On the Application of a Statistical Approximation to the Theory of Turbulent Diffusion

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SUMMARY

The phenomenon of turbulent diffusion is generally studied theoretically from the Lagrangian standpoint in which the calculations aim to evaluate the probability that two particles in an initially specified configuration should, at a later time t , be separated by a distance $\mathbf{x}$. In this paper the problem is discussed from the Eulerian standpoint in which the ultimate goal is the evaluation of the probability that one particle at a certain point in space at the zero of time should, at a later time t , have been carried to a distance $\mathbf{x}$ from its starting point. It is shown that, by using a reasonable statistical hypothesis relating the fourth statistical moments to those of lower order, a knowledge of the spectrum of the turbulent velocity field suffices, in principle, to determine this Eulerian probability uniquely. Various approximations are developed and discussed.

1. Introduction. One of the most fruitful approaches to the theory of turbulent diffusion has been that of representing the effect of the fluid motion in the transport equation by fictitious *coefficients of eddy diffusion* whose exact form is supposed to depend on the character of the turbulent velocities. The implication of this idea is that the molecules in a turbulent fluid undergo a Brownian motion which is associated not only with the random microscopic motions but also with the random macroscopic turbulent motions. In the absence of fluid motions, molecular diffusion produces in a time t a displacement x_m in the x -direction which has the statistical properties

$$(1) \quad \bar{x}_m = 0, \quad \overline{x_m^2} = 2D_m t,$$

where D_m is the coefficient of molecular diffusion. It is supposed that the effect of random turbulent motions in a time t produces a displacement x_f with the properties

$$(2) \quad \bar{x}_f = U_x t, \quad \overline{x_f^2} = 2D_{fx} t,$$

where U_x is the mean velocity and D_{fx} the *coefficient of eddy diffusion* in the x -direction. The sum of these two effects produces therefore a displacement x satisfying

$$(3) \quad \bar{x} = U_x t, \quad \overline{x^2} = 2D_x t,$$

where

$$(4) \quad D_x = D_m + D_{fx}.$$

Over intervals of time so long that the Brownian motion can be considered as one of random flights, any property χ carried by the molecules must then satisfy the equation

$$(5) \quad \frac{\partial \chi}{\partial t} + \mathbf{U} \cdot \text{grad } \chi = D_x \frac{\partial^2 \chi}{\partial x^2} + D_y \frac{\partial^2 \chi}{\partial y^2} + D_z \frac{\partial^2 \chi}{\partial z^2},$$

to which the solution appropriate to an instantaneous point source at the origin at the zero of time is

$$(6) \quad \chi = \frac{c}{\sqrt{[D_x D_y D_z t^3]}} \exp \left\{ -\frac{(x - U_x t)^2}{4D_x t} - \frac{(y - U_y t)^2}{4D_y t} - \frac{(z - U_z t)^2}{4D_z t} \right\}.$$

Equation (5) should be compared to the correct equation that it replaces, namely

$$(7) \quad \frac{\partial \chi}{\partial t} + (\mathbf{U} + \mathbf{u}) \cdot \text{grad } \chi = D_m \nabla^2 \chi,$$

where $\mathbf{u}$ represents the fluctuations in the velocity flow from their mean $\mathbf{U}$. Considered in conjunction with the Stokes-Navier equation, there seems little justification for demanding the operational equality

$$(8) \quad \mathbf{u} \cdot \text{grad} = - \left[D_{fx} \frac{\partial^2}{\partial x^2} + D_{fy} \frac{\partial^2}{\partial y^2} + D_{fz} \frac{\partial^2}{\partial z^2} \right],$$

in order to secure a Gaussian type of solution such as (6). In fact, it cannot be accurate to replace the intrinsically complicated turbulent motion by such a simple stochastic process as random flights. It is with this idea in mind that the author has attempted here to solve the problem directly by using the well-known statistical hypothesis which, since its introduction by HEISENBERG [7], has proved of great value in the study of turbulence itself (see BATCHELOR [1, 2]; CHADRASEKHAR [4, 5, 6]; PROUDMAN & REID [8]; TATSUMI [9, 10]). It is hoped in this way that some of the more heuristic elements in the theory may be removed, albeit at the cost of much mathematical simplicity.

The model considered is the following. An incompressible fluid is in a state of stationary homogeneous isotropic turbulence when, at time $t = 0$, an instantaneous point source of the property χ is introduced at the origin. The subsequent statistical behaviour of χ is sought. (The quantity χ might represent temperature difference in the case of the turbulent convection of heat; it might be the density of inertialess particles in the case of a turbulent aerosol.) It is to be noted that

whereas the fluid is in a statistical distribution that is both isotropic and homogeneous, the transport of χ is neither isotropic nor homogeneous and, in fact, fluctuates with time. It is here that algebraic complications arise. These are so considerable that the author contents himself with establishing that a knowledge of the turbulence spectrum in principle determines the distribution of χ uniquely.

For very small times it is found that the transport of χ is essentially wave-like. A consequence of this is that, before χ spreads far from its initial concentration, it moves as nearly solid body with a fluctuating velocity that gives it the statistical property

$$\bar{x} = 0, \quad \overline{x^2} = \bar{u}_x^2 t^2,$$

where $\bar{u}_x$ is the root mean square turbulent velocity in the x -direction. The position at larger times is discussed more tentatively.

2. Basic tensor equations. We shall write the basic equations of transport, fluid motion and continuity in the forms

$$(9) \quad \frac{\partial \chi}{\partial t} + \frac{\partial}{\partial x_i} (\chi u_i) = D_m \nabla^2 \chi,$$

$$(10) \quad \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i} (u_i u_i) = \nu \nabla^2 u_i - \frac{\partial \bar{\omega}}{\partial x_i},$$

$$(11) \quad \frac{\partial u_i}{\partial x_i} = 0,$$

where

$$(12) \quad \bar{\omega} = \frac{p}{\rho},$$

and p is the fluid pressure, ρ its density and ν its kinematic viscosity. We shall have occasion to introduce several tensors and these, for the sake of convenience, are assembled together with their representation in wave-number space in the table below. Bars in the center column denote statistical averages over ensembles. The time dependence is suppressed in the notation and this procedure is adopted throughout the paper except where confusion might arise.

Averaging equation (9) gives immediately

$$(13) \quad \frac{\partial \chi(\mathbf{x})}{\partial t} = - \frac{\partial}{\partial x_i} \bar{Y}_i(\mathbf{x}, 0) + D_m \frac{\partial^2}{\partial x_i^2} \chi(\mathbf{x}),$$

which, in wave-number space, becomes

$$(14) \quad \left[\frac{\partial}{\partial t} + D_m \xi_i^2 \right] K(\xi) = \int \xi_i Y_i(\xi, \kappa) d\kappa,$$

where we have assumed that, at great distances, χ vanishes more rapidly than R^{-2} .

TABLE

Tensor	Definition	Representation
$\chi(\mathbf{x})$	$\overline{\chi(\mathbf{x})}$	$\int K(\xi) \exp \{i\xi \cdot \mathbf{x}\} d\xi$
$\bar{Y}_i(\mathbf{x}, \mathbf{r})$	$\overline{\chi(\mathbf{x})u_i(\mathbf{x} + \mathbf{r})}$	$i \iint Y_i(\xi, \kappa) \exp \{i(\xi \cdot \mathbf{x} + \kappa \cdot \mathbf{r})\} d\xi d\kappa$
$\bar{P}(\mathbf{x}, \mathbf{r})$	$\overline{\chi(\mathbf{x})\bar{\omega}(\mathbf{x} + \mathbf{r})}$	$\iint P(\xi, \kappa) \exp \{i(\xi \cdot \mathbf{x} + \kappa \cdot \mathbf{r})\} d\xi d\kappa$
$\bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}')$	$\overline{\chi(\mathbf{x})u_i(\mathbf{x} + \mathbf{r})u_i(\mathbf{x} + \mathbf{r}')}$	$\iiint Z_{ii}(\xi, \kappa, \kappa') \exp \{i(\xi \cdot \mathbf{x} + \kappa \cdot \mathbf{r} + \kappa' \cdot \mathbf{r}')\} d\xi d\kappa d\kappa'$
$\bar{R}_i(\mathbf{x}, \mathbf{r}, \mathbf{r}')$	$\overline{\chi(\mathbf{x})\bar{\omega}(\mathbf{x} + \mathbf{r})u_i(\mathbf{x} + \mathbf{r}')}$	$i \iiint R_i(\xi, \kappa, \kappa') \exp \{i(\xi \cdot \mathbf{x} + \kappa \cdot \mathbf{r} + \kappa' \cdot \mathbf{r}')\} d\xi d\kappa d\kappa'$
$\bar{F}_{ijk}(\mathbf{x}, \mathbf{r}, \mathbf{r}')$	$\overline{\chi(\mathbf{x})u_i(\mathbf{x} + \mathbf{r})u_j(\mathbf{x} + \mathbf{r}')u_k(\mathbf{x})}$	$i \iiint F_{ijk}(\xi, \kappa, \kappa') \exp \{i(\xi \cdot \mathbf{x} + \kappa \cdot \mathbf{r} + \kappa' \cdot \mathbf{r}')\} d\xi d\kappa d\kappa'$
$\bar{G}_{ijk}(\mathbf{x}, \mathbf{r}, \mathbf{r}')$	$\overline{\chi(\mathbf{x})u_i(\mathbf{x} + \mathbf{r})u_j(\mathbf{x} + \mathbf{r}')u_k(\mathbf{x} + \mathbf{r}')}$	$i \iiint G_{ijk}(\xi, \kappa, \kappa') \exp \{i(\xi \cdot \mathbf{x} + \kappa \cdot \mathbf{r} + \kappa' \cdot \mathbf{r}')\} d\xi d\kappa d\kappa'$
$Q_{ii}(\mathbf{r}, \mathbf{r}')$	$\overline{u_i(\mathbf{x})u_i(\mathbf{x} + \mathbf{r})}$	$\int \Phi_{ii}(\kappa) \exp \{i\kappa \cdot \mathbf{r}\} d\kappa$
$T_{ijk}(\mathbf{r}, \mathbf{r}')$	$\overline{u_i(\mathbf{x})u_j(\mathbf{x} + \mathbf{r})u_k(\mathbf{x} + \mathbf{r}')}$	$i \iint \Psi_{ijk}(\kappa, \kappa') \exp \{i(\kappa \cdot \mathbf{r} + \kappa' \cdot \mathbf{r}')\} d\kappa d\kappa'$

Multiplying equation (9), applied at the point $\mathbf{x}$, by $u_i(\mathbf{x} + \mathbf{r})$ and taking averages, we obtain

$$(15) \quad \overline{\frac{\partial \chi(\mathbf{x})}{\partial t} u_i(\mathbf{x} + \mathbf{r})} = -\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial r_i}\right) \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, 0) + D_m \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial r_i}\right)^2 \bar{Y}_i(\mathbf{x}, \mathbf{r}).$$

Multiplying equation (10), applied at the point $\mathbf{x} + \mathbf{r}$, by $\chi(\mathbf{x})$ and averaging, we derive

$$(16) \quad \chi(\mathbf{x}) \overline{\frac{\partial u_i(\mathbf{x} + \mathbf{r})}{\partial t}} = -\frac{\partial}{\partial r_i} \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}) - \frac{\partial \bar{P}(\mathbf{x}, \mathbf{r})}{\partial r_i} + \nu \frac{\partial^2}{\partial r_i^2} \bar{Y}_i(\mathbf{x}, \mathbf{r}).$$

Addition of the results (15) and (16) yields

$$(17) \quad \begin{aligned} \frac{\partial \bar{Y}_i(\mathbf{x}, \mathbf{r})}{\partial t} = & -\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial r_i}\right) \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, 0) - \frac{\partial}{\partial r_i} \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}) - \frac{\partial \bar{P}(\mathbf{x}, \mathbf{r})}{\partial r_i} \\ & + \left[D_m \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial r_i}\right)^2 + \nu \frac{\partial^2}{\partial r_i^2} \right] \bar{Y}_i(\mathbf{x}, \mathbf{r}). \end{aligned}$$

Taking the divergence of equation (17) with respect to r_i , remembering that, in virtue of equation (11),

$$(18) \quad \frac{\partial \bar{Y}_i(\mathbf{x}, \mathbf{r})}{\partial r_i} = 0,$$

we obtain

$$(19) \quad \nabla^2 \bar{P}(\mathbf{x}, \mathbf{r}) = -\frac{\partial^2}{\partial r_i \partial r_i} \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}),$$

which, in wave-number space, is equivalent to

$$(20) \quad P(\xi, \kappa) = -\frac{\kappa_\alpha \kappa_i}{\kappa^2} \int Z_{\alpha i}(\xi, \kappa - \kappa', \kappa') d\kappa'.$$

Now convert equation (17) to wave-number space and make use of expression (20) to eliminate $P(\xi, \kappa)$. We obtain the result

$$(21) \quad \begin{aligned} \left[\frac{\partial}{\partial t} + \nu \kappa^2 + D_m(\kappa^2 + \xi^2 - 2\kappa\xi\theta) \right] Y_i(\xi, \kappa) \\ = -(\xi_i - \kappa_i) \int Z_{ij}(\xi, \kappa, \kappa') d\kappa' - \kappa_i \Delta_i^\alpha(\kappa) \int Z_{\alpha i}(\xi, \kappa - \kappa', \kappa') d\kappa', \end{aligned}$$

where

$$(22) \quad \kappa\xi\theta = \kappa_i \xi_i$$

and $\Delta_i^\alpha(\kappa)$ is the symbol introduced by PROUDMAN & REID [8] and is defined by

$$(23) \quad \Delta_i^\alpha(\kappa) = \delta_{i\alpha} - \frac{\kappa_i \kappa_\alpha}{\kappa^2}.$$

Multiply equation (9), applied at the point $\mathbf{x}$, by $u_i(\mathbf{x} + \mathbf{r})u_i(\mathbf{x} + \mathbf{r}')$ and average over ensembles. We find

$$(24) \quad \overline{\frac{\partial \chi(\mathbf{x})}{\partial t} u_i(\mathbf{x} + \mathbf{r})u_i(\mathbf{x} + \mathbf{r}')} = -\left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial r_k} - \frac{\partial}{\partial r'_k}\right) \bar{F}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}') \\ + D_m \left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial r_k} - \frac{\partial}{\partial r'_k}\right)^2 \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}').$$

On multiplying equation (10), applied at the point $\mathbf{x} + \mathbf{r}$, by $\chi(\mathbf{x})u_i(\mathbf{x} + \mathbf{r}')$ and averaging, we obtain

$$(25) \quad \overline{\chi(\mathbf{x}) \frac{\partial u_i(\mathbf{x} + \mathbf{r})}{\partial t} u_i(\mathbf{x} + \mathbf{r}')} = -\frac{\partial}{\partial r_i} \bar{R}_i(\mathbf{x}, \mathbf{r}, \mathbf{r}') - \frac{\partial}{\partial r'_k} \bar{G}_{iik}(\mathbf{x}, \mathbf{r}, \mathbf{r}') \\ + \nu \frac{\partial^2}{\partial r_k^2} \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}').$$

On making the interchange $i \rightleftharpoons j$, $\mathbf{r} \rightleftharpoons \mathbf{r}'$ we obtain

$$(26) \quad \overline{\chi(\mathbf{x})u_i(\mathbf{x} + \mathbf{r}) \frac{\partial u_j(\mathbf{x} + \mathbf{r}')}{\partial t}} = -\frac{\partial}{\partial r'_i} \bar{R}_i(\mathbf{x}, \mathbf{r}', \mathbf{r}) - \frac{\partial}{\partial r'_k} \bar{G}_{iik}(\mathbf{x}, \mathbf{r}, \mathbf{r}') \\ + \nu \frac{\partial^2}{\partial r_k'^2} \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}').$$

Adding (24), (25), (26) we find

$$(27) \quad \left[\frac{\partial}{\partial t} - \nu \left(\frac{\partial^2}{\partial r_k^2} + \frac{\partial^2}{\partial r_k'^2} \right) - D_m \left(\frac{\partial}{\partial r_k} + \frac{\partial}{\partial r'_k} \right)^2 \right] \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}') \\ = -\frac{\partial}{\partial r_i} \bar{R}_i(\mathbf{x}, \mathbf{r}, \mathbf{r}') - \frac{\partial}{\partial r'_i} \bar{R}_i(\mathbf{x}, \mathbf{r}', \mathbf{r}) - \left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial r_k} - \frac{\partial}{\partial r'_k} \right) \bar{F}_{iik}(\mathbf{x}, \mathbf{r}, \mathbf{r}') \\ - \frac{\partial}{\partial r_k} \bar{G}_{iik}(\mathbf{x}, \mathbf{r}, \mathbf{r}') - \frac{\partial}{\partial r'_k} \bar{G}_{iik}(\mathbf{x}, \mathbf{r}', \mathbf{r}).$$

Taking the divergence of equation (27) with respect to r_i , remembering that, in virtue of equation (11),

$$(28) \quad \frac{\partial \bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}')}{\partial r_i} = 0,$$

$$(29) \quad \frac{\partial \bar{F}_{iik}(\mathbf{x}, \mathbf{r}, \mathbf{r}')}{\partial r_i} = \frac{\partial \bar{G}_{iik}(\mathbf{x}, \mathbf{r}', \mathbf{r})}{\partial r_i} = 0,$$

we obtain

$$(30) \quad \nabla_i^2 \bar{R}_i(\mathbf{x}, \mathbf{r}, \mathbf{r}') = -\frac{\partial^2}{\partial r_i \partial r_k} \bar{G}_{iik}(\mathbf{x}, \mathbf{r}, \mathbf{r}'),$$

which, in wave-number space, is equivalent to

$$(31) \quad R_i(\xi, \kappa, \kappa') = -\frac{\kappa_\alpha \kappa_k}{\kappa^2} G_{\alpha ik}(\xi, \kappa, \kappa').$$

Transforming equation (27) to wave-number space and using expression (31) to eliminate $R_i(\xi, \kappa, \kappa')$, we obtain

$$(32) \quad \left[\frac{\partial}{\partial t} + \nu(\kappa^2 + \kappa'^2) + D_m(\xi^2 + \kappa^2 + \kappa'^2 + 2\kappa\kappa'\mu) \right. \\ \left. - 2\kappa\xi\theta - 2\kappa'\xi\theta' \right] Z_{ij}(\xi, \kappa, \kappa') \\ = (\xi_k - \kappa_k - \kappa'_k) F_{ijk}(\xi, \kappa, \kappa') + \kappa_k \Delta_i^\alpha(\kappa) G_{\alpha ik}(\xi, \kappa, \kappa') \\ + \kappa'_k \Delta_i^\beta(\kappa') G_{i\beta k}(\xi, \kappa', \kappa),$$

where

$$(33) \quad \kappa\kappa'\mu = \kappa_i \kappa'_i, \quad \kappa'\xi\theta' = \kappa'_i \xi_i.$$

We have now derived three basic equations: equation (14), which expresses $K(\xi)$ in terms of $Y_i(\xi, \kappa)$; equation (21), which relates $Y_i(\xi, \kappa)$ to $Z_{ij}(\xi, \kappa, \kappa')$; equation (32), which expresses $Z_{ij}(\xi, \kappa, \kappa')$ in terms of $F_{ijk}(\xi, \kappa, \kappa')$ and $G_{ijk}(\xi, \kappa, \kappa')$. The process could be continued indefinitely, the determination of each tensor involving others previously unconsidered. To "close" the equations it is necessary to make some approximation. We will use a statistical hypothesis which relates $F_{ijk}(\xi, \kappa, \kappa')$ and $G_{ijk}(\xi, \kappa, \kappa')$ to $K(\xi)$ and $Y_i(\xi, \kappa)$. The hypothesis is the following: If $\omega_1, \omega_2, \omega_3, \omega_4$ are four quantities subject to statistical fluctuations in a field of homogeneous turbulence, then

$$(34) \quad \overline{\omega_1 \omega_2 \omega_3 \omega_4} = \overline{\omega_1 \omega_2} \overline{\omega_3 \omega_4} + \overline{\omega_1 \omega_3} \overline{\omega_4 \omega_2} + \overline{\omega_1 \omega_4} \overline{\omega_2 \omega_3} \\ + \overline{\omega_1 \cdot \omega_2 \omega_3 \omega_4} + \overline{\omega_2 \cdot \omega_3 \omega_4 \omega_1} + \overline{\omega_3 \cdot \omega_4 \omega_1 \omega_2} + \overline{\omega_4 \cdot \omega_1 \omega_2 \omega_3} \\ - 2\overline{\omega_1 \cdot \omega_2 \cdot \omega_3 \omega_4} - 2\overline{\omega_1 \cdot \omega_3 \cdot \omega_4 \omega_2} - 2\overline{\omega_1 \cdot \omega_4 \cdot \omega_2 \omega_3} \\ - 2\overline{\omega_1 \omega_2 \cdot \omega_3 \cdot \omega_4} - 2\overline{\omega_1 \omega_3 \cdot \omega_4 \cdot \omega_2} - 2\overline{\omega_1 \omega_4 \cdot \omega_2 \cdot \omega_3} \\ + 6\overline{\omega_1 \cdot \omega_2 \cdot \omega_4 \cdot \omega_4}.$$

It may easily be verified that if any or several of the ω quantities are independent of the others, equation (34) reduces to an identity. Equation (34) may also be written

$$(35) \quad \overline{\varphi_1 \varphi_2 \varphi_3 \varphi_4} = \overline{\varphi_1 \varphi_2} \overline{\varphi_3 \varphi_4} + \overline{\varphi_1 \varphi_3} \overline{\varphi_4 \varphi_2} + \overline{\varphi_1 \varphi_4} \overline{\varphi_2 \varphi_3},$$

where

$$(36) \quad \varphi_i = \omega_i - \overline{\omega_i}.$$

Remembering that $\bar{\varphi}_i$ is zero, we see that, in the form (35), the hypothesis is identical to hypotheses employed in the theory of turbulence by the workers already cited. For a more detailed appraisal of the hypothesis, the reader is

referred to a paper by TATSUMI [10] who does not, however, adopt a hypothesis of exactly the same form as (34) although, in the present problem, both are equivalent. We shall confine our discussion of the hypothesis to the remark that, since the approximation (35) has been shown experimentally to be remarkably good for hydrodynamic turbulence, it is not unreasonable to assume it is equally good for any quantity (such as χ or its moments) whose statistical behaviour is intimately related to a field of hydrodynamic turbulence. Despite its complexity we will show later that it is the simplest that can be legitimately applied. The hypothesis leads to the following results:

$$(37) \quad \bar{F}_{ijk}(\mathbf{x}, \mathbf{r}, \mathbf{r}') = \chi(\mathbf{x})T_{kij}(\mathbf{r}, \mathbf{r}') + \bar{Y}_k(\mathbf{x}, \mathbf{0})Q_{ij}(\mathbf{r} - \mathbf{r}') \\ + \bar{Y}_i(\mathbf{x}, \mathbf{r})Q_{jk}(\mathbf{r}') + \bar{Y}_j(\mathbf{x}, \mathbf{r}')Q_{ki}(\mathbf{r}),$$

$$(38) \quad \bar{G}_{ijk}(\mathbf{x}, \mathbf{r}, \mathbf{r}') = \chi(\mathbf{x})T_{ijk}(\mathbf{r}' - \mathbf{r}, \mathbf{0}) + \bar{Y}_i(\mathbf{x}, \mathbf{r})Q_{jk}(\mathbf{r} - \mathbf{r}') \\ + \bar{Y}_k(\mathbf{x}, \mathbf{r})Q_{ij}(\mathbf{r} - \mathbf{r}') + \bar{Y}_j(\mathbf{x}, \mathbf{r}')Q_{ki}(\mathbf{0}),$$

which can be written in wave-number space as

$$(39) \quad F_{ijk}(\xi, \kappa, \kappa') = K(\xi)\Psi_{kij}(\kappa, \kappa') + Y_i(\xi, \kappa)\Phi_{jk}(\kappa') \\ + Y_j(\xi, \kappa')\Phi_{ki}(\kappa) + \delta(\kappa'')\Phi_{ij}(\kappa) \int Y_k(\xi, \kappa_1) d\kappa_1,$$

$$(40) \quad G_{ijk}(\xi, \kappa, \kappa') = Y_i(\xi, \kappa')\Phi_{jk}(\kappa') + Y_k(\xi, \kappa'')\Phi_{ij}(\kappa') \\ + \delta(\kappa'')K(\xi) \int \Psi_{ijk}(\kappa, \kappa_1) d\kappa_1 + \delta(\kappa)Y_j(\xi, \kappa') \int \Phi_{ik}(\kappa_1) d\kappa_1,$$

where

$$(41) \quad \kappa'' = \kappa + \kappa'.$$

Consider the δ -function terms that will appear in equation (32) on substitution from expressions (39) and (40). The term in $F_{ijk}(\xi, \kappa, \kappa')$ is

$$\delta(\kappa'')\Phi_{ij}(\kappa) \int Y_k(\xi, \kappa_1) d\kappa_1$$

and clearly gives rise, in equation (32), to the contribution

$$\delta(\kappa'')\Phi_{ij}(\kappa) \left[\frac{\partial}{\partial t} + D_m \xi^2 \right] K(\xi)$$

where we have used equation (14). The effect of the δ -term in $G_{ijk}(\xi, \kappa, \kappa')$ may be most simply assessed by noticing that if χ is constant everywhere, equation (32) becomes the vector form of the von Karman-Howarth equation and the δ -function term in (40) accordingly gives rise to

$$\delta(\kappa'')K(\xi) \left[\frac{\partial}{\partial t} + 2\nu\kappa^2 \right] \Phi_{ij}(\kappa).$$

The sum of the two δ -function terms therefore makes a contribution of

$$\delta(\kappa'') \left[\frac{\partial}{\partial t} + (D_m \xi^2 + 2\nu \kappa^2) \right] K(\xi) \Phi_{ij}(\kappa).$$

The remaining δ -function terms in expression (40) disappear on substitution for $G_{ijk}(\xi, \kappa, \kappa')$. Equation (32) may now be written

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \nu(\kappa^2 + \kappa'^2) + D_m(\xi^2 + \kappa^2 + \kappa'^2 + 2\kappa\kappa'\mu) \right. \\ & \quad \left. - 2\kappa\xi\theta - 2\kappa'\xi\theta' \right] Z_{ij}(\xi, \kappa, \kappa') \\ (42) \quad & = \delta(\kappa'') \left[\frac{\partial}{\partial t} + (D_m \xi^2 + 2\nu \kappa^2) \right] K(\xi) \Phi_{ij}(\kappa) \\ & \quad + (\xi_k - \kappa_k - \kappa'_k) [K(\xi) \Psi_{kij}(\kappa, \kappa') + Y_i(\xi, \kappa) \Phi_{jk}(\kappa') + Y_j(\xi, \kappa') \Phi_{ik}(\kappa)] \\ & \quad + \Delta_i^\alpha(\kappa) \kappa_k [Y_\alpha(\xi, \kappa'') \Phi_{jk}(\kappa') + Y_k(\xi, \kappa'') \Phi_{\alpha j}(\kappa')] \\ & \quad + \Delta_j^\beta(\kappa') \kappa'_k [Y_\beta(\xi, \kappa'') \Phi_{ik}(\kappa) + Y_k(\xi, \kappa'') \Phi_{\beta i}(\kappa)] \end{aligned}$$

Equations (14), (21) and (42) form a closed set for the three entities $K(\xi)$, $Y_i(\xi, \kappa)$ and $Z_{ij}(\xi, \kappa, \kappa')$ in terms of the two tensors $\Phi_{ij}(\kappa)$, $\Psi_{ijk}(\kappa, \kappa')$. By the use of the same statistical hypothesis (35) as we have used, other workers [8, 9, 10] have shown that $\Psi_{ijk}(\kappa, \kappa')$ can be determined from $\Phi_{ij}(\kappa)$ which, in turn, is uniquely defined by a knowledge of the turbulence spectrum. The boundary conditions developed in section 4 show that a unique solution can then in principle be found for $K(\xi)$ after which Fourier's integral theorem can be used to evaluate $\chi(\mathbf{x})$. Thus, we have shown that the turbulence spectrum uniquely determines the turbulent diffusion of χ .

3. Reduction to scalar equations. Conditions (18), (28) and various symmetry properties permit the representation of the basic tensors of the last section in terms of defining scalars in the following manner:

$$(43) \quad Y_i(\xi, \kappa) \equiv \Delta_i^\alpha(\kappa) Y(\xi, \kappa) \xi_\alpha,$$

$$(44) \quad \begin{aligned} Z_{ij}(\xi, \kappa, \kappa') \equiv & \Delta_i^\alpha(\kappa) \Delta_j^\beta(\kappa') [Z_1 \delta_{\alpha\beta} + Z_2 \xi_\alpha \xi_\beta \\ & + Z_3 \xi_\alpha \kappa_\beta + Z_4 \kappa'_\alpha \xi_\beta + Z_5 \kappa'_\alpha \kappa_\beta], \end{aligned}$$

$$(45) \quad \Phi_{ij}(\kappa) \equiv \Delta_i^j(\kappa) \phi(\kappa),$$

$$(46) \quad \begin{aligned} \Psi_{ijk}(\kappa, \kappa') \equiv & \Delta_i^\alpha(\kappa) \Delta_j^\beta(\kappa') \Delta_k^\gamma(\kappa'') [\Phi \kappa_\gamma \delta_{\alpha\beta} + \Phi_1 \kappa'_\alpha \delta_{\gamma\beta} \\ & + \Phi_2 \kappa_\beta \delta_{\alpha\gamma} + \Psi \kappa'_\alpha \kappa_\beta \kappa_\gamma], \end{aligned}$$

where

$$(47) \quad \begin{cases} Z_\mu(\xi, \kappa, \kappa') \equiv Z_\mu(\xi, \kappa', \kappa), & \mu = 1, 2, 5, \\ Z_3(\xi, \kappa, \kappa') \equiv Z_4(\xi, \kappa', \kappa), \end{cases}$$

and

$$(48) \quad \begin{cases} \Phi(\kappa, \kappa', -\kappa'') \equiv -\Phi(\kappa', \kappa, -\kappa''), \\ \Psi_1(\kappa, \kappa', -\kappa'') \equiv \Phi(\kappa', -\kappa'', \kappa) \\ \Phi_2(\kappa, \kappa', -\kappa'') \equiv -\Phi(-\kappa'', \kappa, \kappa'), \\ \Psi(\kappa, \kappa', -\kappa'') \equiv \Psi(\kappa', -\kappa'', \kappa) = -\Psi(\kappa', \kappa, -\kappa''). \end{cases}$$

Results (46) and (48) are taken from the paper by PROUDMAN & REID that has already been cited [8]. In deriving the forms (43) to (46) frequent use has been made of the idea of gauge invariance introduced into the theory of turbulence by CHANDRASEKHAR [3].

In terms of these defining scalars, equations (14) and (42) may be written

$$(49) \quad \left[\frac{\partial}{\partial t} + D(\xi, \mathbf{0}, \mathbf{0}) \right] K(\xi) = 2\pi\xi^2 \int_0^\infty \int_{-1}^1 Y(\xi, \kappa, \theta) \kappa^2 (1 - \theta^2) d\kappa d\theta,$$

$$(50) \quad \left[\frac{\partial}{\partial t} + D(\xi, \kappa, \kappa') \right] Z_1 = (\xi_i \kappa_\alpha \Delta_i^\alpha(\kappa'')) \Phi(\kappa, \kappa') + (\xi_i \kappa_\alpha \Delta_i^\alpha(\kappa'')) Y(\xi, \kappa') \phi(\kappa') \\ + (\xi_i \kappa'_\alpha \Delta_i^\alpha(\kappa'')) Y(\xi, \kappa'') \phi(\kappa) + \left[\frac{\partial}{\partial t} + (D_m \xi^2 + 2\nu\kappa^2) \right] \delta(\kappa'') \kappa(\xi) \phi(\kappa),$$

$$(51) \quad \left[\frac{\partial}{\partial t} + D(\xi, \kappa, \kappa') \right] Z_2 = Y(\xi, \kappa) \phi(\kappa') + Y(\xi, \kappa') \phi(\kappa),$$

$$(52) \quad \left[\frac{\partial}{\partial t} + D(\xi, \kappa, \kappa') \right] Z_3 = \Phi_2(\kappa, \kappa') K(\xi) - Y(\xi, \kappa) \phi(\kappa') + Y(\xi, \kappa'') \phi(\kappa'),$$

$$(53) \quad \left[\frac{\partial}{\partial t} + D(\xi, \kappa, \kappa') \right] Z_4 = \Phi_1(\kappa, \kappa') K(\xi) - Y(\xi, \kappa') \phi(\kappa) + Y(\xi, \kappa'') \phi(\kappa),$$

$$(54) \quad \left[\frac{\partial}{\partial t} + D(\xi, \kappa, \kappa') \right] Z_5 = \left[(\xi_i \kappa_\alpha \Delta_i^\alpha(\kappa'')) \Psi - \frac{(\xi \cdot \kappa'')}{\kappa'^{1/2}} (\Phi_1 + \Phi_2) \right] K(\xi) \\ - \frac{(\xi \cdot \kappa'')}{\kappa'^{1/2}} Y(\xi, \kappa'') [\phi(\kappa) + \phi(\kappa')],$$

where

$$(55) \quad D(\xi, \kappa, \kappa') = \nu(\kappa^2 + \kappa'^2) + D_m(\xi^2 + \kappa^2 + \kappa'^2 + 2\kappa\kappa'\mu - 2\kappa\xi\theta - 2\kappa'\xi\theta').$$

It is interesting to note that, because of the form of the right-hand side of equations (50)–(54) and because of the symmetry properties (47) and (48), there are several simple relations between the scalars Z . For example, considering only equations (51)–(54) we find

$$(56) \quad \left[\frac{\partial}{\partial t} + D(\xi, \kappa, \kappa') \right] \left[\frac{(\xi \cdot \kappa'')}{\kappa'^{1/2}} (Z_2 + Z_3 + Z_4) + Z_5 \right] \\ = (\xi_i \kappa_\alpha \Delta_i^\alpha(\kappa'')) K(\xi) \Psi(\kappa, \kappa').$$

PROUDMAN & REID [8] have found a similar property for Ψ itself:

$$\left[\frac{\partial}{\partial t} + \nu(\kappa^2 + \kappa'^2 + \kappa''^2) \right] \Psi = 0,$$

showing that it is unaffected by non-linear transfer and that it therefore decays by viscous action alone. Thus in a state of stationary turbulence, for example, we could safely assume that Ψ had completely vanished, equation (56) then giving

$$(57) \quad \frac{(\xi \cdot \kappa'')}{\kappa'^{1/2}} (Z_2 + Z_3 + Z_4) + Z_5 = 0,$$

since the initial values of these quantities are zero (see section 4).

On multiplying equation (21) scalarly by ξ_i , we find the last basic equation is

$$(58) \quad \left[\frac{\partial}{\partial t} + D(\xi, \kappa, 0) \right] \xi^2 (1 - \theta^2) Y(\xi, \kappa, \theta) \\ = -\xi_i (\xi_i - \kappa_i) \int Z_{ij}(\xi, \kappa, \kappa') d\kappa' - \xi_i \kappa_j \Delta_i^\alpha(\kappa) \int Z_{\alpha j}(\xi, \kappa - \kappa', \kappa') d\kappa'.$$

To reduce the algebraic complication, we will now introduce two simplifying assumptions. The first of these is that the viscosity and molecular diffusivity are both zero

$$(59) \quad \nu = D_m = D(\xi, \kappa, \kappa') = 0.$$

This is not a serious restriction as we are chiefly interested in the turbulent transport. On making this assumption, equations (42) and (58) imply

$$(60) \quad \xi^2 (1 - \theta^2) \frac{\partial^2 Y(\xi, \kappa, \theta)}{\partial t^2} + K(\xi) \xi_i \xi_j \xi_k \int \Psi_{ijk}(\kappa, \kappa') d\kappa' \\ + \int [\xi_i (\xi_i - \kappa_i) (\xi_k - \kappa_k) Y_i(\xi, \kappa) \Phi_{jk}(\kappa') \\ + \xi_i \kappa_j \kappa_k \Delta_i^\alpha(\kappa) Y_\beta(\xi, \kappa) \{ \Delta_\alpha^\beta(\kappa') \Phi_{jk}(\kappa') + \Delta_j^\beta(\kappa') \Phi_{\alpha\beta}(\kappa') \} \\ + \xi_i \kappa_j \kappa'_k \Delta_i^\alpha(\kappa) Y_k(\xi, \kappa) \{ \Delta_\alpha^\beta(\kappa') \Phi_{j\beta}(\kappa') + \Delta_j^\beta(\kappa') \Phi_{\alpha\beta}(\kappa') \}] d\kappa' \\ + \int [\{ \xi_i (\xi_i - \kappa_i) (\xi_k - \kappa'_k) + \xi_i \xi_j \xi_k \} Y_j(\xi, \kappa') \Phi_{ik}(\kappa) \\ + \xi_i (\xi_i - \kappa_i) \kappa'_k \Delta_j^\beta(\kappa') \{ Y_\beta(\xi, \kappa') \Phi_{ik}(\kappa) + Y_k(\xi, \kappa') \Phi_{i\beta}(\kappa) \}] d\kappa' \\ + 2 \int \xi_i (\xi_i - \kappa_i) \kappa'_k \Delta_j^\beta(\kappa) \{ Y_\beta(\xi, \kappa') \Phi_{ik}(\kappa') + Y_k(\xi, \kappa') \Phi_{i\beta}(\kappa') \} d\kappa' \\ = 0.$$

The second assumption is that the hydrodynamic turbulence has reached such an advanced state of decay that there is little transfer of energy between eddies of different sizes, so that

$$(61) \quad \Psi_{ijk}(\kappa, \kappa') \equiv 0.$$

This assumption is not essential to the work that follows but produces enormous algebraic simplifications.

With these approximations and after some heavy algebra, equations (49)—(54) and (59) may be combined to give one equation for $Y(\xi, \kappa, \theta)$ namely

$$(62) \quad \left[\frac{\partial^2}{\partial t^2} + A(\xi, \kappa, \theta) \right] Y(\xi, \kappa, \theta) \\ = E(\kappa) \int_0^\infty \int_{-1}^1 B(\xi, \kappa, \kappa', \theta, \theta') Y(\xi, \kappa', \theta') \kappa'^2 d\kappa' d\theta' \\ + \int_0^\infty \int_{-1}^1 \int_{\theta_2'}^{\theta_1'} \frac{C(\xi, \kappa, \kappa', \mu, \theta, \theta') E(\kappa') Y(\xi, \kappa', \theta') \kappa'^2 d\kappa' d\theta' d\mu}{\kappa'^{1/2} \sqrt{[1 - \mu^2 - \theta^2 - \theta'^2 + 2\mu\theta\theta']}},$$

where

$$(63) \quad \begin{cases} \theta_1' = \theta\mu + \sqrt{[(1 - \theta^2)(1 - \mu^2)]} \\ \theta_2' = \theta\mu - \sqrt{[(1 - \theta^2)(1 - \mu^2)]} \end{cases}$$

and

$$(64) \quad E(\kappa) = 4\pi\kappa^2\phi(\kappa),$$

$$(65) \quad A(\xi, \kappa, \theta) = \frac{2}{3} (\xi^2 + \kappa^2 - 2\xi\kappa\theta) \int_0^\infty E(\kappa) d\kappa + \frac{5}{12} \kappa^2 \int_0^\infty E(\kappa) d\kappa$$

$$- \frac{1}{4} \int_0^\infty \left[\frac{7}{3} - \frac{(\kappa^2 + \kappa'^2)^2}{2\kappa^2\kappa'^2} + \frac{(\kappa^2 - \kappa'^2)^3}{4\kappa^3\kappa'^3} \log \frac{|\kappa - \kappa'|}{(\kappa + \kappa')} \right] \kappa'^2 E(\kappa') d\kappa',$$

$$B(\xi, \kappa, \kappa', \theta, \theta') = -\frac{\xi}{\kappa^2} (\xi - \kappa\theta)(1 - \theta'^2)$$

$$+ \frac{1}{8\kappa^3\kappa'} \cdot \frac{[(\kappa^2 - 2\xi\kappa\theta) - (\kappa'^2 - 2\xi\kappa'\theta')]}{(1 - \theta^2)}$$

$$(66) \quad \cdot \left\{ \frac{[\theta'(\kappa^2 + \kappa'^2) - 2\kappa\kappa'\theta][\theta(\kappa^2 + \kappa'^2) - 2\kappa\kappa'\theta]}{\sqrt{[\kappa^2 + \kappa'^2 - 2\kappa\kappa'\theta\theta']^2 - [2\kappa\kappa']^2(1 - \theta^2)(1 - \theta'^2)}} \right. \\ \left. + 2\kappa\kappa'(\theta^2 + \theta'^2) - \theta\theta'(\kappa^2 + \kappa'^2 + 2\kappa\kappa'\theta\theta') \right\},$$

$$\begin{aligned}
(67) \quad C(\xi, \kappa, \kappa', \mu, \theta, \theta') = & -\frac{\kappa\xi}{2\pi}(\theta - \mu\theta') \\
& - \frac{\kappa\kappa'}{2\pi\kappa'^{1/2}} \frac{(\theta - \mu\theta')(\theta' - \mu\theta)}{(1 - \theta^2)} [\kappa(\kappa + \mu\kappa') - \xi(\kappa\theta + \kappa'\theta')] \\
& + \frac{\kappa\kappa'}{2\pi\kappa'^{1/2}} [(\kappa' + \mu\kappa)(\kappa - \xi\theta) \\
& - (\kappa + \mu\kappa')(\kappa\mu - \xi\theta')] \frac{[1 - \theta^2 - \theta'^2 + \theta\theta'\mu]}{(1 - \theta^2)}.
\end{aligned}$$

4. The character of the solutions. Choose the origin of $\mathbf{x}$ to be the point at which an instantaneous point source of χ is placed. We will suppose that χ then has the form

$$(68) \quad \chi(\mathbf{x}) = \Lambda \delta(\mathbf{x}), \quad t = 0,$$

where Λ is a constant. Clearly, at the zero of time, $\bar{Y}_i(\mathbf{x}, \mathbf{r})$ is zero everywhere except possibly at the point $\mathbf{x} = \mathbf{0}$ but, since $\bar{u}_i(\mathbf{r})$ is zero, $\bar{Y}_i(\mathbf{0}, \mathbf{r})$ is also zero. We therefore have the second condition

$$(69) \quad \bar{Y}_i(\mathbf{x}, \mathbf{r}) = 0, \quad t = 0.$$

In a similar manner, $\bar{Z}_{ij}(\mathbf{x}, \mathbf{r}, \mathbf{r}')$ is zero except possibly at the origin of $\mathbf{x}$ and, by considering this point separately in conjunction with equation (68), we derive

$$(70) \quad \bar{Z}_{ij}(\mathbf{x}, \mathbf{r}, \mathbf{r}') = \Lambda \delta(\mathbf{x}) Q_{ij}(\mathbf{r} - \mathbf{r}'), \quad t = 0.$$

In wave-number space, conditions (68)–(70) become

$$(71) \quad K(\xi) = \frac{\Lambda}{8\pi^3}, \quad t = 0,$$

$$(72) \quad Y(\xi, \kappa, \theta) = 0, \quad t = 0,$$

$$(73) \quad Z_{ij}(\xi, \kappa, \kappa') = \frac{\Lambda}{8\pi^3} \delta(\kappa') \phi_{ij}(\kappa), \quad t = 0.$$

Using equation (21), the last of these conditions may be replaced by

$$(74) \quad \frac{\partial Y(\xi, \kappa, \theta)}{\partial t} = -\frac{\Lambda}{32\pi^4 \kappa^2} E(\kappa), \quad t = 0.$$

Conditions (71), (72), (74) suffice to determine the solution uniquely. It is possible to develop series expansions of $K(\xi)$ and $Y(\xi, \kappa, \theta)$:

$$(75) \quad K(\xi, t) = \sum_{n=1}^{\infty} t^{2n} K_n(\xi),$$

$$(76) \quad Y(\xi, \kappa, \theta, t) = \sum_{n=0}^{\infty} t^{2n+1} Y_n(\xi, \kappa, \theta),$$

where $K_n(\xi)$ is a polynomial in ξ^2 whose highest power is ξ^{2n} and whose lowest (for positive n) is ξ^2 . The solution therefore implies that $K(0, t)$ is constant as indeed it must be if χ is to be conserved. Note also that all odd moments vanish since $K(\xi)$ is even in ξ . The behaviour of the solutions for small t is governed by the highest powers of ξ and may best be examined by considering only that part $K^*(\xi)$ of $K(\xi)$ which involves powers of (ξt) . Equations (21), (42), (49), (60), (61) then combine to give

$$(77) \quad \frac{\partial^2 K^*}{\partial t^2} + \xi^2 \bar{u}^2 K^* = 0,$$

where $\bar{u}^2$ is the mean square velocity of the turbulent flow and satisfies

$$(78) \quad \bar{u}^2 = 2 \int_0^\infty E(\kappa) d\kappa.$$

It follows from condition (71) that

$$(79) \quad K^*(\xi, t) = \frac{\Lambda}{8\pi^3} \cos \xi \bar{u} t$$

corresponding, in physical space, to an outgoing wave whose velocity is $\bar{u}$. The form for χ implied by equation (79) is

$$(80) \quad \chi = \frac{\Lambda}{4\pi\gamma^2} \delta(r - \bar{u}t).$$

The appearance of the δ -function in this result is not really surprising. All eddies, no matter what their size, help in moving the initial concentration of χ from its starting point but it is only the smallest eddies that can disrupt and disperse χ in the early stages. Thus, for very small times, the main centre of χ moves as a solid body with the root mean square velocity.

It seems worth pointing out at this point that a statistical approximation could not have been applied to any of the moments of lower order than the fourth. For example, had the analogous hypothesis to (34) been applied to the third order moment $\bar{Z}_{ii}(\mathbf{x}, \mathbf{r}, \mathbf{r}')$, we would have found from equation (58) that

$$(81) \quad \frac{\partial Y(\xi, \kappa, \theta)}{\partial t} = -K(\xi)\phi(\kappa)$$

leading, *for all times*, to an equation similar to (77) whose solution would be almost identical to (80). It is clear that, for large times, such a solution is untenable from the physical standpoint.

Equation (62) does not appear to possess the mathematical character of a diffusion equation, *i.e.* it does not seem possible to approximate in any valid manner to obtain the equation

$$(82) \quad \left[\frac{\partial}{\partial t} + \lambda \xi^2 \right] K(\xi) = 0, \quad \lambda = \text{constant}.$$

However, the following argument suggests that the solution of equation (62) may nevertheless possess properties similar to those of (82). Imagine a spherical surface of radius R centred at the point of release. The flux of χ out of this surface will depend primarily on the density of χ in its neighbourhood and on the mixing rate due to the turbulence of eddies whose radii are larger than R . The first of these effects is time dependent and the second, in stationary turbulence, is not. In mathematical language, this statement, if true, may be written

$$(83) \quad \bar{Y}_i(\mathbf{x}, \mathbf{0}, t) = \chi(\mathbf{x}, t) D_i(\mathbf{x}),$$

where $D_i(\mathbf{x})$ depends on the character of the turbulent motion. This, in turn, implies that we may write

$$(84) \quad Y(\xi, \kappa, \theta, t) = K(\xi, t) D(\xi, \kappa, \theta).$$

The argument must break down in regions where the gradients of χ are large and the approximation can therefore be expected to become valid only after a great time has elapsed after the release of χ . The form of equation implied by the approximation (84) is

$$(85) \quad \frac{\partial K(\xi, t)}{\partial t} = -f(\xi) K(\xi, t),$$

where

$$(86) \quad f(\xi) = -2\pi\xi^2 \int_0^\infty \int_{-1}^1 D(\xi, \kappa, \theta) \kappa^2 (1 - \theta^2) d\kappa d\theta.$$

The form of equation (82) is similar to that of equation (86) and both possess an important property which, in the case of equation (85), takes the form

$$(87) \quad \chi(\mathbf{x}, t + \tau) = \iiint \chi(\mathbf{x}', t) G(\mathbf{x} - \mathbf{x}', \tau) d\mathbf{x}',$$

where

$$(88) \quad G(\mathbf{x}, t) = G(|\mathbf{x}|, t) = \frac{1}{2\pi^2 |\mathbf{x}|} \int_0^\infty e^{-t f(\xi)} \sin(\xi |\mathbf{x}|) \xi d\xi.$$

Presented in this manner, the solution exhibits χ at the time $t + \tau$ as the diffusion from individual parts of the distribution at the time t rather in the same manner that HUYGENS pictured a wave front to be constructed as a superposition of waves generated from an earlier position of the front. The Green's function $G(|\mathbf{x}|, t)$ itself is the solution to the instantaneous point source problem. It should be pointed out again, from another point of view, that the result (87) cannot be expected to be accurate except for large times since it implies that the spread of χ from each point of a previous distribution is a process quite independent of the spread from other points. This can hardly be true since a transport in one direction at one point must necessarily imply transport in almost parallel directions at neighbouring points and times. Further, from the

equation of continuity there must be transport in the reverse direction in other regions of the distribution at the same time.

It seems dangerous to pursue this line further. However, it may be desirable to present a plausible line of development. If we use the similarity theory of KOLMOGOROFF, we might expect $f(\xi)$ to be of the form

$$(89) \quad f(\xi) = c\epsilon^{\frac{1}{3}}\xi^{\frac{2}{3}},$$

where ϵ is the rate of dissipation of turbulent energy per unit mass of fluid and c is a positive dimensionless constant. However, even admitting the strength of KOLMOGOROFF's arguments in their application to homogeneous turbulence, it is debatable whether the same principles have as wide a validity in the field of turbulent diffusion. It might, for example, be anticipated that $\bar{u}$ should enter into the description of the distribution of χ . Equation (89) would then be replaced by

$$(90) \quad f(\xi) = c\epsilon^{\frac{1}{3}}\xi^{\frac{2}{3}}g\left(\frac{\bar{u}\xi^{\frac{1}{3}}}{\epsilon^{\frac{1}{3}}}\right).$$

As we have already seen, for small times

$$g(z) \sim z.$$

For large times, the form suggested by the basic equations does not seem to be that of equation (89). This may be seen in the following way. If we confine our attention to a sphere $|\mathbf{x} - \mathbf{x}_0| = R$ well within the main part of the distribution of χ , the anisotropy of that distribution is small. The character of $\bar{Y}_i(\mathbf{x}, \mathbf{r})$ and $\varphi_{1,i}(\mathbf{r})$ must then become very similar (where 1 denotes any particular component) in that region; this is confirmed when we notice that

$$(91) \quad D = \frac{1}{\xi\bar{u}}\phi(\kappa)H(\xi)$$

satisfies equation (62) in virtue of the equation satisfied by $\phi(\kappa)$ itself (see TATSUMI [10], equation 5.1). From KOLMOGOROFF's similarity principles, we expect that

$$(92) \quad E(\kappa) = d\epsilon^{\frac{2}{3}}\kappa^{-5/3}, \quad d \text{ constant},$$

which indicates, in conjunction with equation (91), that

$$(93) \quad f(\xi) = \frac{c\epsilon^{\frac{1}{3}}\xi^{\frac{2}{3}}}{\bar{u}}$$

or

$$g(z) \sim \frac{1}{z}$$

for large times.

Generally, if we take

$$(94) \quad g(z) = z^n,$$

where n exceeds -2 (since $f(0)$ must be zero), we find

$$(95) \quad G(|\mathbf{x}|, t) = \frac{4\pi}{|\mathbf{x}|^3} \int_0^\infty \exp \left\{ -\left(\frac{\xi}{\beta}\right)^{\frac{1}{2}(n+2)} \right\} \xi \sin \xi \, d\xi,$$

where

$$(96) \quad \beta = |\mathbf{x}| (ct)^{-3/(n+2)} \epsilon^{(n-1)/(n+2)} \bar{u}^{-3n/(n+2)}.$$

From equation (95) we derive the asymptotic results

$$(97) \quad G(|\mathbf{x}|, t) \sim \frac{1}{t^{9/(n+2)}} \quad \left(\frac{|\mathbf{x}|}{t} \rightarrow 0 \right),$$

$$(98) \quad G(|\mathbf{x}|, t) \sim \begin{cases} \frac{t}{|\mathbf{x}|^{\frac{1}{2}(n+2)}}, & n < 4 \\ \frac{1}{t^{\frac{1}{2}}} \exp \{ -|\mathbf{x}|^2/4\mu t \}, & n = 4 \end{cases} \quad \left(\frac{|\mathbf{x}|}{t} \rightarrow \infty \right)$$

where μ is a constant. Four interesting cases are: (a) $n = 0$, eq. 89; (b) $n = -1$, eq. 93; (c) $n = \frac{5}{2}$, the Holtzmark distribution; (d) $n = 4$, classical diffusion. The first three of these cases give a concentration diminishing more slowly at large distances than that occurring in classical diffusion (d). We may tentatively expect the same to be true of the exact solution to the problem.

It might be possible from equation (62) to determine the best value of n , if one exists, or, in any case, the form of $\chi(\mathbf{x}, t)$. The labour involved is great and a more feasible plan would be to evaluate the second moment $\overline{\mathbf{x}^2(t)}$ of $\chi(\mathbf{x})$. This is the most important one in determining the character of the distribution at moderate but not perhaps (if the earlier discussion is valid) at large times. This moment is defined by

$$(99) \quad \overline{\mathbf{x}^2(t)} = \frac{\iiint \chi(\mathbf{x}) \mathbf{x}^2 \, d\mathbf{x}}{\iiint \chi(\mathbf{x}) \, d\mathbf{x}},$$

which may be written as

$$\overline{\mathbf{x}^2(t)} = -\frac{1}{K(\mathbf{0}, 0)} [\nabla_{\xi}^2 K(\xi, t)]_{\xi=\mathbf{0}},$$

or

$$(100) \quad \overline{\mathbf{x}^2(t)} = -\frac{3}{K_0} \left(\frac{\partial^2 K(\xi)}{\partial \xi^2} \right)_{\xi=\mathbf{0}}; \quad K_0 = \frac{\Lambda}{8\pi^3}.$$

It follows that

$$(101) \quad \frac{\partial \overline{\mathbf{x}^2(t)}}{\partial t} = -\frac{12\pi}{K_0} \int_0^\infty \int_{-1}^1 Y(0, \kappa, \theta, t) \kappa^2 (1 - \theta^2) d\kappa d\theta,$$

where

$$(102) \quad \overline{\mathbf{x}^2(0)} = 0, \quad Y(0, \kappa, \theta, 0) = 0, \quad \frac{\partial Y(0, \kappa, \theta, 0)}{\partial t} = -K_0 \phi(\kappa),$$

and $Y(0, \kappa, \theta, t)$ may be determined from equation (62) much more easily than $Y(\xi, \kappa, \theta, t)$ since the simplification achieved by setting ξ zero is not insignificant. For small times, equation (101) and the boundary conditions (102) imply

$$(103) \quad \overline{\mathbf{x}^2(t)} = \bar{u}^2 t^2$$

as we have found before. It is noteworthy that, since

$$(104) \quad Y(0, \kappa, \theta, t) = c\phi(\kappa) = \text{independent of } t \text{ in stationary turbulence}$$

is a solution to equation (62), it is not inconceivable that $Y(0, \kappa, \theta, t)$ approaches a stationary value as t becomes large, and that consequently $\overline{\mathbf{x}^2(t)}$ becomes proportional to t . If so, the distribution would share this property with the classical diffusion distribution. However, its behaviour might still be very different because of the predominance of other moments.

5. Conclusions and acknowledgements. It has been established that, by use of a statistical hypothesis which proved successful in the theory of hydrodynamic turbulence, a knowledge of the spectrum of a turbulent velocity field enables the diffusion of any property χ carried by the fluid to be uniquely determined. To solve the problem in any particular case would involve more labour than the author is willing to undertake although the problem is soluble in principle. It has been shown that, for very small times, the equation governing χ has the character of the wave equation. For larger times, with suitable approximations, a method of characteristics was developed by means of which the distribution at any time could be calculated as the diffusion from the configuration at any previous time. Some particular cases were given and asymptotic expansions of these suggest that at large distances the distribution will fall off more slowly than for the classical diffusion. Nevertheless, it has been shown likely that the second moments of both classical and turbulent diffusion distributions are, at large times, proportional to time, the difference in the two distributions being in their higher order even moments.

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