*On Coefficient Problems of Solutions of Partial Differential Equations of the Fourth Order**

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1. The problem under consideration. We consider linear partial differential equations in two variables which are of the fourth order and have analytic coefficients. S. BERGMAN [1] proved that various properties of the solutions of such equations can be derived from the coefficients of the power series development of these solutions. The basic tool for such investigations are Bergman operators (definition below) which transform analytic functions of a complex variable into solutions of the above equations. In this way theorems on these solutions can be obtained from theorems on analytic functions. By means of BERGMAN's results J. MITCHELL [2] has considered some coefficient problems concerning solutions of those partial differential equations, especially the behavior of the solutions on the boundary of the domain of regularity. In consequence of the properties of Bergman operators the relations between analytic functions and solutions of the partial differential equations under consideration become relative simple.

As can be seen from [1] and [2] only certain subsequences of the above mentioned coefficients enter into the conditions of the theorems thus obtained, while the other coefficients do not occur in these considerations. This is a consequence of the properties of Bergman operators. It is obvious that information on the behavior of the solutions should be obtained also from other subsets of those coefficients. Hence we meet the following basic problem: to find relations between different subsequences of these coefficients. This problem will be considered in the present paper.

Let us now formulate this idea in a more precise manner and introduce suitable notation.

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2. Notation, Bergman operators. We consider partial differential equations of the form

$$(2.1) \quad \Delta\varphi + \alpha\varphi_{xx} + 2\beta\varphi_{xy} + \gamma\varphi_{yy} + \kappa\varphi_x + \lambda\varphi_y + \mu\varphi = 0$$

whose coefficients are real analytic functions of the real variables x and y . If we introduce the complex variables $z_1 = x + iY$, $z_2 = y + iX$ we may continue these coefficients to complex values. We assume that the functions $\alpha(z_1, z_2)$, $\beta(z_1, z_2)$, $\dots$, $\mu(z_1, z_2)$ are entire. If we introduce the new variables $z = z_1 + iz_2$, $z^* = z_1 - iz_2$ and use

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

we may write (2.1) in the form

$$(2.2) \quad u_{zz^*z^*} + au_{zz} + bu_{zz^*} + cu_{z^*z^*} + du_z + eu_{z^*} + hu = 0$$

where $\varphi(z_1, z_2) = u(z, z^*)$ and

$$\begin{aligned} \alpha &= 4(a + b + c), & \beta &= 4i(-a + c), & \gamma &= 4(-a + b - c), \\ \kappa &= 8(d + e), & \lambda &= 8i(-d + e), & \mu &= 16h \end{aligned}$$

as can easily be verified.

We note that, in general, (2.2) is equivalent to a system of two partial differential equations of the fourth order for the real and imaginary parts of the complex solution. For real values of z_1 and z_2 the two equations are completely independent of each other and of the same form if and only if

$$(2.3) \quad a = \bar{c}, \quad b = \bar{b}, \quad d = \bar{e}, \quad h = \bar{h}.$$

Detailed information about Bergman operators can be obtained from several papers of BERGMAN, cf. [3–6]. We restrict ourselves to some remarks which are necessary in order to understand what will follow.

We denote complex solutions by lower case letters and real solutions (*i.e.* solutions which are real for real values of the variables) by capital letters. In the case of partial differential equations of the second order and with analytic coefficients,

$$(2.4) \quad v_{zz^*} + Av_z + Bv_{z^*} + Cv = 0,$$

every complex solution can be represented in the form

$$(2.5) \quad v(z, z^*) = P(f, 0) \equiv \int_{-1}^1 E(z, z^*, t) f(\tfrac{1}{2}z(1 - t^2))(1 - t^2)^{-\frac{1}{2}} dt$$

and every real solution in the form

$$(2.6) \quad V(z, z^*) = \operatorname{Re} P(f, 0).$$

Cf. BERGMAN [3]. In (2.5) the *generating function* E of the Bergman operator P depends only on the coefficients of (2.4) but is independent of the choice of the

associated function $f(z)$ which is an analytic function regular at the origin. Obviously (2.5) is a generalization of the well known relation between harmonic and analytic functions.

In the case of partial differential equations (2.2) there exists a similar representation of the solutions, namely

$$(2.7) \quad \begin{aligned} u(z, z^*) = P(f, g; 0) = & \int_{-1}^1 E_1(z, z^*, t) f(\tfrac{1}{2}z(1-t^2))(1-t^2)^{-\frac{1}{2}} dt \\ & + \int_{-1}^1 E_2(z, z^*, t) g(\tfrac{1}{2}z^*(1-t^2))(1-t^2)^{-\frac{1}{2}} dt, \end{aligned}$$

and in the case of real solutions we have $U(z, z^*) = \operatorname{Re} P(f, g, 0)$. Obviously this is a generalization of the representation of real solutions of the equation $\Delta\Delta\varphi = 0$, namely $\varphi = \operatorname{Re} (f(z) + z^*g(z))$. Cf. BERGMAN [1].

In the case of the partial differential equation (2.4) Bergman operators (2.5) yield a one-to-one correspondence between the solutions v and the analytic functions $f(z)$. In the case of the partial differential equation (2.2) to every solution there corresponds a pair of analytic functions of one complex variable.

If the coefficients of the series developments

$$(2.8) \quad v(z, z^*) = \sum_{m, n=0}^{\infty} v_{mn} z^m z^{*n}$$

and

$$(2.9) \quad u(z, z^*) = \sum_{m, n=0}^{\infty} u_{mn} z^m z^{*n}$$

of solutions of (2.4) and (2.2), respectively, are known, various properties of these solutions can be obtained. In the case of partial differential equations (2.4) several results of this type occur in the papers of BERGMAN [4–6] and other authors [2, 7, 8]. Similar coefficient problems concerning solutions of partial differential equations (2.2) were considered by J. MITCHELL [2].

Because of the properties of the Bergman operator (2.5) the subsequence (v_{m0}) of the coefficients in (2.8) plays a special role; all properties of $v(z, z^*)$ can be derived if this subsequence is known. Clearly, it should be possible to obtain information about $v(z, z^*)$ also from other subsequences (v_{mn}) , $n > 0$ and fixed. This problem was considered by several authors [7–9] and relations were obtained between the subsequences (v_{m0}) and (v_{mn}) , $n > 0$ and fixed.

In the present paper we consider the corresponding problem in the case of solutions $u(z, z^*)$ of partial differential equations (2.2). We mention first an essential difference between the problem in the case of partial differential equations of the second order and those of the fourth order: While in the first case knowledge of the subsequence (v_{m0}) suffices in order to determine the properties of the solution $v(z, z^*)$, in the second case the two subsequences (u_{m0}) and (u_{m1}) must be known for the same purpose, cf. BERGMAN [1], MITCHELL [2]. Also other differences occur as can be seen by comparing this paper with [8].

We have to find relations between a sequence (u_{mn}) , $n > 1$ and fixed, and the sequences (u_{m0}) , (u_{m1}) , cf. (2.9). For this purpose we shall write (2.9) in the form

$$(2.10) \quad u(z, z^*) = \sum_{n=0}^{\infty} u_n(z) z^{*n}$$

and investigate the functions

$$(2.11) \quad u_n(z) = \sum_{m=0}^{\infty} u_{mn} z^m, \quad n = 0, 1, \dots,$$

occurring in (2.10), since each of the above sequences corresponds to just one of these functions.

3. Derivation of a system of ordinary differential equations involving the functions $u_n(z)$. Relations between the functions $u_0(z)$, $u_1(z)$, ... can be obtained from a system of linear ordinary differential equations of the second order which we shall now derive. We may write the power series developments

$$(3.1) \quad \begin{aligned} a(z, z^*) &= \sum_{m, n=0}^{\infty} a_{mn} z^m z^{*n}, & b(z, z^*) &= \sum_{m, n=0}^{\infty} b_{mn} z^m z^{*n}, \dots, \\ h(z, z^*) &= \sum_{m, n=0}^{\infty} h_{mn} z^m z^{*n} \end{aligned}$$

of the coefficients of the partial differential equation (2.2) as series developments in one variable, namely

$$(3.2) \quad \begin{aligned} a(z, z^*) &= \sum_{n=0}^{\infty} a_n(z) z^{*n}, & b(z, z^*) &= \sum_{n=0}^{\infty} b_n(z) z^{*n}, \dots, \\ h(z, z^*) &= \sum_{n=0}^{\infty} h_n(z) z^{*n}. \end{aligned}$$

In these representations,

$$(3.3) \quad a_n(z) = \sum_{m=0}^{\infty} a_{mn} z^m, \quad b_n(z) = \sum_{m=0}^{\infty} b_{mn} z^m, \dots,$$

$$h_n(z) = \sum_{m=0}^{\infty} h_{mn} z^m, \quad n = 0, 1, \dots$$

If we introduce (2.11) and (3.2) into (2.2) we obtain an expression which is a series development in powers of z^* . In order that the resulting equation be satisfied for every value of z^* , the coefficients of the powers of z^* must vanish. We thus obtain the following system of ordinary differential equations:

$$(3.4) \quad \begin{aligned} u_n'' + c_0 u_n &= G_n(u_0, u_0', u_0'', u_1, u_1', u_1'', \dots, u_{n-1}, u_{n-1}'), \\ n &= 2, 3, \dots, \end{aligned}$$

where

$$G_n = -\frac{1}{n(n-1)} \left\{ (n-1)[b_0 u'_{n-1} + ((n-2)c_1 + e_0)u_{n-1}] + \sum_{\nu=0}^{n-2} \{a_{n-\nu-2} u''_{\nu} + (\nu b_{n-\nu-1} + d_{n-\nu-2})u'_{\nu} + (\nu(\nu-1)c_{n-\nu} + \nu e_{n-\nu-1} + h_{n-\nu-2})u_{\nu}\} \right\}.$$

While in the case of the partial differential equation (2.4) one obtains a system of n ordinary differential equations of the first order relating $v_0(z), v_1(z), \dots, v_n(z)$ we now have a system of $n-1$ ordinary differential equations of the second order relating the functions $u_0(z), u_1(z), \dots, u_n(z)$. Cf. [8]. Another important difference between the two cases is the following. If we set $n=2$ we obtain from (3.4)

$$(3.5) \quad u_2'' + c_0 u_2 = -\frac{1}{2}(a_0 u_0'' + d_0 u_0' + h_0 u_0 + b_0 u_1' + c_0 u_1),$$

i.e. in general $u_2(z)$ depends on $u_0(z)$ as well as on $u_1(z)$ and the derivatives of these functions. In the case of the partial differential equation (2.4) the ordinary differential equation corresponding to (3.5) takes the form

$$v_1' + B_0 v_1 = -(A_0 v_0' + C_0 v_0)$$

where

$$A(z, z^*) = \sum_{m, n=0}^{\infty} A_{mn} z^m z^{*n} \equiv \sum_{n=0}^{\infty} A_n(z) z^{*n}, \quad v(z, z^*) = \sum_{m, n=0}^{\infty} v_{mn} z^m z^{*n} \equiv \sum_{n=0}^{\infty} v_n(z) z^{*n}$$

$$B(z, z^*) = \sum_{m, n=0}^{\infty} B_{mn} z^m z^{*n} \equiv \sum_{n=0}^{\infty} B_n(z) z^{*n},$$

i.e. $v_1(z)$ is uniquely determined by just one function, namely $v_0(z)$.

From (3.4) we shall now derive relations between $u_n(z)$, n fixed, on the one hand and $u_0(z), u_1(z)$ on the other, i.e. relations between the sequences (u_{mn}) , $n > 1$ and fixed, and $(u_{m0}), (u_{m1})$, cf. (2.9).

4. Behavior of $u_0(z), u_1(z)$ at singular points of $u_n(z)$. We have the following simple but important fact:

Theorem 1. *If $u_n(z)$, $n > 1$, has a singularity at a point $z = z_0 (\neq 0)$ then at least one of the two functions $u_0(z), u_1(z)$ is singular at this point.*

Proof. Since the coefficients of the partial differential equation (2.2) are entire, the functions (3.3) occurring in the equations (3.4) are also entire. Since the equations (3.4) are linear, every solution of the corresponding homogeneous equations is thus entire. If the functions $u_{\alpha}(z)$, $\alpha = 0, 1, \dots, n-1$, were all entire the solutions of the non-homogeneous equations (3.4) would also be entire; indeed, if one derives these solutions by the usual Lagrange method the Wronskian occurring in the denominator of some expressions is an entire function without zeros. But, since we assumed $u_n(z)$ to be singular at $z = z_0$, at least one of the functions $u_{\alpha}(z)$, $\alpha = 0, \dots, n-1$, must therefore be singular at

this point. By repeating this argument we see that at least one of the two functions $u_0(z)$ and $u_1(z)$ must be singular at $z = z_0$.

In consequence of Theorem 1 the number $\omega(n)$ of singularities of $u_n(z)$ in a fixed domain B (each singular point counted just once) cannot exceed the number of points in B at which at least one of the functions $u_0(z)$, $u_1(z)$ is singular. In particular, if $u_0(z)$ and $u_1(z)$ are regular in B the functions $u_2(z)$, $u_3(z)$, $\dots$ are also regular in B . If we do not impose special restrictions on the form of the coefficients of (2.2) we cannot determine whether a singularity of $u_n(z)$, $n > 1$, corresponds to a singularity of $u_0(z)$ or of $u_1(z)$. This can easily be seen from (3.4).

While at a singular point of $u_n(z)$ at least one of the functions $u_0(z)$, $u_1(z)$ must be singular, the function $u_\alpha(z)$, $n > \alpha > 1$, can be regular at such a point. For example in the case of the equation

$$u_{zzz^*z^*} + z^*u_{zz} + zz^*u = 0$$

every singular point of $u_3(z)$ is a regular point of $u_2(z)$ but a singular point of $u_0(z)$ or $u_1(z)$. Similar examples can easily be obtained. Hence, in general, $\omega(n)$ is not a monotone function of n . In the case of partial differential equations of the second order we have a similar situation, cf. [8].

If we impose restrictions on the form of the coefficients of the partial differential equation (2.2) the situation may be simpler. From (3.4) we obtain

Lemma 2. *If the series developments (3.2) of the coefficients $a(z, z^*)$, $d(z, z^*)$, and $h(z, z^*)$ of (2.2) do not involve constant terms and terms depending only on z and if z^{*n-2} is the smallest power of z^* occurring in these series, then at any point where $u_n(z)$ is singular $u_1(z)$ must also be singular.*

For the statement of Lemma 2 to be true for all functions $u_n(z)$, $n = 2, 3, \dots$, the partial differential equation (2.2) must be of the form

$$u_{zzz^*z^*} + b(z, z^*)u_{zz^*} + c(z, z^*)u_{z^*z^*} + e(z, z^*)u_{z^*} = 0,$$

i.e. it reduces to a partial differential equation of the third order for the function u_{z^*} . For all the functions $u_n(z)$, $n = 2, 3, \dots$, to be independent of $u_0(z)$ as well as of $u_1(z)$ equation (2.2) must reduce to the form

$$u_{zzz^*z^*} + c(z, z^*)u_{z^*z^*} = 0$$

which is an ordinary differential equation for the function $u_{z^*z^*}$. If for every solution $u(z, z^*)$ the function $u_n(z)$, n fixed, is independent of $u_1(z)$ and if the coefficients of (2.2) are such that $a_{n-2}(z) \equiv 0$, $d_{n-2}(z) \equiv 0$, $h_{n-2}(z) \equiv 0$ then $u_n(z)$ is independent of $u_0(z)$ also and, consequently, is an entire function. These and similar facts can be derived from (3.4).

5. Regular points of $u_n(z)$ at which $u_0(z)$ and $u_1(z)$ are regular. The preceding statements illustrate the basic fact that at singular points of $u_0(z)$ or $u_1(z)$ the function $u_n(z)$, $n > 1$, may be regular. In the above cases this happened in consequence of the fact that the function $u_n(z)$ under consideration

was completely independent of $u_0(z)$ and $u_1(z)$. There may arise another possibility: It may happen that $u_0(z)$ (or $u_1(z)$) has a singularity at a certain point $z = z_0$ ($\neq 0$) but nevertheless the function G_n on the right hand side of (3.4) is regular since that singularity is cancelled, in consequence of the special form of the coefficients of (3.2). Examples of this type can easily be obtained. From this fact and Theorem 1 we have the following situation: The domain of regularity B_n of $u_n(z)$ contains the intersection B_{01} of the domains of regularity B_0 of $u_0(z)$ and B_1 of $u_1(z)$, but B_{01} may be a proper subdomain of B_n .

Since the domain of regularity of a solution $u(z, z^*)$ of (2.2) can be determined from the domains of regularity of $u_0(z)$ and $u_1(z)$, *cf.* BERGMAN [1], MITCHELL [2], it is necessary to investigate the relations between B_n and B_0, B_1 in more detail.

In the case of $u_2(z)$ information can be obtained from (3.5). In the case of $u_n(z)$, $n > 2$, the situation becomes more complicated, as can be seen from (3.4). In general the system (3.4), which we denote by S_n^* , can be reduced to a single linear ordinary differential equation involving only the functions $u_n(z)$, n fixed, $u_0(z)$, $u_1(z)$ and derivatives of these functions. Since the procedure for eliminating the functions $u_2(z)$, $u_3(z)$, $\dots$, $u_{n-1}(z)$ from (3.4) is similar to that considered in [8] we may restrict ourselves to a brief description of the method and stress the modification of the procedure which is necessary in the present case of partial differential equations of the fourth order.

In the case of partial differential equations of the second order we were able to obtain an ordinary differential equation involving $v_n(z)$ and $v_0(z)$ only. It is obvious from the above considerations that in the present case the desired equation—if it exists (*cf.* the remark below)—in general must contain $u_0(z)$ as well as $u_1(z)$.

For our purpose we may consider S_n^* as a system of homogeneous equations in the “unknowns” 1, $u_\alpha(z)$, $\alpha = 2, 3, \dots, n-1$, and their first and second derivatives; the coefficient of the “unknown” 1 consists of the terms involving $u_0(z)$, $u_1(z)$, $u_n(z)$ (n fixed) and their derivatives. In this interpretation S_n^* represents a system of $n-1$ equations in $1 + 3(n-2) = 3n-5$ “unknowns”. If we differentiate these $n-1$ equations a suitable number of times we obtain an enlarged system S_n of linear differential equations of higher order whose number of equations is equal to the number of “unknowns”. Indeed, if we differentiate each equation of the system S_n^* k times we obtain $k(n-1)$ new equations which involve $k(n-2)$ new “unknowns”, namely $u_2^{(k)}, u_3^{(k)}, \dots, u_{n-1}^{(k)}$, $u_2^{(k+1)}, u_3^{(k+1)}, \dots, u_{n-1}^{(k+1)}$. Hence in order that the number of equations of the system S_n be equal to the number of “unknowns” we have to differentiate each of the equations of S_n^* $2n-4$ times. The system S_n thus obtained consists of $2n^2 - 5n + 3$ equations in the same number of “unknowns”. Since this system has a non-trivial solution its determinant D_n must vanish. (Of course the form of D_n depends on the value of the index n of the function $u_n(z)$ under consideration.) The equation

$$(5.1) \quad D_n = 0$$

is just the desired ordinary differential equation involving $u_0(z)$, $u_1(z)$, and $u_n(z)$. Obviously this equation is a linear ordinary differential equation whose order cannot exceed $2n - 2$.

However, two types of exceptional cases may occur: (a) It may happen that $u_n(z)$ is completely independent of $u_0(z)$ and $u_1(z)$, *cf.* the examples in the preceding section. (b) There may exist a relation between $u_n(z)$ on the one hand and $u_0(z)$, $u_1(z)$ on the other, but D_n vanishes identically. In the latter case the method of elimination can be applied to a system of fewer equations which is obtained from the system S_n ; the procedure is the same in principle.

We now consider the equation (5.1) for non-exceptional cases. We arrange each of the equations of S_n so that the terms corresponding to the "unknown" 1, *i.e.* the terms involving

$$u_0, u'_0, \dots, u_0^{(2n-2)}, u_1, u'_1, \dots, u_1^{(2n-2)}, u_n, \dots, u_n^{(2n-2)}$$

come first and then the terms corresponding to the other "unknowns"

$$u_2, u'_2, \dots, u_2^{(2n-2)}, u_3, u'_3, \dots, u_3^{(2n-2)}, \dots, u_{n-1}, u'_{n-1}, \dots, u_{n-1}^{(2n-2)}$$

follow in this order. Let $\langle p, 1 \rangle$ denote the equation of the original system S_n^* , *cf.* (3.4), corresponding to $n = p$, and $\langle p, q \rangle$ the equation obtained by differentiating that equation $q - 1$ times. Let $r_{pq}(\lambda)$ be the coefficient of $u_0^{(\lambda)}$ and let $s_{pq}(\lambda)$ be the coefficient of $u_1^{(\lambda)}$ in this equation. Since in the system S_n^* the function $u_n(z)$ occurs only in the equation $\langle n, 1 \rangle$, in the system S_n this function and its derivatives occur only in the equations $\langle n - 1, q \rangle$, $q = 1, 2, \dots, 2n - 3$. Let $t_q(\lambda)$ denote the coefficient of $u_n^{(\lambda)}$ in the equation $\langle n - 1, q \rangle$. Explicit expressions for all these coefficients can be easily obtained. If we develop D_n by its first column, the equation (5.1) takes the form

$$(5.2) \quad \sum_{\lambda=0}^{2n-2} (\rho_\lambda u_0^{(\lambda)} + \sigma_\lambda u_1^{(\lambda)} + \tau_\lambda u_n^{(\lambda)}) = 0$$

where

$$(5.3) \quad \rho_\lambda = \sum_{p=1}^{n-1} \sum_{q=1}^{2n-3} r_{pq}(\lambda) D_{n,pq}, \quad \sigma_\lambda = \sum_{p=1}^{n-1} \sum_{q=1}^{2n-3} s_{pq}(\lambda) D_{n,pq},$$

$$\tau_\lambda = \sum_{q=1}^{2n-3} t_q(\lambda) D_{n,n-1,q}.$$

In (5.3), $D_{n,pq}$ is the cofactor in D_n of the element in the first column corresponding to the equation $\langle p, q \rangle$.

We may now have two exceptional cases, (E_1) and (E_2) , two special cases, (S_1) and (S_2) , and the general case (G) as follows.

(E_1) If $\tau_\lambda \equiv 0$ for all values of $\lambda = 0, 1, \dots, 2n - 2$ then all coefficients ρ_λ and σ_λ must vanish identically; otherwise a relation would hold between $u_0(z)$ and $u_1(z)$ which would be the same for every solution $u(z, z^*)$ of (2.2) corresponding to any function $u_n(z)$ whatsoever. Hence $D_n \equiv 0$; this is an exceptional case which we excluded from our consideration.

(E₂) If all the coefficients ρ_λ and σ_λ vanish identically but at least one of the functions τ_λ does not vanish identically then $u_n(z)$ is independent of $u_0(z)$ and $u_1(z)$, and (5.2) reduces to a differential equation involving only $u_n(z)$. This is again one of the above exceptional cases.

We exclude these cases. We may assume that there exists at least one value of λ for which $\tau_\lambda \not\equiv 0$.

(S₁) If all the functions ρ_λ vanish identically but $\sigma_\lambda \not\equiv 0$ for at least one value of λ then $u_n(z)$ depends only on $u_1(z)$. (5.2) thus takes the form

$$(5.4) \quad u_1^{(m)} + \sum_{\lambda=0}^{m-1} \frac{\sigma_\lambda}{\sigma_m} u_1^{(\lambda)} + P = 0$$

where

$$P = \sum_{\lambda=0}^k \frac{\tau_\lambda}{\sigma_m} u_n^{(\lambda)};$$

in (5.4) m and k are the largest values of λ for which the respective functions σ_λ and τ_λ are not identically zero.

(S₂) If all the functions σ_λ vanish identically but at least one of the functions ρ_λ does not vanish identically, the function $u_n(z)$ depends only on $u_0(z)$. Then (5.2) takes the form

$$(5.5) \quad u_0^{(l)} + \sum_{\lambda=0}^{l-1} \frac{\rho_\lambda}{\rho_l} u_0^{(\lambda)} + Q = 0$$

where

$$Q = \sum_{\lambda=0}^j \frac{\tau_\lambda}{\rho_l} u_n^{(\lambda)},$$

and l and j are the largest values of λ for which the respective functions ρ_λ and τ_λ do not vanish identically.

(G) If at least one function of each of the three sets of functions ρ_λ , σ_λ , and τ_λ does not vanish identically, the function $u_n(z)$ depends on $u_0(z)$ as well as on $u_1(z)$. Then (5.2) takes the form

$$(5.6) \quad \sum_{\lambda=0}^l \rho_\lambda u_0^{(\lambda)} + \sum_{\lambda=0}^m \sigma_\lambda u_1^{(\lambda)} + \sum_{\lambda=0}^p \tau_\lambda u_n^{(\lambda)} = 0$$

where l , m , and p are the largest values of λ for which the respective functions ρ_λ , σ_λ , and τ_λ do not vanish identically. Instead of (5.6) we may write

$$(5.7) \quad u_0^{(l)} + \sum_{\lambda=0}^{l-1} \frac{\rho_\lambda}{\rho_l} u_0^{(\lambda)} + R = 0$$

where

$$R = \sum_{\lambda=0}^m \frac{\sigma_\lambda}{\rho_l} u_1^{(\lambda)} + \sum_{\lambda=0}^p \frac{\tau_\lambda}{\rho_l} u_n^{(\lambda)}$$

or

$$(5.8) \quad u_1^{(m)} + \sum_{\lambda=0}^{m-1} \frac{\sigma_\lambda}{\sigma_m} u_1^{(\lambda)} + S = 0$$

where

$$S = \sum_{\lambda=0}^l \frac{\rho_\lambda}{\sigma_m} u_0^{(\lambda)} + \sum_{\lambda=0}^p \frac{\tau_\lambda}{\sigma_m} u_n^{(\lambda)}.$$

We may now evaluate the different forms of the above differential equation (5.2) by means of the well known

Theorem 2. *Every solution of a linear non-homogeneous differential equation*

$$w^{(n)} + \sum_{\alpha=0}^{n-1} q_\alpha(z) w^{(\alpha)} = r(z),$$

where $q_\alpha(z)$ and $r(z)$ are analytic functions, is regular in a neighborhood of a point $z = z_0$ in which the functions $q_0(z), \dots, q_{n-1}(z)$ and $r(z)$ are regular.

The coefficients of the partial differential equation (2.2) are assumed to be entire. Since in (5.2) the coefficients ρ_λ , σ_λ , and τ_λ involve only the functions defined by (3.3) these coefficients are thus entire.

Let us consider first the cases (S₁) and (S₂). From the preceding theorem we obtain

Theorem 3. *I. If the conditions stated in (S₁) hold and if B is a domain such that $\sigma_m(z) \neq 0$ and $u_n(z)$ is regular in B , the function $u_1(z)$ is also regular in B . II. If the conditions stated in (S₂) hold and if B' is a domain such that $\rho_l(z) \neq 0$ and $u_n(z)$ is regular in B' , the function $u_0(z)$ is also regular in B' .*

The situation in the special cases (S₁) and (S₂) shows a formal similarity to that in the general case of a partial differential equation of the second order. Indeed, the differential equations (5.4) and (5.5) involve only $u_n(z)$ and $u_1(z)$ or $u_0(z)$ and derivatives of these functions; however, no information about $u_0(z)$ or $u_1(z)$, respectively, can be obtained in these cases.

Let us consider the general case (G). If we apply the above theorem to (5.7) and (5.8) we obtain the following

Theorem 4. *I. Under the conditions stated in (G) the function $u_0(z)$ is regular in a domain in which $\rho_l(z) \neq 0$ and $u_1(z)$ and $u_n(z)$ are regular. II. Under the conditions stated in (G) the function $u_1(z)$ is regular in a domain in which $\sigma_m(z) \neq 0$ and $u_0(z)$ and $u_n(z)$ are regular.*

Let now B be a domain in which the functions $\rho_l(z)$ and $\sigma_m(z)$ do not vanish at all. Then the functions ρ_λ/ρ_l , σ_λ/ρ_l , τ_λ/ρ_l , ρ_λ/σ_m , σ_λ/σ_m , and τ_λ/σ_m occurring in (5.7) and (5.8) are all regular in B . If $u_1(z)$ is regular in a neighborhood of a point $z = \zeta$ of B then, in consequence of (5.7), the function $u_0(z)$ is also regular in this neighborhood. If $u_1(z)$ is singular at a point $z = \xi$ then $u_0(z)$ must be

also singular at this point, as follows from (5.8). A similar result about $u_1(z)$ holds if $u_0(z)$ is assumed to be regular (or singular) at a certain point of B . We thus obtain the following

Corollary. *If the function $u_n(z)$ is regular in a domain B and the functions $\rho_1(z)$ and $\sigma_m(z)$, cf. (G), are different from zero everywhere in B , then $u_0(z)$ and $u_1(z)$ are either regular in B or singular at exactly the same points of B .*

6. Special cases of partial differential equations (2.2). If no restrictions on the form of the coefficients of (2.2) are imposed the conditions occurring in Theorem 4 may be relatively complicated, in particular if a function $u_n(z)$ corresponding to a large value of n is considered. However, for many types of equations (2.2), which are of practical importance, the relations between $u_n(z)$, $n > 1$ and fixed, on the one hand and $u_0(z)$, $u_1(z)$ on the other are simple. It is sometimes possible (and preferable) to base the whole consideration upon the system S_n^* in its original form (3.4) and avoid the procedure of elimination developed in the preceding section. It is worthwhile to consider some special forms of (2.2).

In the case of the partial differential equation

$$\Delta\Delta\varphi + \varphi = 0$$

i.e.

$$u_{zzz*zz*} + \frac{1}{16}u = 0$$

or in the more general case

$$(6.1) \quad u_{zzz*zz*} + h(z)u = 0$$

we have the following remarkable fact: *There is no connection between any two functions $u_\alpha(z)$ and $u_\beta(z)$ if α is odd and β is even. The functions $u_{2m}(z)$, $m = 1, 2, \dots$, depend only on $u_0(z)$ while the functions $u_{2m+1}(z)$, $m = 1, 2, \dots$, depend only on $u_1(z)$.* Indeed we have from (3.4) the following simple recurrence relations:

$$(6.2) \quad u_{2m}'' = -\frac{h(z)}{2m(2m-1)} u_{2m-2}, \quad m = 1, 2, \dots,$$

and

$$(6.3) \quad u_{2m+1}'' = -\frac{h(z)}{(2m+1)2m} u_{2m-1}, \quad m = 1, 2, \dots.$$

From these relations we immediately find: If $u_0(z)$ has a pole of r^{th} order at a point $z = z_0$ ($\neq 0$) and $h(z_0) \neq 0$ then $u_{2m}(z)$, $m > 1$, has at $z = z_0$ a pole of the order $r - 2m$ (> 0) or a logarithmic singularity (if $r \leq 2m$). If any two subsequent functions $u_n(z)$ and $u_{n+1}(z)$ are regular in a domain B which does not contain zeros of the coefficient $h(z)$ of (6.1) then both functions $u_0(z)$ and $u_1(z)$ must be regular in B .

The most general partial differential equation (2.2) for which the functions $u_{2m}(z)$ and $u_{2m+1}(z)$, $m = 0, 1, \dots$, form two independent sets is obtained if the coefficients $a(z, z^*)$, $\dots$, $h(z, z^*)$ of (2.2), considered as functions of z^* for constant values of z , satisfy the following condition: b and e are odd functions of z^* while the other coefficients are even functions of z^* . This is a consequence of (3.4).

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