*Rising Plane Bubbles**

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1. Introduction. We describe below the approximate determination of the plane potential flow past a symmetrical bubble rising under gravity, as in Fig. 1,

FIGURE 1

in a vertical tube with parallel walls. As explained in [1], this flow has some interest in connection with Taylor instability.

Our method consists in the numerical solution of an appropriate one-parameter family of non-linear integral equations. Similar integral equations have been used previously in theoretical discussions of ideal plane flows under gravity,** but this seems to be the first case in which a serious attempt is made at effective solution.

Though the solution given below is probably adequate for physical applica-

*Work done at the Los Alamos Scientific Laboratory, under the auspices of the Atomic Energy Commission.

**References are given in [3, pp. 200-204], together with a brief discussion of related problems.

tions, our analysis brought to light several unexpected difficulties. We present a discussion of these, in the hope that they may be resolved by further analysis.

2. Mathematical problem. It is easy to describe the mathematical problem to be solved. We use the notation of [1] and [3]: $W = U + iV$ for the complex potential, $z = x + iy$ for position, and $\zeta = u + iv = dW/dz$ for the conjugate velocity; $W(z)$ is a complex analytic function. We assume steady motion, relative to axes moving with constant velocity u_0 (the unknown bubble rise velocity), put the origin at the stagnation point, and direct the x -axis vertically downwards.

If D is the tube diameter, clearly the stream function $V(z)$ has a jump of $u_0 D$ across the upstream branch, and equal jumps of $-\frac{1}{2}u_0 D$ across the downstream branches, by symmetry. Assuming only one stagnation point, $W(z) - W(0)$ therefore maps the flow one-one and conformally onto the strip $-\frac{1}{2}u_0 D < V < \frac{1}{2}u_0 D$, cut along the positive U -axis.

The condition of constant pressure in the bubble is equivalent, by Bernoulli's Theorem, to

$$(1) \quad |\zeta^2(z)| = \zeta\zeta^* = 2gx$$

on the bubble wall, g being the acceleration of gravity. Finally, we have the condition of uniform velocity across the stream at infinity,

$$(2) \quad \lim_{|x| \rightarrow \infty} |\zeta(x, y) - \zeta(x, y')| = 0.$$

This implies that $\zeta(x, y) \rightarrow u_0$ as $x \rightarrow -\infty$. The implications as $x \rightarrow +\infty$ are less obvious and will be discussed in §4.

3. Parametrization. One can adapt ideas of LEVI-CIVITA [3, Ch. 6] to various plane flows under gravity involving straight fixed boundaries, by mapping them conformally onto the unit semicircle $\Gamma: |t| < 1, \text{Im } \{t\} > 0$ in the t -plane. We now do this, as in Fig. 2, so that the upstream "source" ∞_0 goes onto $t = 0$,

FIGURE 2

the channel walls onto the real diameter, the bubble vertex onto $t = i$, and the descending jets ∞_1, ∞_2 onto $t = \pm 1$. This amounts to setting

$$(3) \quad W = A \operatorname{Ln} \frac{t}{1 - t^2}, \quad \text{where} \quad A = \frac{u_0 D}{\pi} > 0.$$

As a function of t , ζ is clearly real on the real axis and continuous on the boundary of Γ except at $t = \pm 1$, where it becomes infinite. Hence, by the Schwarz reflection principle, $\zeta(t)$ can be expanded in a power series with real coefficients, convergent in the unit circle $|t| < 1$. The analytic continuation of $W(t)$ around the branch point of (3), at $t = 0$, gives an infinite periodic sequence of rising bubbles, separated by descending jets.

We have assumed that $t = i$ (or $z = 0$) is the only stagnation point in Γ . By reflection symmetry, ζ will also have a simple zero at $t = -i$ and no other zero in $|t| \leq 1$. A short calculation of $\zeta = (dW/dt) (dt/dz)$, based on (3), gives

$$(4) \quad \frac{\zeta}{1 + t^2} \rightarrow \frac{A}{2B} \quad \text{as} \quad t \rightarrow i,$$

where $B = iz'(i) \neq 0$.

A theorem of LEWY* shows that ζ is, moreover, *analytic* in $|t| \leq 1$, except at $t = \pm 1$ and (possibly) $t = \pm i$. We shall now discuss the singularities at $t = \pm 1$, corresponding to the asymptotic behavior on the descending jets.

4. Asymptotic jet behavior. The asymptotic behavior of $\zeta(t)$ in Γ , near $t = \pm 1$, is suggested by (1). The flow balance equation $\eta(x)\bar{u}(x) = \frac{1}{2}u_0 D$ shows that the asymptotic jet width $\eta(x) \sim u_0 D / \sqrt{8gx}$, and the slope $-\eta'(x) \sim u_0 D / \sqrt{32gx^3}$. The trial guess

$$(5) \quad \zeta = \sqrt{2gz}$$

gives, on $z = x + i\eta(x)$,

$$|\zeta^2| = |2gz| = 2g(x^2 + \eta^2)^{\frac{1}{2}} = 2gx + O\left(\frac{1}{x^{\frac{1}{2}}}\right).$$

Thus, it satisfies (1) to a good approximation; note also that it satisfies (2).

The approximation (5) implies also

$$(5') \quad W - W_1 = \int \zeta dz \sim \frac{1}{3} \sqrt{8g} z^{\frac{3}{2}}.$$

Solving for z and using (3), we get to a first approximation

$$(6) \quad z \sim \left(\frac{3W}{\sqrt{8g}} \right)^{\frac{2}{3}} \sim \left[\frac{-3A}{\sqrt{8g}} \operatorname{Ln} (1 \mp t) \right]^{\frac{2}{3}}$$

*Proc. Am. Math. Soc. **3** (1952), 111-13.

and

$$(6') \quad \zeta \sim \sqrt{2gz} \sim [-3Ag \operatorname{Ln}(1 \mp t)]^{\frac{1}{2}}.$$

To a higher approximation, near $t = 1$,

$$(7) \quad z \sim \left[\frac{-3A}{\sqrt{8g}} \{ \operatorname{Ln}(1 - t) + W_1 + \operatorname{Ln} 2 \} \right]^{\frac{2}{3}}$$

and

$$(7') \quad \zeta \sim \sqrt{2gz} \sim [3Ag \{ \operatorname{Ln}(1 - t) + W_1 + \operatorname{Ln} 2 \}]^{\frac{1}{2}}.$$

In the higher approximation W_1 appears as a further unknown, corresponding monotonely to arc-length along the cavity wall from the stagnation point to a given depth.

5. Levi-Civita functions $\Omega(t)$. We now define a class of functions analogous to the one introduced by LEVI-CIVITA.* Let $F(t)$ be any complex analytic function, real when t is real and having the same zeros and singularities in Γ as $\zeta(t)$, up to a bounded continuous factor. In the present instance, the preceding analysis shows that the functions

$$(8) \quad F(t, C) = (1 + t^2)[- \operatorname{Ln} C(1 - t^2)]^{\frac{1}{2}}, \quad 0 < C < 0.5,$$

have the desired property.

In any such case, one may define a *Levi-Civita function* $\Omega(t)$ by the equivalent formulas

$$(9) \quad \zeta(t) = F(t)e^{i\Omega(t, C)} \quad \text{or} \quad \Omega(t, C) = \operatorname{Ln} \left[\frac{\zeta(t)}{F(t, C)} \right].$$

It will follow that $\Omega(t)$ must be *analytic and bounded* in $|t| < 1$, continuous on $|t| \leq 1$, and real on the real diameter. Such functions can also be constructed for the other ideal plane flows under gravity described in [3, p. 202, Figs. 3a-3b].

In the present case, defining $\Omega(t, C)$ by (8) and (9), we see that Ω is real on both real and imaginary axes. In view of this four-fold symmetry, we can write

$$(10) \quad \Omega(t, C) = a_0(C) + a_2(C)t^2 + a_4(C)t^4 + \dots,$$

the series being absolutely convergent in $|t| < 1$ and Abel summable on $|t| = 1$. Our problem will be solved if we can determine the *real* coefficients $a_{2k}(C)$.

Since any choice of $\Omega(t, C)$ defines an ideal plane flow

$$(9^*) \quad \zeta(t) = (1 + t^2)[- \operatorname{Ln} C(1 - t^2)]^{\frac{1}{2}} e^{i\Omega(t)},$$

having the desired asymptotic properties, the problem will be solved if the free boundary condition (1) is satisfied.

**Rend. cir. mat. Palermo* 23 (1907), 1-37; for a *résumé* see [3, p. 132]. In the present application, the location of the fixed and free boundaries in Γ is the opposite of that used by LEVI-CIVITA, and so $\exp \Omega(t)$ plays the role of LEVI-CIVITA's $\exp i\Omega(t)$.

6. Free boundary condition. To interpret the free boundary condition (1), we write $\zeta = ve^{i\phi}$. Then (1) is $v^2 = 2gx$ on $t = e^{i\sigma}$, $0 < \sigma < \pi$. Differentiating with respect to arc-length s along the free boundary, we get $v dv/ds = g dx/ds$. Since the fluid velocity $\zeta^* = ve^{-i\phi}$ is tangential to the free boundary, $dz/ds = e^{-i\phi}$ or $dx/ds = \cos \phi$. Moreover, we may write $dv/ds = (dv/d\sigma) \cdot (d\sigma/dW)(dW/ds)$, where $dW/ds = v$, and, by (3), $dW/d\sigma = -A \cot \sigma$. In conclusion, (1) is equivalent to

$$(11) \quad \frac{1}{3} \tan \sigma \frac{d(v^3)}{d\sigma} = -gA \cos \phi, \quad \zeta = ve^{i\phi},$$

plus the condition $v = 0$ at $\sigma = \frac{1}{2}\pi$.

Condition (11) becomes singular at $\sigma = \frac{1}{2}\pi$, but if it is rewritten as

$$\sin \sigma \left(\frac{v}{\cos \sigma} \right)^2 \frac{dv}{d\sigma} = -gA \frac{\cos \phi}{\cos \sigma},$$

it tends in the limit $\sigma \rightarrow \frac{1}{2}\pi$ to

$$(11') \quad \left(\frac{dv}{d\sigma} \right)^3 = -gA \frac{d\phi}{d\sigma} \quad \text{at} \quad \sigma = \frac{1}{2}\pi.$$

Still following the analysis of LEVI-CIVITA and VILLAT, by analogy, we shall now express the free boundary condition (11) as a non-linear integral equation. Though our specific formulas are based on the choice (8), straightforward substitutions would yield analogous formulas for any Levi-Civita function.

We begin by writing

$$(12) \quad \Omega(t, C) = \tau(t, C) + i\theta(t, C)$$

and observe that (by symmetry) it suffices to satisfy (11) on $0 < \sigma < \frac{1}{2}\pi$ (i.e., on one descending branch).

On this interval $\cos \sigma > 0$, $\sin \sigma > 0$, and

$$(1 + t^2) = 2 \cos \sigma e^{i\sigma}, \quad (1 - t^2) = 2 \sin \sigma e^{i(\sigma - \frac{1}{2}\pi)}.$$

Hence, introducing the convenient abbreviation

$$(13a) \quad L(\sigma) = -\text{Ln}(2C \sin \sigma), \quad 0 < 2C < 1,$$

we get by complex trigonometry

$$(13b) \quad -\text{Ln} C(1 - t^2) = \sqrt{M(\sigma)} e^{i \arctan [(\frac{1}{2}\pi - \sigma)/L(\sigma)]},$$

where

$$M(\sigma) = L^2(\sigma) + (\frac{1}{2}\pi - \sigma)^2.$$

Equating absolute values and arguments in (9*), we get

$$(14a) \quad v = 2 |\cos \sigma| \sqrt[6]{M(\sigma)} e^\tau$$

and

$$(14b) \quad \phi = \theta + P(\sigma),$$

where

$$(14c) \quad P(\sigma) = \sigma + \frac{1}{3} \arctan [(\frac{1}{2}\pi - \sigma)/L(\sigma)].$$

We are now ready to get an integral equation equivalent to (11) on $0 < \sigma < \frac{1}{2}\pi$. Substituting from (14a) into the left side of (11), we get after some simplification

$$(15a) \quad Q(\sigma)e^{3\tau}\{-\sin \sigma + [N(\sigma) + \tau'(\sigma)] \cos \sigma\},$$

where

$$(15b) \quad N(\sigma) = [-L(\sigma) \cot \sigma - (\frac{1}{2}\pi - \sigma)]/3M(\sigma)$$

and

$$(15c) \quad Q(\sigma) = 4\sqrt{M(\sigma)} \sin 2\sigma.$$

The right side of (11) is, by (14b), simply

$$(16) \quad -gA \cos [\theta + P(\sigma)].$$

Again, as $t \rightarrow 1$, an elementary calculation based on (6') gives

$$1 \sim \left[\frac{8}{3Ag} \right]^{\frac{1}{3}} e^{\Omega(1)},$$

whence

$$(16') \quad gA = \frac{8}{3}e^{3\Omega(1)} = \frac{8}{3}e^{3\tau(0)}.$$

The free boundary condition (11) is equivalent to the assertion that (15a) and (16) are equal on $0 < \sigma < \frac{1}{2}\pi$.

We can express this equation in terms of the single unknown function $\lambda(\sigma) = \tau'(\sigma)$, which is analytic by Lewy's Theorem, except at $\sigma = \frac{1}{2}n\pi$. By [3, p. 136] we have $\tau(\sigma) = \Omega(0) + \mathbf{J}\lambda(\sigma)$ and $\theta(\sigma) = \mathbf{D}\lambda(\sigma)$, where $\mathbf{J}$ and $\mathbf{D}$ are integral operators; see also (18a)—(18c) below, with $\Omega(0) = a_0$. Hence the free boundary condition (11) is equivalent to the non-linear, singular integral equation

$$(17) \quad Q(\sigma)e^{3\mathbf{J}\lambda}\{-\sin \sigma + [N(\sigma) + \lambda(\sigma)] \cos \sigma\} = -\frac{8}{3}e^{3\mathbf{J}\lambda(0)} \cos (\mathbf{D}\lambda + P(\sigma)).$$

7. Coefficient relations and dimensions. Some of the preceding relations have simple interpretations in terms of the series (10), which implies

$$(18a) \quad \tau(\sigma) = a_0 + a_2 \cos 2\sigma + a_4 \cos 4\sigma + a_6 \cos 6\sigma + \cdots,$$

$$(18b) \quad \lambda(\sigma) = -2a_2 \sin 2\sigma - 4a_4 \sin 4\sigma - \cdots = \tau'(\sigma),$$

$$(18c) \quad \theta(\sigma) = a_2 \sin 2\sigma + a_4 \sin 4\sigma + a_6 \sin 6\sigma + \cdots.$$

To use these series it is necessary first to express the various physical constants involved in terms of A , C , and the a_{2k} .

At $t = 0$, clearly $\zeta = u_0$; hence, by (9*) and (10),

$$(19a) \quad u_0 = (-\text{Ln } C)^{\frac{1}{2}} e^{a_0}.$$

Again, as $t \rightarrow 1$, an elementary calculation based on (6') gives $1 \sim [8/3Ag]^{\frac{1}{2}} e^{\eta(1)}$, whence

$$(19b) \quad a_0 + a_2 + a_4 + \dots = \frac{1}{2} \text{Ln } \left(\frac{3}{8} Ag\right).$$

Finally, near the bubble vertex, as $t \rightarrow i$, by (9*),

$$\frac{\zeta}{1+t^2} \rightarrow (-\text{Ln } 2C)^{\frac{1}{2}} e^{a_0 - a_2 + a_4 - \dots}.$$

Comparing with equation (4), we get

$$B = iz'(i) = \text{Lim} \frac{(1+t^2)A}{2\zeta} = \frac{Ae^{-\eta(i)}}{2(-\text{Ln } 2C)^{\frac{1}{2}}}$$

or

$$(19c) \quad B = \frac{Ae^{-a_0 + a_2 - a_4 + \dots}}{2(-\text{Ln } 2C)^{\frac{1}{2}}}.$$

It is evident that if g and D are transformed in any ratio, then a simple change of scale, preserving the dimensionless ratios B/D , $u_0/\sqrt{gD}$, and A/u_0D , will carry solutions of the problem stated in §2 into solutions of the transformed problem. To express these ratios in terms of the coefficients a_{2k} we note first that A/u_0D is given by (3) as

$$(20a) \quad \frac{A}{u_0D} = \frac{1}{\pi};$$

combining this with (16') and (19a) we find that

$$(20b) \quad \frac{u_0^2}{gD} = \left[\frac{-3 \text{Ln } C}{8\pi} - 3(a_2 + a_4 + a_6 + \dots) \right].$$

Finally, a combination of (19a), (19c), and (20a) yields B/D as

$$(20c) \quad \frac{B}{D} = \frac{1}{2\pi} \left[\frac{\text{Ln } C}{\text{Ln } 2C} \right]^{\frac{1}{2}} e^{a_2 - a_4 + a_6 - \dots}.$$

Clearly, the last two ratios are uniquely determined, if (17) has a unique solution; cf. §9. The ratio u_0^2/gD is especially interesting, since $u_0/\sqrt{gD}$ is the dimensionless rate of bubble rise.

Another interesting dimensionless ratio is R/D , the ratio of the radius of curvature R of the bubble at the vertex to the tube diameter D . Since the surface

inclination is $-\phi$, the surface curvature $1/R$ is $|d\phi/ds|$. Writing $|d\phi/ds| = |d\phi/d\sigma|/|ds/d\sigma| = |d\phi/d\sigma|/|dz/dt| = |d\phi/d\sigma|/B$ at $t = i$, we find that

$$(21a) \quad \frac{R}{D} = \frac{B/D}{\left| \frac{d\phi}{d\sigma} \right|}.$$

To find R/D from (21a), we may take B/D from (20c) and $|d\phi/d\sigma| = d\phi/d\sigma$ from the equation

$$(21b) \quad \frac{d\phi}{d\sigma} = 1 + \frac{1}{3 \operatorname{Ln}(2C)} - 2a_2 + 4a_4 - \dots, \quad \text{at } \sigma = \tfrac{1}{2}\pi,$$

which follows from (14b), (14c), and (18c).

Finally, the free boundary condition at the vertex (11') gives an equation on our coefficients. To find this equation we first use (20a) to express (11') in dimensionless form:

$$(22a) \quad \pi \left(\frac{u_0^2}{gD} \right) \left(\frac{dv}{u_0 d\sigma} \right)^3 = -\frac{d\phi}{d\sigma} \quad \text{at } \sigma = \tfrac{1}{2}\pi.$$

One can find the derivative of v/u_0 from (14a), (18a), and (19a) as

$$(22b) \quad \frac{d}{d\sigma} \frac{v}{u_0} = -2 \left[\frac{\operatorname{Ln}(2C)}{\operatorname{Ln} C} \right]^{\frac{1}{2}} e^{-a_2 + a_4 - \dots} \quad \text{at } \sigma = \tfrac{1}{2}\pi.$$

Substituting (20b), (21b), and (22b) into (22a) yields the required equation:

$$(22c) \quad (-3 \operatorname{Ln} 2C) e^{-6(a_2 + a_4 + a_6 + \dots)} = 1 + \frac{1}{3 \operatorname{Ln}(2C)} - 2a_2 + 4a_4 - \dots.$$

8. Approximate numerical results. The formulas of §§6 and 7 are very similar to those for determining ideal plane flows with free boundaries past curved obstacles, gravity being neglected. We therefore tried to adapt the procedure, known to be successful* for this case. This procedure is based on equal-spaced interpolation in σ at N points, matched against truncation of the series (18a)–(18c) after a finite number of N terms. For given N , one then tries to solve the resulting discrete approximation to (17) by a suitable iterative procedure.

We followed various modifications of this procedure; our results are discussed in some detail in [1, pp. 38–46]. The “more accurate” calculations gave values of $u_0/\sqrt{gD}$ consistently in the range 0.215–0.25, from which it may be concluded with some confidence that

$$(23) \quad u_0/\sqrt{gD} = 0.23 \pm 0.01,$$

which is probably sufficient for physical applications.

*[3, pp. 215–19]. For a detailed discussion, see G. BIRKHOFF, H. H. GOLDSTINE, & E. H. ZARANTONELLO, *Rend. sem. mat. Torino* **13** (1954), 205–23.

However, the convergence of the procedures used—unlike the corresponding calculations for free boundary flows without gravity*—was not really satisfactory. Moreover, we did not get consistent predictions of R/D .

The remainder of this paper contains our analysis of how better results might be obtained. The discussion is based largely on hindsight, but partly on a fresh analysis, and has been influenced by the interesting ideas suggested in [2].

The problem is certainly interesting, because the approach described in §§1–7 seems very promising, in principle, for attacking a variety of unsolved problems in classical hydrodynamics. It also bears on the philosophical question as to how much mathematical rigor is needed in planning research computations.

9. Existence and uniqueness. From the start we were vexed by the absence of any satisfactory existence and uniqueness theory for our problems. As the existence and uniqueness theory for the analogous free boundary problem without gravity is difficult and highly technical [3, Ch. VII], we decided to calculate without waiting for its extension to the present case.**

A few facts can be established rigorously. First, uniqueness cannot be inferred without making assumptions about the flow topology and symmetry (see Figs. 3a–3b). Second, the existence and uniqueness theory is equivalent to

FIGURE 3

that of the non-linear integral equation (17). Third, it is the same† for all C in $0 < C < 0.5$, since the formula

$$(24) \quad \Omega(t, C) - \Omega(t, C_1) = \frac{1}{3} \text{Ln} \left\{ \frac{\text{Ln } C_1(1 - t^2)}{\text{Ln } C(1 - t^2)} \right\}$$

carries solutions for C_1 into solutions for C . In (24)

$$\text{Ln } [C(1 - t^2)] = \text{Ln } [2C | \sin \sigma |] + i(\sigma - \frac{1}{2}\pi)$$

*See the preceding footnote.

**An attempt at this extension by one of us, based on the Leray-Schauder technique, indicated that a solution existed for (17).

†This was not realized in [1].

on $0 < \theta < \pi$, and similarly for $\text{Ln } [C_1(1 - t^2)]$. Since

$$\Omega(t, C) - \Omega(t, C_1) = \frac{1}{3} \text{Ln} \left\{ \frac{1 + \text{Ln } C_1 / \text{Ln } (1 - t^2)}{1 + \text{Ln } C / \text{Ln } (1 - t^2)} \right\},$$

the asymptotic difference near $t = \pm 1$ is only $O(1/|\text{Ln } \Delta t|)$. However, the effect on $\lambda(\sigma)$ is $O(1/\Delta t \text{Ln}^2 \Delta t)$, which is infinitely large! Very likely, this explains the scatter in our calculated values.

In [2], GARABEDIAN has adapted the Comparison Method [3, pp. 90–96] to show that there is only one flow for a given $u_0/\sqrt{gD}$. Though some details are lacking, the argument seems convincing. It follows that (17) has at most a one-parameter family of solutions.

GARABEDIAN also asserts, after a heuristic study of asymptotic series for the descending jets*, that there *exists* a flow for each $u_0/\sqrt{gD}$ below a stable maximum value. We do not find his proof convincing. However, stimulated by his discussion, we present some other heuristic arguments.

We originally thought [1, §6] that variations in C were related, through variations in W_1 , to variations in the arc-length along the free boundary from the stagnation point to a given depth x . The analogous theory of the plane flow under a sluice gate** suggests that this arc-length might be variable—and that the variation might be associated with waviness of the free boundary. This idea seems connected with GARABEDIAN's, though it is now clear that there is no connection with the choice of C .

On the other hand, the experimental stability observed for the axially symmetric analog [3, p. 235] has made most authors assume that the problem is well-determined. (Above a certain size ripples are observed on the wall downstream, however.) GARABEDIAN [2, p. 1] suggests that the fastest rising bubbles should be the most stable, so that experimental stability need not imply uniqueness. His idea is plausible—but in the plane case his positive argument would imply a tendency to asymmetry (see Fig. 3b): symmetry should be unstable.

Some experimental evidence is given by the photographic observations of RUSSELL DUFF [1, §12]. He did not observe any tendency to asymmetry in a 4:1 rectangular tube. Though his measured value of $u_0/\sqrt{gD} \simeq 0.29$ was about 25% greater than our calculated value, this might have been largely a side-wall effect. Moreover, DUFF did not consider his accuracy very great (say ± 0.02).

Further experimental evidence, involving the rise of air columns between parallel tongues of liquid and parallel glass walls, is contained in the Ph. D. Thesis of BARRY WATSON.† WATSON reports values of $u_0/\sqrt{gD}$ in the range 0.22–0.23.

*These are based on results of H. LEWY, *Univ. Calif. Publ. Math.* 1 (1950) 247–80, and *this Journal* 6 (1957), 91–107.

**A. M. BINNIE, *Quar. J. Mech. Appl. Math.* 5 (1952), 395–407.

†“An experimental study of accelerated fluids (the Taylor problem),” Harvard Doctoral Thesis, 1957.

10. Convergence. The theory of the convergence, as $N \rightarrow \infty$, of the solutions of the system of N equations is even more difficult.* (We do not here discuss the convergence of iterative procedures for fixed N .)

Our numerical experience confirmed the expectation that the singularities at $t = \pm 1$ should be the main source of trouble. It seems likely that improved results could be obtained by using (7') instead of (6'). To do this W_1 would have to be estimated, which would make the computational procedures more cumbersome. However, without this refinement it appears from (7') that $F(t, C)$ will ordinarily have an infinite derivative at $t = \pm 1$. Hence, the convergence of equal-spaced interpolation is not clearly indicated**.

In this connection, it should be noted that fairly good estimates of $u_0/\sqrt{gD}$ can be obtained using drastically erroneous asymptotic formulas.† Analytical smoothness on the downstream jets seems far more important than faithful satisfaction of the free boundary condition. Any moderate device which would make $\Omega'(\pm 1) = 0$ (i.e., give differentiability and four-fold symmetry) would probably greatly improve the smoothness of predicted results. It would seem sufficient to choose the "right" C .

For any C and N , the discretization formulated in §8 guarantees that the two sides of (1) will have the same first derivative at $N - 1$ equally spaced values σ_i of σ and the same second derivative at $\sigma = \frac{1}{2}\pi$ (where the first derivative of both sides vanishes). It would have been instructive to plot the difference between the first derivatives of the two sides of (1) at intermediate values of σ . The variation with C in the calculated approximations to $u_0/\sqrt{gD}$ can be explained as due to the truncation of the Fourier series for (24) after N terms. This affects (1) at intermediate values of σ , changing the truncation error there.

REFERENCES

- [1] G. BIRKHOFF & D. CARTER, *Appendix D of LA-1927*, "Taylor instability. Appendices to Report LA-1862," Los Alamos, 1955.
- [2] P. GARABEDIAN, "On steady state bubbles generated by Taylor instability," *Tech. Rep.* 59, Appl. Math. Statistics Laboratory, Stanford, 1957.
- [3] GARRETT BIRKHOFF & E. H. ZARANTONELLO, "Jets, wakes and cavities," Academic Press, 1957.

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*E. H. ZARANTONELLO, *Collectanea Mathematica, Barcelona*, 5 (1952) 175-225. For a brief résumé, see [3, pp. 215-19].

**D. JACKSON, "Theory of Approximation," p. 123. Near $t = 1$, $F(t, C) - F(1, C) = O(1/\ln(1 - t))$; hence $|f(x) - S_n(x)| = O(1)$ is possible in JACKSON's Cor. 2.

†D. LAYZER, *Astrophysical J.* 122 (1955), 1-12. See [1, Appendix E] for a discussion of the asymptotic error.