Uniqueness and Representation Theorems for Solutions of $\Delta u + k^2 u = 0$ in Infinite Domains

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1. Introduction. In this paper we will discuss uniqueness and representation theorems for solutions u of the reduced wave equation

$$(1) \quad \Delta u + k^2 u = 0, \quad k > 0,$$

when the domain D of definition of u is infinite. B denotes the boundary of D .

HISTORY OF THE PROBLEM.

We will discuss first the case in which B is bounded. In 1912 SOMMERFELD [8] introduced the radiation and boundedness conditions for u . These are respectively

$$(2) \quad \lim_{r \rightarrow \infty} \frac{1}{2}r(n-1) \left| \frac{\partial u}{\partial r} - iku \right| = 0,$$

$$(3) \quad \lim_{r \rightarrow \infty} \frac{1}{2}r(n-1)|u| < \text{constant}.$$

n is the dimension and r is a radial polar coordinate measured from some fixed point in space. Using (2) and (3) and a special assumption concerning the Green's function for (1), SOMMERFELD proved the uniqueness of the exterior boundary value problem for u . In 1942 W. MAGNUS [2] deduced the same result using only (2) and (3). In 1943 RELICH [5] obtained the uniqueness theorem using only the radiation condition and, moreover, in the following weakened form:

$$(4) \quad \lim_{R \rightarrow \infty} \iint_{\Sigma_R} \left| \frac{\partial u}{\partial R} - iku \right|^2 dS = 0$$

where Σ_R is the surface of a sphere of radius R with some fixed point in space as center and dS is a surface element. He obtained his proof by showing that

$$(5) \quad \lim_{R \rightarrow \infty} \iint_{\Sigma_R} |u|^2 dS \neq 0$$

unless $u \equiv 0$. This result is referred to below as the Rellich growth estimate. RELICH's result seems to have settled the uniqueness problem in the case of bounded B .

The idea of representing u as a sum of layers dates back at least to POINCARÉ. This representation reads

$$(6) \quad u(P) = \frac{1}{4\pi} \iint_B \left(\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{e^{ik|P-Q|}}{|P-Q|} \right) dS$$

where $\partial/\partial n$ is differentiation in the exterior normal direction. The result follows trivially from the radiation and boundedness conditions and so must have been known to SOMMERFELD in 1912. In 1948 W. MANGUS [3] showed that the radiation condition implied the boundedness condition. Thus implicitly contained in MAGNUS' paper is a proof of the representation theorem using only the radiation condition. An explicit proof is found below and in [9].

We will now discuss the history of the problem for the case when B is infinite.

In RELICH's 1943 paper, he deduced the growth estimate (5) for u in the case that B had roughly the shape of a paraboloid. More precisely every plane perpendicular to a fixed direction is to cut off at most a finite portion of D , and the angle made by this fixed direction with every exterior normal to B is not less than $\frac{1}{2}\pi$. Using the growth estimate he showed that the radiation condition implied uniqueness in this case. The strange restriction on B has physical significance. A domain with pinches in it is capable of trapping modes. For example, radiation of appropriate wavelength set up in a box with a long thin pipe attached to it cannot escape to infinity down the pipe. In 1953 D. S. JONES [1] showed that the radiation condition implies uniqueness in the case that D is semi-infinite and approaches a cone at infinity. In this paper of JONES and in [6] are theorems concerning the existence of eigensolutions of (1) when D is semi-infinite and does not "open up" too rapidly at infinity. These conditions explain the physical difficulties just mentioned. If the opening to infinity is wide enough (a cone) a mode cannot be trapped in the above-mentioned box but will leak out of the opening and escape to infinity (*i.e.* radiate).

In 1954 A. PETERS & J. J. STOKER [4] deduced a uniqueness theorem for two dimensions when B is a straight line outside of some circle. (This situation was tailored to provide a uniqueness theorem for the famous Sommerfeld problem of diffraction of a plane wave by a half plane.) PETERS & STOKER point out that the radiation condition, since it involves the dimension, is inappropriate if the domain admits a lower dimensional solution, because this lower dimensional solution cannot obey the condition. For example, in Sommerfeld's problem, the diffracted solution consists in part of an optical reflection of the incoming plane wave, e^{ikhz} , which is a one-dimensional quantity and does not satisfy the radiation condition. (It satisfies a one-dimensional radiation condition.) Thus PETERS & STOKER conclude that to obtain a uniqueness theorem one must decompose u into two parts (each discontinuous). One part does not satisfy the radiation

condition and must be fixed by considerations involving the requirements of the specific problem. The remaining part satisfies the radiation condition. Their theorem states that the radiating part is determined uniquely.

The results of RELICH and JONES on the one hand and PETERS & STOKER on the other foretell the difficulties to be encountered in attempting to prove a general uniqueness theorem in the case in which B is infinite..

Apart from boundaries of special shapes, there exists no representation theorem for u when B is infinite which uses only the radiation condition.

DESCRIPTION OF RESULTS.

In section 2 we consider the case of bounded B . We present a new proof of the Rellich growth estimate and show how the uniqueness theorem follows from it. Our proof makes no use of expansions in spherical harmonics and seems to be simpler than RELICH's original proof. The method of proof is similar to one which is used in deriving the mean value theorem for solutions of (1) (see [7]). By means of the same method, we obtain the well known theorem concerning the representation of u as a sum of single and double layers. Our proof of this theorem has the advantage that, apart from the radiation condition, it makes no use of asymptotic properties of u .

In section 3 we consider the case of unbounded B . By exploiting the method introduced in section 2, we deduce the representation theorem and a uniqueness theorem for u . These results have not hitherto been proved.

Our results are given in three dimensions because of the physical significance. However, similar results are valid for any number of dimensions.

2. Domains with a Bounded Boundary. Let D be the complement of a bounded domain with boundary B . Let Σ_R be the surface of a sphere of radius R and let B be such that Green's theorem can be applied to $D_R = D \cap$ (interior Σ_R), for Σ_R containing B . Let u be a solution in D of

$$(1) \quad \Delta u + k^2 u = 0.$$

The radiation condition is

$$(4) \quad \lim_{R \rightarrow \infty} \iint_{\Sigma_R} \left| \frac{\partial u}{\partial R} - iku \right|^2 dS = 0.$$

dS denotes a surface element and dV will denote a volume element. The letters P and Q will denote points in D . Our first result is given in the following

Theorem. 1. (*Rellich growth estimate.*)

Let u satisfy (1). Then

$$(5) \quad \lim_{R \rightarrow \infty} \iint_{\Sigma_R} |u|^2 dS \neq 0$$

unless u is identically zero.

Proof: Let P be the center of Σ_R . Let P be outside of B and let Σ_R contain B . Then apply Green's theorem

$$(7) \quad \iiint (u\Delta v - v\Delta u) dV = \iint \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

to D_R . First let v be $(\cos k | P - Q |) / | P - Q |$ and then $(\sin k | P - Q |) / | P - Q |$. If ρ is the distance from P to the point of integration on B , we obtain respectively

$$(8) \quad \begin{aligned} u(P) &= \frac{1}{4\pi} \iint_{\Sigma_R} \left[\frac{\cos kR}{R} \frac{\partial u}{\partial R} - u \frac{\partial}{\partial R} \left(\frac{\cos kR}{R} \right) \right] dS \\ &\quad + \frac{1}{4\pi} \iint_B \left[\frac{\cos k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\cos k\rho}{\rho} \right) \right] dS \end{aligned}$$

$$(9) \quad \begin{aligned} 0 &= \frac{1}{4\pi} \iint_{\Sigma_R} \left[\frac{\sin kR}{R} \frac{\partial u}{\partial R} - u \frac{\partial}{\partial R} \left(\frac{\sin kR}{R} \right) \right] dS \\ &\quad + \frac{1}{4\pi} \iint_B \left[\frac{\sin k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\sin k\rho}{\rho} \right) \right] dS. \end{aligned}$$

Now multiply (8) by $(\sin kR)/R$ and (9) by $-(\cos kR)/R$, and add. We obtain

$$(10) \quad \begin{aligned} \frac{\sin kR}{R} u(P) &= \frac{1}{4\pi} \left[\frac{\cos kR}{R} \frac{\partial}{\partial R} \left(\frac{\sin kR}{R} \right) - \frac{\sin kR}{R} \frac{\partial}{\partial R} \left(\frac{\cos kR}{R} \right) \right] \iint_{\Sigma_R} dS \\ &\quad + \frac{\sin kR}{4\pi R} \iint_B \left[\frac{\cos k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\cos k\rho}{\rho} \right) \right] dS \\ &\quad - \frac{\cos kR}{4\pi R} \iint_B \left[\frac{\sin k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\sin k\rho}{\rho} \right) \right] dS. \end{aligned}$$

Rewrite (10) as

$$(11) \quad \begin{aligned} &\frac{\sin kR}{R} \left\{ u(P) - \frac{1}{4\pi} \iint_B \left[\frac{\cos k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\cos k\rho}{\rho} \right) \right] dS \right\} \\ &\quad + \frac{\cos kR}{R} \left\{ \frac{1}{4\pi} \iint_B \left[\frac{\sin k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\sin k\rho}{\rho} \right) \right] dS \right\} = \frac{k}{4\pi R^2} \iint_{\Sigma_R} u dS. \end{aligned}$$

We introduce the abbreviations α and β by rewriting (11) as

$$(12) \quad \alpha \frac{\sin kR}{R} + \beta \frac{\cos kR}{R} = \frac{k}{4\pi R^2} \iint_{\Sigma_R} u dS.$$

α and β are independent of R . Apply Schwarz' inequality to (12). We obtain

$$(13) \quad \left| \alpha \frac{\sin kR}{R} + \beta \frac{\cos kR}{R} \right| \leq \frac{k}{2\sqrt{\pi R}} \sqrt{\iint_{\Sigma_R} |u|^2 dS}.$$

If α and β are not both zero for all choices of P , then, for some P , the left member of (13) is exactly $O(1/R)$ as R tends to infinity. Then (13) yields

$$(14) \quad \lim_{R \rightarrow \infty} \iint_{\Sigma_R} |u|^2 dS > 0.$$

In the case that α and β are zero for all choices of P , we must show that $u \equiv 0$. Suppose then that $\alpha \equiv \beta \equiv 0$. Add $\pm i\beta$ to α . We obtain

$$(15) \quad u = \frac{1}{4\pi} \iint_B \left[\frac{e^{\pm i k \rho}}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{e^{\pm i k \rho}}{\rho} \right) \right] dS.$$

We rewrite (15) as

$$(16) \quad u = u_s + u_d.$$

Let r and r' be the radial polar coordinates of P and the point of integration Q in (15) respectively. Let O be the origin of these polar coordinates, and let $\cos \chi$ be the cosine of the angle between $\overline{PO}$ and $\overline{QO}$. We have then

$$(17) \quad u_s(P)e^{\pm i k r} = \frac{1}{4r} \iint_S \frac{\partial u}{\partial n} \exp \{ \pm i [\sqrt{r^2 - 2rr' \cos \chi + r'^2} - r] \} \frac{dS}{\sqrt{r^2 - 2rr' \cos \chi + r'^2}}.$$

If r is sufficiently large, the quantity under the radicals in (17) cannot vanish since B and, therefore, r' are bounded. Thus the kernel in (17) is an analytic function of r in a neighborhood of $r = \infty$. Multiplication by the continuous function $\partial u / \partial n$ and integration over B preserves this analyticity. (If $\partial u / \partial n$ is not continuous, replace B by a sphere containing it.) Thus the right member of (17) is an analytic function of r in a neighborhood of $r = \infty$. In a similar manner this same statement holds for $e^{\mp i k r} u_d$. Thus, from (16), we have

$$(18) \quad u e^{\mp i k r} = A^{\mp}(r)$$

where A^+ and A^- are analytic functions of r in a neighborhood of $r = \infty$.

If u is not identically zero, we may divide one relation in (18) by the other to get

$$e^{-2ikr} = \frac{A^-(r)}{A^+(r)}.$$

But this is absurd since the left member has an essential singularity at $r = \infty$, while the right member has at most a pole there. Thus $u \equiv 0$ and the theorem is proved.

To prove the uniqueness theorem we will need the following

Lemma 1. *Let u satisfy (1) and (4), and let u or $\partial u/\partial n$ vanish on B . Then*

$$\lim_{R \rightarrow \infty} \iint |u|^2 dS = 0.$$

Proof: Let $\bar{u}$ denote the complex conjugate of u . From Green's theorem we have

$$(19) \quad \iiint_{D_R} [\bar{u}(\Delta u + k^2 u) - u(\Delta \bar{u} + k^2 \bar{u})] dV = \iint_B \left[\bar{u} \frac{\partial u}{\partial n} - \frac{\partial \bar{u}}{\partial n} \right] dS + \iint_{\Sigma_R} \left[\bar{u} \frac{\partial u}{\partial R} - u \frac{\partial \bar{u}}{\partial R} \right] dS.$$

Since $\Delta u + k^2 u = 0$ and $\Delta \bar{u} + k^2 \bar{u} = 0$ in D_R , and $u = \bar{u} = 0$ or $\partial u/\partial n = \partial \bar{u}/\partial n = 0$ on B , (19) reduces to

$$(20) \quad \iint_{\Sigma_R} \left[\bar{u} \frac{\partial u}{\partial R} - u \frac{\partial \bar{u}}{\partial R} \right] dS = 0.$$

Expanding the integrand in the radiation condition (4), we get

$$(21) \quad 0 = \lim_{R \rightarrow \infty} \iint_{\Sigma_R} \left[\left| \frac{\partial u}{\partial R} \right|^2 + k^2 |u|^2 - ik \left(u \frac{\partial \bar{u}}{\partial R} - \bar{u} \frac{\partial u}{\partial R} \right) \right] dS.$$

Inserting (20) into (21), we obtain

$$0 = \lim_{R \rightarrow \infty} \iint_{\Sigma_R} \left[\left| \frac{\partial u}{\partial R} \right|^2 + k^2 |u|^2 \right] dS,$$

which contains the assertion of the lemma.

We can now state and prove

Theorem 2. (*Uniqueness Theorem.*) *There exists at most one solution of (1) which satisfies (4) and has prescribed values or prescribed values of its normal derivative on B .*

Proof (RELLICH): Let u_1 and u_2 be two solutions. Let $u = u_1 - u_2$. An application of the Schwarz inequality shows that u satisfies the radiation condition (4). Thus since $u = 0$ or $\partial u/\partial n = 0$ on B , Lemma 1 implies that

$$\lim_{R \rightarrow \infty} \iint_{\Sigma_R} |u|^2 dS = 0.$$

But then Theorem 1 implies that $u \equiv 0$.

We will now prove

Theorem 3. (*Representation Theorem.*) *Let u be a solution of (1) which satisfies (4). Then*

$$(6) \quad u(P) = \frac{1}{4\pi} \iint_B \left(\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{e^{ik|P-Q|}}{|P-Q|} \right) dS.$$

Proof: Let α and β be the same as in (12). Then the assertion of the representation theorem is expressed by the equation

$$\alpha - i\beta = 0.$$

Now multiply (8) by $(\partial/\partial R)[(\sin kR)/R]$ and (9) by $-(\partial/\partial R)[(\cos kR)/R]$ and add. After some simplification we get

$$(22) \quad \frac{\partial}{\partial R} \left(\frac{\sin kR}{R} \right) \alpha + \frac{\partial}{\partial R} \left(\frac{\cos kR}{R} \right) \beta = \frac{k}{4\pi R^2} \iint_{\Sigma_R} \frac{\partial u}{\partial R} dS.$$

Now multiply (22) by $-ik$ and add this result to (12). We obtain

$$\begin{aligned} \alpha \left[\frac{\partial}{\partial R} \left(\frac{\sin kR}{R} \right) - ik \frac{\sin kR}{R} \right] + \beta \left[\frac{\partial}{\partial R} \left(\frac{\cos kR}{R} \right) - ik \frac{\cos kR}{R} \right] \\ = \frac{k}{4\pi R^2} \iint_{\Sigma_R} \left(\frac{\partial u}{\partial R} - iku \right) dS. \end{aligned}$$

Applying Schwarz' inequality to this equation yields

$$\begin{aligned} (23) \quad \left| (\alpha - i\beta) \frac{k \cos kR}{R} + (i\alpha + \beta) \left(-\frac{k \sin kR}{R} \right) - \alpha \frac{\sin kR}{R^2} - \beta \frac{\cos kR}{R^2} \right| \\ \leq \frac{k}{2\sqrt{\pi}R} \sqrt{\iint_{\Sigma_R} \left| \frac{\partial u}{\partial R} - iku \right|^2 dS}. \end{aligned}$$

The radiation condition implies that the right member of (23) vanishes faster than $1/R$ as R tends to infinity. α and β are independent of R . The coefficient of $\alpha - i\beta$ in the left member of (23) is exactly $O(1/R)$. The remaining terms in the left member of (23) are $O(1/R^2)$. Thus since the right member tends to zero faster than $1/R$ as R tends to infinity, we must have that $\alpha - i\beta = 0$. This proves the theorem.

We conclude this section with

Corollary 1. Let u be a solution of $\Delta u + k^2 u = f$ which satisfies (4) and for which

$$\iiint_D f(Q) \frac{e^{ik|P-Q|}}{|P-Q|} dV$$

exists. Then

$$\begin{aligned} u(P) = \frac{1}{4\pi} \iint_B \left(\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{e^{ik|P-Q|}}{|P-Q|} \right) dS \\ - \frac{1}{4\pi} \iiint_D f(Q) \frac{e^{ik|P-Q|}}{|P-Q|} dV. \end{aligned}$$

The proof of Corollary 1 is similar to the proof of Theorem 4 and will therefore be omitted.

3. Unbounded Domains with Unbounded Boundaries. Let D be a domain whose boundary B is unbounded. Let Σ_R be the surface of a sphere of radius R , let $D_R = D \cap (\text{interior } \Sigma_R)$, let $B_R = B \cap (\text{interior } \Sigma_R)$, and let E_R be that part of Σ_R which is contained in D . Let $A(R)$ be the area of E_R . Let B be such that we can apply Green's theorem to D_R . The radiation condition (4) becomes

$$(24) \quad \lim_{R \rightarrow \infty} \iint_{E_R} \left| \frac{\partial u}{\partial R} - iku \right|^2 dS = 0.$$

Our first result of section 3 is

Theorem 4. (*Representation Theorem.*) Let u be a solution in D of (1) which satisfies (24). Let the following integrals exist

$$(25) \quad \begin{aligned} & \iint_B \left(\frac{\cos k |P - Q|}{|P - Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{\cos k |P - Q|}{|P - Q|} \right) dS \\ & \iint_B \left(\frac{\sin k |P - Q|}{|P - Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{\sin k |P - Q|}{|P - Q|} \right) dS. \end{aligned}$$

Then

$$(26) \quad u(P) = \frac{1}{4\pi} \iint_B \left(\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{e^{ik|P-Q|}}{|P-Q|} \right) dS.$$

Remark. The integrals (25) exist if the values of u on B and the values of the normal derivative of u on B satisfy the Sommerfeld boundedness condition (3).

Proof of Theorem 4: The proof proceeds exactly as the proof of Theorem 3, and the assertion of Theorem 4 is expressed by the equation $\alpha - i\beta = 0$. In this case we obtain in place of (23) the equation

$$(27) \quad \begin{aligned} & \left| (\alpha_R - i\beta_R) \frac{k \cos kR}{R} + (i\alpha_R + \beta_R) \left(-\frac{k \sin kR}{R} \right) \right. \\ & \quad \left. - \alpha_R \frac{\sin kR}{R^2} - \beta_R \frac{\cos kR}{R^2} \right| \\ & \leq \frac{k \sqrt{A(R)}}{4\pi R^2} \sqrt{\iint_{E_R} \left| \frac{\partial u}{\partial R} - iku \right|^2 dS}. \end{aligned}$$

Here

$$(28) \quad \alpha_R = u(P) - \frac{1}{4\pi} \iint_{B_R} \left(\frac{\cos k |P - Q|}{|P - Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{\cos k |P - Q|}{|P - Q|} \right) dS$$

and

$$(29) \quad \beta_R = \frac{1}{4\pi} \iint_{B_R} \left(\frac{\sin k |P - Q|}{|P - Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{\sin k |P - Q|}{|P - Q|} \right) dS.$$

Since $A(R) \leq O(R^2)$, the radiation condition (24) implies that the right member of (27) tends to zero faster than $1/R$. By hypothesis the following limits exist.

$$(30) \quad \lim_{R \rightarrow \infty} \alpha_R = \alpha, \quad \lim_{R \rightarrow \infty} \beta_R = \beta, \quad \lim_{R \rightarrow \infty} (\alpha_R - i\beta_R) = \alpha - i\beta.$$

Thus, by arguments similar to those used in the proof of Theorem 3, (27) implies that $\alpha - i\beta = 0$ and this proves the theorem.

Corollary 2. Let u be a solution in D of $\Delta u + k^2 u = f$ which satisfies (24). Let the integrals

$$\iiint_D f(Q) \frac{\sin k |P - Q|}{|P - Q|} dV$$

and

$$\iiint_D f(Q) \frac{\cos k |P - Q|}{|P - Q|} dV$$

exist. Then, under the hypothesis of Theorem 4,

$$u(P) = \frac{1}{4\pi} \iint_B \left(\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{e^{ik|P-Q|}}{|P-Q|} \right) dS - \frac{1}{4\pi} \iiint_D f(Q) \frac{e^{ik|P-Q|}}{|P-Q|} dV.$$

The proof of Corollary 2 is similar to the proof of Theorem 4 and will therefore be omitted.

The influence of the shape of B on the question of uniqueness is exhibited by

Theorem 5. Let u be a solution in D of (1) which satisfies (24). Let $u = 0$ on B and let the integrals which appear in (31) and (32) below exist. Then

$$(31) \quad u(P) = \frac{1}{4\pi} \iint_B \frac{\cos k |P - Q|}{|P - Q|} \frac{\partial u}{\partial n} dS$$

and

$$(32) \quad 0 = \frac{1}{4\pi} \iint \frac{\sin k |P - Q|}{|P - Q|} \frac{\partial u}{\partial n} dS.$$

If B were such that the integral equation (32) for $\partial u / \partial n$ possessed only the trivial solution, then Theorem 4 would show that the radiation condition implies uniqueness.

Proof of Theorem 5: The proof proceeds as the proof of Theorem 1. In this case we obtain in place of (13) the equation

$$(33) \quad \left| \alpha_R \frac{\sin kR}{R} + \beta_R \frac{\cos kR}{R} \right| \leq \frac{k \sqrt{A(R)}}{2 \sqrt{\pi R}} \sqrt{\iint_{B_R} |u|^2 dS}.$$

Here, because u vanishes on B , we have

$$(34) \quad \alpha_R = u(P) - \frac{1}{4\pi} \iint_{B_R} \frac{\cos k |P - Q|}{|P - Q|} \frac{\partial u}{\partial n} dS$$

and

$$(35) \quad \beta_R = \frac{1}{4\pi} \iint_{B_R} \frac{\sin k |P - Q|}{|P - Q|} \frac{\partial u}{\partial n} dS.$$

By hypothesis the following limits exist

$$\lim_{R \rightarrow \infty} \alpha_R = \alpha, \quad \lim_{R \rightarrow \infty} \beta_R = \beta.$$

Thus the left member of (33) is exactly $O(1/R)$ unless $\alpha = 0$ and $\beta = 0$. Now, exactly as in the proof of Lemma 1, the radiation condition and the fact that u vanishes on B imply that $\iint_{B_R} |u|^2 dS$ tends to zero as R tends to infinity. Thus the right member of (32) tends to zero faster than $1/R$. Combining these facts, we see by arguments already used that (33) implies that α and β are zero. From (34) which defines α_R , we see that $\alpha = 0$ is equivalent to (31). Similarly (35) and $\beta = 0$ imply (32). Thus the theorem is proved.

Our final result is a uniqueness theorem in the spirit of the assertion of PETERS & STOKER (see the introduction), namely that within a non-radiating part the radiation condition implies uniqueness. Thus let

$$(36) \quad u_1 = f + g_1, \quad u_2 = f + g_2$$

be two solutions of (1) with prescribed boundary values. f is a fixed non-radiating solution and g_1 and g_2 satisfy (24). Let

$$(37) \quad u = u_1 - u_2 = g_1 - g_2.$$

Since u_1 and u_2 are continuous solutions of (1), then (37) implies that u is also. Furthermore, Schwarz' inequality implies that u satisfies the radiation condition (24) since g_1 and g_2 do. The identical vanishing of u is the subject of

Theorem 6. *Let u be a solution in D of (1) which satisfies the radiation condition (24). Let $u = 0$ and $|R \partial u / \partial n|$ be bounded on B . Let D contain a right circular cone of half angle θ . If $\theta > \frac{1}{2}\pi$ then $u \equiv 0$.*

Proof: The proof proceeds as the proof of Theorem 5. From the assertions $\alpha = \beta = 0$ of that proof we obtain $\alpha \pm i\beta = 0$. From this in turn we obtain

$$(38) \quad u(P)e^{i k r} = \frac{1}{4\pi} \iint_B \frac{\partial u}{\partial n} \frac{\exp \{ \pm i [\sqrt{r^2 - 2rr' \cos \chi + r'^2} - r] \}}{\sqrt{r^2 - 2rr' \cos \chi + r'^2}} dS$$

where r and r' are respectively the radial polar coordinates of P and the point of integration Q measured from an origin O . $\cos \chi$ is the cosine of the angle between $\overline{OP}$ and $\overline{OQ}$.

The remainder of the proof consists of showing that (38) implies that $u \equiv 0$. In the proof of Theorem 1, $u(P)e^{*ikr}$ was observed to be analytic in a full neighborhood of $r = \infty$, and from this a simple contradiction completed the proof. In the present situation, since B is unbounded, we do not have analyticity in a full neighborhood of infinity. The hypothesis concerning the cone will enable us to prove that we have analyticity in a half plane. Finally, we will use NEVANLINNA'S form [10] of the Phragmén-Lindelöf Theorem to complete the proof.

Let the vertex of the cone be placed at the pole O from which r and r' are measured. Now let P be restricted to the axis of the cone. Since

$$(39) \quad \Delta \equiv r^2 - 2rr' \cos \chi + r'^2 = (r - r'e^{ix})(r - r'e^{-ix}),$$

Δ , considered as a function of the complex variable r , cannot vanish if $|\operatorname{amp} r| < \theta$. Thus the kernel in (38), viz.

$$(40) \quad K \equiv \frac{e^{*i(\sqrt{\Delta}-r)}}{\sqrt{\Delta}},$$

is analytic in the sector $|\operatorname{amp} r| < \theta$ in the complex r -plane. In particular, since $\theta > \frac{1}{2}\pi$, K is analytic in a domain which contains the right half plane. Now rewrite (38) as

$$(41) \quad u(P)e^{*ikr} = \frac{1}{4\pi} \iint \frac{\partial u}{\partial n} \frac{\exp \left\{ \pm ikr'r \left[\left(\frac{1}{r'} - \frac{1}{r} e^{ix} \right)^{\frac{1}{2}} \left(\frac{1}{r'} - \frac{1}{r} e^{-ix} \right)^{\frac{1}{2}} - \frac{1}{r'} \right] \right\}}{rr' \left(\frac{1}{r'} - \frac{1}{r} e^{ix} \right)^{\frac{1}{2}} \left(\frac{1}{r'} - \frac{1}{r} e^{-ix} \right)^{\frac{1}{2}}} dS.$$

From this we observe since $|(\partial u / \partial n)r'| < \text{constant}$ that A^* is analytic where K is, and furthermore that $|A^+| \leq O(1/|r|)$ in the upper half plane and $|A^-| \leq O(1/|r|)$ in the lower half plane. Thus

$$(42) \quad \begin{aligned} |u(r)| &\leq O\left(\frac{e^{-k\operatorname{Im}r}}{|r|}\right) & \operatorname{Im} r \geq 0 \\ &\leq O\left(\frac{e^{k\operatorname{Im}r}}{|r|}\right) & \operatorname{Im} r \leq 0. \end{aligned}$$

Now NEVANLINNA'S form of the Phragmén-Lindelöf Theorem states that if $f(z)$ is analytic and bounded in the right half plane and if

$$(43) \quad \liminf_{|z| \rightarrow \infty} \frac{1}{|z|} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log f(|z|e^{i\operatorname{amp} z}) \cos \varphi \, d\varphi = -\infty,$$

then $f(z) = 0$.

To apply this result, let ζ be a complex variable such that

$$(44) \quad \zeta = r \frac{\pi}{2\theta}.$$

This map takes the sector $|\operatorname{amp} r| < \theta$ into the right half ζ -plane. Since $\theta > \frac{1}{2}\pi$, there exists a $\delta > 0$ such that

$$(45) \quad r = \zeta^{1+\delta}.$$

If $f(\zeta) = u(\zeta^{1+\delta})$, then $f(\zeta)$ is analytic and bounded in the right half ζ -plane. Moreover, from (42), there exists a positive constant c such that

$$(46) \quad |f(\zeta)| \leq c \exp \{-k|\zeta|^{1+\delta} |\sin [(1+\delta) \operatorname{amp} \zeta]| \} / |\zeta|^{1+\delta}.$$

Thus, applying (43), we are led to

$$(47) \quad \liminf_{|\zeta| \rightarrow \infty} \frac{1}{|\zeta|} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \{-k|\zeta|^{1+\delta} |\sin [(1+\delta) \operatorname{amp} \zeta]| \cos \varphi\} d\varphi.$$

This quantity is clearly equal to $-\infty$ and Theorem 6 is proved.

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