5) \quad \{a^r z b^s\}, \quad r, s > 0,$$

where $z \in S_0$ and has $\mathfrak{S}$ -height $m - 2$. We shall call these ab -elements of S_0 . If m is odd, then the a -elements of S_0 of $\mathfrak{S}$ -height m are the elements

$$(3.6) \quad \begin{aligned} & \text{(i)} \quad a^r y a^r, \quad r > 0, \quad y \text{ a } b\text{-element of } \mathfrak{S}\text{-height } m - 2 \text{ in } S_0. \\ & \text{(ii)} \quad a^r z a^r, \quad r > 0, \quad z \text{ an } ab\text{-element of } \mathfrak{S}\text{-height } m - 1 \text{ in } S_0. \end{aligned}$$

If m is odd, then the b -elements of S_0 of $\mathfrak{S}$ -height m are the elements

$$(3.7) \quad \begin{aligned} & \text{(i)} \quad b^s x b^s, \quad s > 0, \quad x \text{ an } a\text{-element of } \mathfrak{S}\text{-height } m - 2 \text{ in } S_0. \\ & \text{(ii)} \quad b^s z b^s, \quad s > 0, \quad z \text{ an } ab\text{-element of } \mathfrak{S}\text{-height } m - 1 \text{ in } S_0. \end{aligned}$$

Lemma 1. *Every element of $\mathfrak{S}$ is a linear combination of the elements of S_0 .*

Proof. $1 \in S_0$. We proceed by induction and assume that every element of the form (3.2) of height $< 2k$ is a linear combination of ab -elements of S_0 of $\mathfrak{J}$ -height $< 2k$. Consider

$$x = a^{r_1} b^{s_1} \dots a^{r_k} b^{s_k} + b^{s_k} a^{r_k} \dots b^{s_1} a^{r_1}, \quad r_i, s_i > 0.$$

Express $y = a^{r_k} b^{s_{k-1}} \dots a^{r_2} b^{s_1} + b^{s_1} a^{r_2} \dots b^{s_{k-1}} a^{r_k} = \sum \alpha_i u_i$, where the u_i are ab -elements in S_0 of $\mathfrak{J}$ -height $< 2k$. Then

$$\{a^{r_1} y b^{s_k}\} = \sum \alpha_i \{a^{r_1} u_i b^{s_k}\},$$

a linear combination of elements of S_0 of $\mathfrak{J}$ -height $\leq 2k$. On the other hand

$$\{a^{r_1} y b^{s_k}\} = x + a^{r_1+r_k} b^{s_{k-1}} \dots a^{r_2} b^{s_1+s_k} + b^{s_1+s_k} a^{r_2} \dots b^{s_{k-1}} a^{r_1+r_k},$$

and the sum of the last two terms is an element of the form (3.2) of height $2k - 2$. Hence x is a linear combination of ab -elements of S_0 of $\mathfrak{J}$ -height $\leq 2k$. We consider next the elements of (3.3) and (3.4). The elements in this set which are of height 1 are the elements a^r, b^s and these are in S_0 . Assume that the elements of (3.3) and (3.4) of height $\leq 2k - 1$ are expressible as linear combinations of a -elements and b -elements respectively of $\mathfrak{J}$ -heights $\leq 2k - 1$. Consider an element x of the form (3.3i). We can write this as $a^r y a^r$, where y is of the form (3.4i) and y is of height $2k - 1$. Then y is a linear combination of b -elements of S_0 of $\mathfrak{J}$ -height $\leq 2k - 1$ so that, by (3.6i), x is a linear combination of a -elements of S_0 of $\mathfrak{J}$ -height $\leq 2k + 1$. A similar argument applies to the elements x of the form (3.3ii) for which $r_1 = r_{k+1}$. Next assume we have an x of the form (3.3ii) with $r_1 \neq r_{k+1}$. We may assume that $r_{k+1} = r_1 + r, r > 0$, and write

$$x = a^{r_1} (b^{s_1} a^{r_2} \dots b^{s_{k+1}} a^{r_k} b^{s_k} a^r + a^r b^{s_k} a^{r_k} b^{s_{k-1}} \dots a^{r_2} b^{s_1}) a^{r_1}.$$

The element in the parenthesis is an ab -element of the form (3.2) in S_0 of $\mathfrak{J}$ -height $2k$. Hence it is a linear combination of ab -elements of S_0 of $\mathfrak{J}$ -heights $\leq 2k$. Then, by (3.6ii), x is a linear combination of a -elements of S_0 of $\mathfrak{J}$ -heights $\leq 2k + 1$. A similar argument applies to the b -elements (3.4). This completes the proof of Lemma 1.

Lemma 2. *The elements of S_0 are linearly independent.*

Proof. It is easy to see by induction that if $x \in S_0$ has $\mathfrak{J}$ -height m then x has height m . It therefore suffices to show that the elements of S_0 of every fixed $\mathfrak{J}$ -height are linearly independent. Suppose this height is $2k$. We assume $k > 1$. Then the elements in our set are of the form $\{a^r z b^s\}$, $r, s > 0$, z in S_0 of $\mathfrak{J}$ -height $2(k - 1)$. The expression for $\{a^r z b^s\}$ in terms of the monomial basis for $\mathfrak{F}$ involves only monomials which begin with a^r and end with b^s or begin with b^s and end with a^r . Hence if we have a non-trivial relation in these elements then we must have one for fixed r, s . Assuming the corresponding z 's are linearly independent, we are led to a contradiction. Next assume the common height is $2k + 1$. There are four types of elements of this height, namely, the a -elements of (3.6) and the b -elements of (3.7). If we have a non-trivial linear relation

among these elements then we have one involving only a -elements or only b -elements. The elements (3.6i) are expressible in terms of monomials of $\mathfrak{F}$ which begin and end with a^r while those in (3.6ii) are expressible in terms of monomials which begin with a^r and end with $a^{r'}$, $r \neq r'$. Hence we have a non-trivial linear relation connecting the elements of (3.6i) or one connecting the elements of (3.6ii). In either case we are led to a contradiction as in the even case, assuming the elements of lower $\mathfrak{F}$ -height are linearly independent.

Evidently $S_0 \subset \mathfrak{F}_0$. Hence Lemmas 1 and 2 have the following consequence:

Theorem 1. $\mathfrak{F} = \mathfrak{F}_0$ and S_0 is a basis for this space.³

It has been shown by COHN [2] that if the characteristic is not two then for any ideal $\mathfrak{R}$ in the Jordan algebra $\mathfrak{F}_0 (= \mathfrak{F})$ we have $\mathfrak{F}_0 \wedge \mathfrak{R}^* = \mathfrak{R}$, where $\mathfrak{R}^*$ is the ideal generated by $\mathfrak{R}$ in the associative algebra $\mathfrak{F}$. This result implies that every homomorphic image of $\mathfrak{F}_0$ of characteristic not two is special. We remark that the corresponding result fails in the characteristic two case even for one generator. Here one defines an ideal $\mathfrak{R}$ in a special Jordan algebra $\mathfrak{A}$ as a subspace of $\mathfrak{A}$ such that if $a \in \mathfrak{A}$, $k \in \mathfrak{R}$ then $\{ak\} = ak + ka \in \mathfrak{R}$, $k^2 \in \mathfrak{R}$, $aka \in \mathfrak{R}$, $kak \in \mathfrak{R}$ (see [6], p. 491). Now let $\mathfrak{A}$ be the polynomial algebra in an indeterminate a over a field of characteristic two. Let $\mathfrak{R}$ be the subspace with basis $a^2, a^n, n \geq 4$. Then $\mathfrak{R}$ is an ideal. On the other hand, the associative ideal $\mathfrak{R}^*$ generated by $\mathfrak{R}$ has the basis $a^m, m \geq 2$. Since $\mathfrak{R}^* \subset \mathfrak{A}$, $\mathfrak{R}^* \cap \mathfrak{A} = \mathfrak{R}^* \supset \mathfrak{R}$.

4. A basis for the free Jordan algebra with two generators. Let $\mathfrak{F}$ be the free Jordan algebra generated by two elements a, b over a field Φ of characteristic not two. We denote the product in $\mathfrak{F}$ by xy and, as before, we set $\{xyz\} = xyz + zyx - zxy$.

By analogy with the basis S_0 for the free special Jordan algebra $\mathfrak{F}_0$ we shall define a subset S of $\mathfrak{F}$, heights for elements of S , a -elements, b -elements and the notions of a -degree and b -degree for the elements of S . We proceed inductively and list the elements of S in the following tabular form:

(4.1) ab -elements (even heights)

- (i) Height 0 : 1; a -degree = 0; b -degree = 0.
- (ii) Height 2: $a^r b^s$, $r, s > 0$; a -degree = r ; b -degree = s .
- (iii) Height $2k$: $\{a^r z b^s\}$, $r, s > 0$, z of height $2(k-1)$; a -degree = $r + a$ -degree of z ; b -degree = $s + b$ -degree of z .

(4.2) a -elements (odd heights)

- (i) Height 1: a^r , $r > 0$; a -degree = r ; b -degree = 0.
- (ii) Height $2k+1$: $\{a^r z a^r\}$, $r > 0$, z of height $2k$; a -degree = $2r + a$ -degree of z ; b -degree = b -degree of z .
- (iii) Height $2k+1$: $\{a^r y a^r\}$, $r > 0$, yab -element of height $2k-1$; a -degree = $2r + a$ -degree of y ; b -degree = b -degree of y .

³The equality $\mathfrak{F} = \mathfrak{F}_0$ in the characteristic not two case was proved by COHN [2].

(4.3) b -elements (odd heights)

Same as (4.2) with a and b interchanged.

If T is a subset of $\mathfrak{J}$ then we write $[T]$ for the subspace spanned by T . We write A, B, E for the subsets of S of a -elements, b -elements and ab -elements respectively, and if K is one of these sets then we let $K_{r,s}^{(m)}$ denote the subset of K of elements of height m and a -degree r , b -degree s . Some of these subsets are vacuous. For example, if m is even then $A_{r,s}^{(m)}$ is vacuous and $E_{r,s}^{(2m)}$ is vacuous if $r < m$ or $s < m$ since the a -degree and the b -degree of an ab -element is at least half of the height of the element.

We wish to prove that $S\mathfrak{J} \subset [S]$ and hence that $[S] = \mathfrak{J}$.

Lemma 3. *A necessary and sufficient condition that $S\mathfrak{J} \subset [S]$ is that $Sx \subset [S]$ for $x = a, b, a^2, b^2$ and ab .*

Proof. The necessity is obvious. For the sufficiency it is clear that we need consider only products su , s in S and u a monomial in a and b . By induction, we assume that $su \in [S]$ for all s and every u which is a product of $k < n$ a 's and b 's. If u is a product of $n \geq 3$ a 's and b 's then, since $\mathfrak{J}$ is commutative, $u = (u_1 u_2) u_3$, where each u_i is a product of less than n a 's and b 's. Then by (2.1)

$$su = s(u_1 u_2 u_3) = -su_2 u_3 u_1 - su_1 u_3 u_2 + su_1 (u_2 u_3) + su_2 (u_1 u_3) + su_3 (u_1 u_2).$$

The inductive assumption and the hypothesis imply that $su \in [S]$.

Lemma 4. *Necessary and sufficient conditions that $S\mathfrak{J} \subset [S]$ are (i) $Sa \subset [S]$, (ii) $Sb \subset [S]$, (iii) $\{aSb\} \subset [S]$.*

Proof. It follows from (2.10) and the definition of S that $\{a^r S a^r\} \subset [S]$ and $\{b^r S b^r\} \subset [S]$. Also we have

$$\begin{aligned} sa^2 &= \{asa\} - 2saa, & sb^2 &= \{bsb\} - 2sbb, \\ s(ab) &= -\{asb\} + sab + sba. \end{aligned}$$

Hence (i), (ii) and (iii) imply $Sa^2 \subset [S]$, $Sb^2 \subset [S]$, $S(ab) \subset [S]$. The result therefore follows from Lemma 3.

With the notations introduced previously we can now formulate our main lemma as follows:

Lemma 5. *The following inclusions hold for $t, u > 0$:*

$$\begin{aligned} \text{(i)} \quad & \{a^t E_{r,s}^{(n)} b^u\} \subset E_{r+t,s+u}^{(n+2)}; \\ \text{(ii)} \quad & \begin{cases} E_{r,s}^{(n)} a^t \subset X_{r+t,s}^{(n+1)} \equiv [E_{r+t,s}^{(n)}] + \sum_{k \leq n+1} [A_{r+t,s}^{(k)}]; \\ E_{r,s}^{(n)} b^t \subset Y_{r,s+t}^{(n+1)} \equiv [E_{r,s+t}^{(n)}] + \sum_{k \leq n+1} [B_{r,s+t}^{(k)}]; \end{cases} \\ \text{(iii)} \quad & \begin{cases} A_{r,s}^{(n+1)} a^t \subset U_{r+t,s}^{(n+1)} \equiv \sum_{k \leq n+1} [A_{r+t,s}^{(k)}]; \\ B_{r,s}^{(n+1)} b^t \subset V_{r,s+t}^{(n+1)} \equiv \sum_{k \leq n+1} [B_{r,s+t}^{(k)}]; \end{cases} \end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad & \begin{cases} \{a^t A_{r,s}^{(n)} b^u\} \subset Z_{r+t,s+u}^{(n+1)} \equiv \sum_{k \leq n+1} [E_{r+t,s+u}^{(k)}]; \\ \{a^t B_{r,s}^{(n)} b^u\} \subset Z_{r+t,s+u}^{(n+1)}; \end{cases} \\
\text{(v)} \quad & \begin{cases} A_{r,s}^{(n)} b^t \subset Z_{r,s+t}^{(n+1)}; \\ B_{r,s}^{(n)} a^t \subset Z_{r+t,s}^{(n+1)}. \end{cases}
\end{aligned}$$

Proof. (i) is evident from the definition of the ab -elements. The remaining relations will be proved by induction on the height n . It is easy to check that these hold for $n = 0, 1$. Hence we may assume they hold for $0 \leq n < m$ and we shall prove they are valid for $n = m$. We remark that

$$(4.4) \quad \begin{cases} \{a^r X_{k,h}^{(n+1)} a^r\} \subset U_{k+2r,h}^{(n+1)} \\ \{a^r Y_{k,h}^{(n-1)} a^r\} \subset U_{k+2r,h}^{(n+1)} \\ \{a^r Z_{k,h}^{(n)} a^r\} \subset U_{k+2r,h}^{(n+1)} \end{cases}$$

are consequences of (2.10) and the definition of the a -elements. Similar relations hold for b with V replacing U . We suppress these for it is clear that we can obtain the results of the lemma for multiplication by b^t from those on multiplication by a^t by a "dual" argument. We remark next that the induction assumption together with the validity of (ii) for $n = m$ and a particular t implies (iii) for $n = m$ and the same t . There is nothing to prove if m is odd. Hence we assume m is even and let $x \in A_{r,s}^{(m+1)}$ so that either $x = \{a^k y a^k\}$ or $x = \{a^k z a^k\}$ with $k > 0$ and y in $B_{r-2k,s}^{(m-1)}$, z in $E_{r-2k,s}^{(m)}$. In the first case, $x a^t = \{a^k (y a^t) a^k\}$ by (2.9), and $y a^t \in Z_{r-2k+t,s}^{(m)}$ by the induction assumption. Hence $x a^t \in U_{r+t,s}^{(m+1)}$ by (4.4). If $x = \{a^k z a^k\}$, z in $E_{r-2k,s}^{(m)}$, then $x a^t = \{a^k (z a^t) a^k\}$ and $z a^t \in X_{r-2k+t,s}^{(m+1)}$ by assumption. Hence $x a^t \in U_{r+t,s}^{(m+1)}$ by (4.4). We now proceed to the proofs of (ii)–(v) for $n = m$.

(ii) and (iii). If m is odd there is nothing to prove. Hence we assume m even. Let $z \in E_{r,s}^{(m)}$ and write $z = \{a^k w b^h\}$, where $k, h > 0$ and $w \in E_{r-k,s-h}^{(m-2)}$. By (2.12) and (2.13),

$$(4.5) \quad za = \{a^k w b^h\} a = \frac{1}{2} \{a \{a^{k-1} w b^h\} a\} + \frac{1}{2} \{a^{k+1} w b^h\}.$$

Since

$$\{a^{k+1} w b^h\} \in E_{r+1,s}^{(m)}, \quad za \equiv \frac{1}{2} \{a \{a^{k-1} w b^h\} a\} \pmod{X_{r+1,s}^{(m+1)}}.$$

If $k > 1$, then

$$\{a^{k-1} w b^h\} \in E_{r-1,s}^{(m)} \subset X_{r-1,s}^{(m+1)} \quad \text{and} \quad \{a \{a^{k-1} w b^h\} a\} \in U_{r+1,s}^{(m+1)} \subset X_{r+1,s}^{(m+1)}.$$

Hence $za \in X_{r+1,s}^{(m+1)}$ as required. If $k = 1$, then

$$\{a \{w b^h\} a\} \in \{a Y_{r-1,s}^{(m-1)} a\} \subset U_{r+1,s}^{(m+1)} \subset X_{r+1,s}^{(m+1)},$$

by (4.4). This proves (ii) for $n = m$ and $t = 1$. By the remark above we also have (iii) for $n = m$ and $t = 1$.

If $t > 1$ we use (2.4) with $k = 1$ to write

$$(4.6) \quad za^t = \{a(za^{t-2})a\} - 2(za^{t-1})a.$$

We may assume that $za^{t-2} \in X_{r+t-2,s}^{(m+1)}$ and $za^{t-1} \in X_{r+t-1,s}^{(m+1)}$. Then by (4.4), $\{a(za^{t-2})a\} \in X_{r+t,s}^{(m+1)}$. Moreover, the validity of (ii) and (iii) for $t = 1$ implies that $X_{r+t-1,s}^{(m+1)} \subset X_{r+t,s}^{(m+1)}$. Hence $za^t \in X_{r+t,s}^{(m+1)}$ as required. Thus (ii) holds for any t and consequently (iii) holds for any t .

(iv). We may assume $m = 2\mu + 1$, $\mu \geq 1$. Since the minimum a -degree of an a -element of height $2\mu + 1$ is $\mu + 1$, $A_{r,s}^{(m)}$ is vacuous if $r \leq \mu$ and so (iv) is trivially satisfied if $r \leq \mu$. We shall now use induction on the a -degree r in proving the first half of (iv). Let $x \in A_{r,s}^{(m)}$ so that either $x = \{a^k ya^k\}$ with $k > 0$ and y in $B_{r-2k,s}^{(m-2)}$ or $x = \{a^k za^k\}$, $k > 0$, z in $E_{r-2k,s}^{(m-1)}$. In the first case, by (2.14),

$$(4.7) \quad \{a^t xb^u\} = \{a^t \{a^k ya^k\} b^u\} = 2\{a^{t+u}(a^k y)b^u\} - \{a^{t+2k} y b^u\}.$$

By the induction assumption, $\{a^{t+2k} y b^u\} \in Z_{r+t,s+u}^{(m-1)} \subset Z_{r+t,s+u}^{(m+1)}$. Also $a^k y \in Z_{r-k,s}^{(m-1)}$ by the induction assumption on (v). Hence $\{a^{t+k}(a^k y)b^u\} \in Z_{r+t,s+u}^{(m+1)}$ by definition of the sets Z . This proves that $\{a^t xb^u\} \in Z_{r+t,s+u}^{(m+1)}$ in the first case. Next assume that $x = \{a^k za^k\}$, z in $E_{r-2k,s}^{(m-1)}$. Then

$$(4.8) \quad \{a^t xb^u\} = 2\{a^{t+k}(a^k z)b^u\} - \{a^{t+2k} z b^u\}.$$

The second term in (4.8) is in $E_{r+t,s+u}^{(m+1)} \subset Z_{r+t,s+u}^{(m+1)}$. By the induction assumption on (ii),

$$a^k z \in X_{r-k,s}^{(m)} = E_{r-k,s}^{(m-1)} + \sum_{i \leq m} A_{r-k,s}^{(i)}.$$

Induction on the a -degree r and the definition of the E -sets permits us to conclude that $\{a^{t+k}(a^k z)b^u\} \in Z_{r+t,s+u}^{(m+1)}$. This completes the proof of (iv).

(v). We may assume m is odd. Let $x \in A_{r,s}^{(m)}$ so that either $x = \{a^k ya^k\}$, $k > 0$, $y \in B_{r-2k,s}^{(m-2)}$ or $x = \{a^k za^k\}$, $k > 0$, $z \in E_{r-2k,s}^{(m-1)}$. By (2.11),

$$(4.9) \quad \{a^k ya^k\} b^t = 2\{a^k(a^k y)b^t\} - \{a^{2k} y b^t\}.$$

By induction, $a^k y \in Z_{r-k,s}^{(m-1)}$ so that $\{a^k(a^k y)b^t\} \in Z_{r,s+t}^{(m+1)}$. Also $\{a^{2k} y b^t\} \in Z_{r,s+t}^{(m-1)} \subset Z_{r,s+t}^{(m+1)}$. Hence (v) holds in this case. If $x = \{a^k za^k\}$, z in $E_{r-2k,s}^{(m-1)}$, then

$$(4.10) \quad \{a^k za^k\} = 2\{a^k(a^k z)b^t\} - \{a^{2k} z b^t\}.$$

By (ii) for $n = m - 1$, $a^k z \in X_{r-k,s}^{(m)}$ and $\{a^k(a^k z)b^t\} \in Z_{r,s+t}^{(m+1)}$ by (iv) for $n = m$. Also $\{a^{2k} z b^t\} \in E_{r,s+t}^{(m+1)}$. Hence (v) is valid for $n = m$.

Theorem 2. *The free Jordan algebra $\mathfrak{J}$ with two generators is special and S is a basis for $\mathfrak{J}$.*

Proof. Let $\mathfrak{F}$ be the free associative algebra generated by a_0, b_0 with associative product $x \times y$. We introduce the Jordan product $xy = \frac{1}{2}(x \times y + y \times x)$ in $\mathfrak{F}$ and let $\mathfrak{J}_0$ be the Jordan algebra generated by a_0, b_0 relative to this composition. Then $\mathfrak{J}_0$ is the free special Jordan algebra which we defined in §3

and $\mathfrak{J}_0$ has the basis S_0 (3.5—3.7). Since $\{xyz\} = \frac{1}{2}(x \times y \times z + z \times y \times x)$ in $\mathfrak{J}_0$, it is clear that the homomorphism of $\mathfrak{J}$ onto $\mathfrak{J}_0$ which maps a into a_0 , b into b_0 maps the elements of S in a 1 — 1 way onto non-zero multiples (powers of 2) of the elements of S_0 . It follows that the elements of S are linearly independent. By Lemmas 4 and 5 every element of $\mathfrak{J}$ is a linear combination of the elements in S . Hence S is a basis for $\mathfrak{J}$. Then the homomorphism of $\mathfrak{J}$ onto $\mathfrak{J}_0$ maps this basis into a set of linearly independent elements; consequently this mapping is an isomorphism.

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Yale University
New Haven, Connecticut
and
University of California
Los Angeles, California