The Decay of Waves in Elastic Solids

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1. Introduction. Consider an isotropic elastic solid with displacements from the unstrained position given by $u_i(x, t)$ where the x 's are the coordinates of a dynamically allowable rectangular system and t denotes the time. The equations of motion are

$$(1) \quad \sigma_{ii,i} = \rho \frac{dv_i}{dt},$$

where the quantities σ_{ij} are the components of the symmetric stress tensor such that

$$(2) \quad \sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i}),$$

in which λ and μ are the well known material constants of the medium. Also in (1) the quantity ρ is the density and the v_i are the components of velocity. As customary we assume for simplicity that the v_i are given by the formula

$$(3) \quad v_i = \frac{\partial u_i}{\partial t},$$

which can be applied under the condition that the displacements u and their spatial derivatives are sufficiently small. Indeed, this condition is one of the essential requirements of the linear or classical theory of elasticity with which we are concerned in this article.

It has been known for over a hundred years that two types of waves having different velocities can be propagated within an isotropic elastic medium¹. The names *irrotational* and *equivoluminal* have been given to these waves for reasons which will be apparent from the discussion in the following section. It is the purpose of the present article² to investigate the question of the decay in the strength of the irrotational and equivoluminal waves during propagation on

¹ For reference to the early literature see A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*, 4th. Ed., New York, Dover Publications, 1944.

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the basis of the results of our recent publication on *extended compatibility conditions*³ in THIS JOURNAL.

Since the velocities of propagation are constant the successive positions of an irrotational or an equivoluminal wave will form a family of parallel surfaces (§3). In §5 we have derived the differential equation for the variation of the strength of an irrotational wave and in §6 we have given the corresponding equation for the equivoluminal wave. By integration of these differential equations we obtain the general theorem on wave decay. Briefly stated it is shown that for decay the wave surface must be a developable surface (not a plane) or a surface of positive curvature. The precise result is contained in the italicized statement at the end of §7.

The question of the increase of the wave strength during propagation, corresponding to the situation which can arise in the propagation of sonic waves in gases⁴ where the wave may culminate in an actual shock under certain circumstances, is not considered here since this would necessitate the growth of the spatial derivatives of the displacements u beyond the limits which are proper in the classical theory of elasticity.

2. Velocity of waves. Let us begin by considering the dynamical relations

$$(4) \quad \rho_0 G = \bar{\rho}(G - \bar{v}_n),$$

$$(5) \quad [\sigma_{ij}]v_i = -\rho_0 G[v_i] = -\rho_0 G \bar{v}_i,$$

which are valid over a singular surface $\Sigma(t)$ propagated into an elastic medium at rest in its unrestrained position and having the constant density ρ_0 . We have denoted by v_i the components of the unit normal vector to $\Sigma(t)$, directed into the unstrained portion of the medium, and $G > 0$ is the coordinate velocity of propagation of Σ in this direction. The barred quantities $\bar{\rho}$ and $\bar{v}_i$ denote density and velocity components on the rear side of Σ and the quantity $\bar{v}_n = \bar{v}_i v_i$ is the normal component of this velocity. In the following the use of a bar over a quantity or its derivative will represent a similar evaluation. Finally we denote by the bracket $[]$ the difference in the values on the two sides of $\Sigma(t)$ of the quantity enclosed by the bracket. More precisely it will be supposed in this article that the bracket $[]$ is defined as the value of the quantity in question on the rear of Σ minus its value on the front of this surface. This latter value will usually be zero due to the fact that the medium on the front side of Σ is unstrained and at rest; in such a case the bracket expression will merely give an evaluation of the quantity enclosed on the rear of the surface Σ . Thus (5) is equivalent to the relations

$$\bar{\sigma}_{ij}v_j = -\rho_0 G \bar{v}_i.$$

³ Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics, *Jour. of Math. and Mech.* **6** (1947) pp. 311-312.

⁴ See T. Y. THOMAS, The growth and decay of sonic discontinuities in ideal gases, *Jour. of Math. and Mech.* **6** (1957) pp. 455-469.

We shall assume that $[u_i] = 0$ or, in other words, that there is no *separation* of the medium over the moving surface $\Sigma(t)$. But this means that the displacements u are continuous over $\Sigma(t)$, and under this condition we know that we must have relations of the form

$$(6) \quad [u_{i,i}] = \omega_i \nu_i ; \quad \left[\frac{\partial u_i}{\partial t} \right] = -G \omega_i ,$$

where the ω_i are functions defined over $\Sigma(t)$. Hence from (3) and the second relations (6) we obtain

$$(7) \quad \bar{v}_i = [\nu_i] = \left[\frac{\partial u_i}{\partial t} \right] = -G \omega_i .$$

Using these relations (7) it follows from (2), (5) and (6) that

$$(8) \quad (\lambda + \mu) \omega_k \nu_k \nu_i + \mu \omega_i = \rho_0 G^2 \omega_i .$$

The moving surface $\Sigma(t)$ will be said to be singular of order 1, or to be a *wave surface* of order 1, provided there is a discontinuity in spatial derivatives $u_{i,i}$ over Σ . From the relations (6) we immediately see that the condition for Σ to be singular of order 1 is that not all of the quantities ω_i vanish. In the following discussion it will be assumed that this condition is satisfied.

If $\omega_i \nu_i$ does not vanish on $\Sigma(t)$ let us multiply (8) by ν_i and sum on the repeated index i . This gives

$$(9) \quad G^2 = \frac{\lambda + 2\mu}{\rho_0} , \quad \omega_i \nu_i \neq 0 .$$

On the other hand, if $\omega_i \nu_i = 0$ on $\Sigma(t)$ we see directly from (8) and the condition that not all the ω_i can vanish that

$$(10) \quad G^2 = \frac{\mu}{\rho_0} , \quad \omega_i \nu_i = 0 .$$

Substituting the value of G^2 given by (9) into (8) and taking account of the fact that $\omega_i \nu_i \neq 0$ the resulting relations become

$$(11) \quad \omega_i = \omega \nu_i , \quad \omega = \omega_k \nu_k \neq 0 .$$

Hence from (6) we have

$$(12) \quad [u_{i,i}] = \bar{u}_{i,i} = \omega \nu_i \nu_i ; \quad \left[\frac{\partial u_i}{\partial t} \right] = \frac{\partial \bar{u}_i}{\partial t} = -G \omega \nu_i ,$$

with $\omega \neq 0$. Hence

$$\bar{u}_{i,i} - \bar{u}_{i,i} = 0 ,$$

or, in other words, the *rotation* vanishes immediately behind the wave front in the case of a wave whose velocity is given by (9). Correspondingly for waves whose velocity is given by (10) it follows directly from (6) that the *dilatation* $u_{i,i} = 0$

immediately behind the wave front. Adopting the terminology of *irrotational* and *equivoluminal waves* in accordance with these observations we therefore have:

$$\text{For irrotational waves: } G^2 = \frac{\lambda + 2\mu}{\rho_0}.$$

$$\text{For equivoluminal waves: } G^2 = \frac{\mu}{\rho_0}.$$

For the equivoluminal wave it follows immediately from (7) that $\bar{v}_n = 0$. Hence (4) yields $[\rho] = 0$. However in the case of the irrotational wave we find from (7) that $\bar{v}_n = -G\omega$ and hence (4) becomes $[\rho] = -\bar{\rho}\omega$ with $\omega \neq 0$. Hence the density is continuous across the equivoluminal but discontinuous across the irrotational wave.

Remark. We have

$$(13) \quad v_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_{i,j}v_j$$

as the exact equations corresponding to (3). Evaluating on the rear of the wave surface $\Sigma(t)$ these relations give

$$\bar{v}_i = -G\omega_i + \omega_i\bar{v}_j\nu_j = -(G - \bar{v}_n)\omega_i.$$

Hence

$$(14) \quad \bar{v}_i = -\frac{\bar{\rho}(G - \bar{v}_n)\omega_i}{\bar{\rho}} = -\frac{\rho_0 G \omega_i}{\bar{\rho}}$$

from (4). In place of (8) we now find from (2), (5), (6) and (14) that

$$(15) \quad (\lambda + \mu)\omega_k\nu_k\nu_i + \mu\omega_i = \frac{\rho_0^2 G^2 \omega_i}{\bar{\rho}},$$

and from these relations we obtain

$$(16) \quad G^2 = \frac{\bar{\rho}}{\rho_0} \frac{\lambda + 2\mu}{\rho_0}, \quad \text{if } \omega_k\nu_k \neq 0,$$

$$(17) \quad G^2 = \frac{\bar{\rho}}{\rho_0} \frac{\mu}{\rho_0}, \quad \text{if } \omega_k\nu_k = 0.$$

Substituting the value of G^2 given by (16) into (15) we again deduce the relations (11). As above this leads to the irrotational character of the waves whose velocity is given by (16) while the formula (17) now gives the velocity G of the equivoluminal waves.

From (14) we obtain

$$(18) \quad \bar{v}_n = -\frac{\rho_0 G}{\bar{\rho}} (\omega_k\nu_k).$$

Hence $\bar{v}_n = 0$ for the equivoluminal waves and it follows from (4) that $[\rho] = 0$. Hence $\rho_0 = \bar{\rho}$ and the equation (17) reduces to the previous equation (10) for the velocity G of the equivoluminal wave. In the case of the irrotational wave (18) becomes

$$\bar{v}_n = -\frac{\rho_0 G \omega}{\bar{\rho}}$$

with $\omega \neq 0$ and hence from (4) we have $\bar{\rho} = (1 - \omega)\rho_0$. Hence (16) can be written

$$(19) \quad G^2 = (1 - \omega) \frac{\lambda + 2\mu}{\rho_0}.$$

But when we neglect the small quantity ω in comparison with unity the equation (19) reduces to the above equation (9) for the velocity G of the irrotational wave.

3. Waves as parallel surfaces. The equations (60) of our paper on *extended compatibility conditions*³ state that

$$(20) \quad \frac{\delta \nu_i}{\delta t} = -g^{\alpha\beta} G_{,\alpha} x_{i,\beta},$$

where the $g^{\alpha\beta}$ are the contravariant components of the metric tensor of the wave surface $\Sigma(t)$ and where the comma denotes differentiation relative to the curvilinear coordinates of this surface. But since $G = \text{const.}$ for the irrotational and the equivoluminal waves under consideration it follows that the δ time derivatives $\delta \nu_i / \delta t = 0$ and hence the normals to these wave surfaces must be *straight lines*. Hence the successive positions of the irrotational and equivoluminal wave surfaces $\Sigma(t)$ form a family of parallel surfaces in space⁵. By laying off equal lengths along the normals to the wave surface $\Sigma(t_0)$ in the direction of the propagation one can therefore construct the surface $\Sigma(t)$ at times $t > t_0$.

4. Compatibility conditions of the second order. Since the displacements u are continuous over the wave surface $\Sigma(t)$ the relations (29), (42) and (43) of the paper on *extended compatibility conditions*³ are applicable and we can write

$$(21) \quad [u_{i,jk}] = \hat{\omega}_i \nu_j \nu_k + g^{\alpha\beta} \omega_{i,\alpha} (\nu_j x_{k,\beta} + \nu_k x_{j,\beta}) - \omega_i g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{j,\beta} x_{k,\tau},$$

$$(22) \quad \left[\frac{\partial^2 u_i}{\partial x_j \partial t} \right] = \left(-G \hat{\omega}_i + \frac{\delta \omega_i}{\delta t} \right) \nu_j - G g^{\alpha\beta} \omega_{i,\alpha} x_{j,\beta},$$

$$(23) \quad \left[\frac{\partial^2 u_i}{\partial t^2} \right] = G^2 \hat{\omega}_i - 2G \frac{\delta \omega_i}{\delta t},$$

where the ω_i are functions defined over the surface $\Sigma(t)$ and the $b_{\alpha\beta}$ are the components of the second fundamental form of this surface. In constructing the relations (22) and (23) we have made use of the fact that the velocity G is constant for the irrotational and equivoluminal waves under consideration.

Certain special relations will now be deduced for application in the following sections. Thus if we contract the indices j, k and also the indices i, j in (21) we obtain

$$(24) \quad [\mu_{i,kk}] = \hat{\omega}_i - 2\Omega \omega_i,$$

⁵ See E. P. LANE, *Metric Differential Geometry of Curves and Surfaces*, Univ. of Chicago Press, 1940, p. 209.

where Ω , defined by

$$2\Omega = q^{\alpha\beta} b_{\alpha\beta} ,$$

is the mean curvature of the surface $\Sigma(t)$, and

$$(25) \quad [u_{k,ki}] = (\hat{\omega}_k \nu_k) \nu_i + (\omega_{k,\alpha} \nu_k) g^{\alpha\beta} x_{i,\beta} \\ + g^{\alpha\beta} \omega_{k,\alpha} x_{k,\beta} \nu_i - (\omega_k x_{k,\beta}) g^{\alpha\beta} g^{\sigma\tau} b_{\sigma\alpha} x_{i,\tau} .$$

Also if we multiply (24) and (25) by ν_i and sum on the repeated index i we find that

$$(26) \quad [u_{i,kk}] \nu_i = \hat{\omega}_k \nu_k - 2\Omega \omega_k \nu_k ,$$

$$(27) \quad [\mu_{k,ki}] \nu_i = \hat{\omega}_k \nu_k + g^{\alpha\beta} \omega_{k,\alpha} x_{k,\beta} .$$

From (2) we have

$$(28) \quad [\sigma_{ij,i}] = (\lambda + \mu) [u_{k,ki}] + \mu [u_{i,kk}] .$$

Using these relations and (1), (26) and (27) we now obtain

$$(29) \quad (\lambda + 2\mu) \hat{\omega}_k \nu_k - 2\mu \Omega \omega_k \nu_k + (\lambda + \mu) g^{\alpha\beta} \omega_{k,\alpha} x_{k,\beta} = \left[\rho \frac{dv_i}{dt} \right] \nu_i .$$

In the following section we shall consider the reduction of this relation for the case of the irrotational wave.

5. Application to irrotational waves. Let us define the strength w of the irrotational or the equivoluminal wave by the equation

$$(30) \quad w = \sqrt{\omega_i \omega_i} .$$

Then $w = 0$ implies $\omega_i = 0$ and conversely. Hence if $w = 0$ the wave surface $\Sigma(t)$ will not be singular of order 1 as postulated.

We shall now deduce the differential equation for the variation in the strength w of an irrotational wave during its propagation. For this wave we have $w = |\omega|$ from (11). Also from (11) we have

$$\omega_{k,\alpha} = \omega_{,\alpha} \nu_k + \omega \nu_{k,\alpha} .$$

Hence the last term in the left member of (29) becomes

$$(\lambda + \mu) \omega g^{\alpha\beta} \nu_{k,\alpha} x_{k,\beta} = -(\lambda + \mu) \omega g^{\alpha\beta} g^{\sigma\tau} b_{\sigma\alpha} x_{k,\tau} x_{k,\beta} ,$$

and this reduces to

$$-(\lambda + \mu) \omega g^{\alpha\beta} g^{\sigma\tau} b_{\sigma\alpha} g_{\beta\tau} = -(\lambda + \mu) \omega g^{\alpha\beta} b_{\alpha\beta} = -2(\lambda + \mu) \omega \Omega .$$

Making this substitution and also the substitution $\omega_k \nu_k = \omega$ in (29) we now have

$$(31) \quad (\lambda + 2\mu) (\hat{\omega}_k \nu_k - 2\omega \Omega) = \left[\rho \frac{dv_i}{dt} \right] \nu_i .$$

By differentiation of (3) we obtain

$$\frac{dv_i}{dt} = \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 u_i}{\partial x_j \partial t} \frac{\partial u_j}{\partial t}.$$

But this gives

$$\left[\frac{dv_i}{dt} \right] = G^2 \hat{\omega}_i - 2G \frac{\delta \omega_i}{\delta t} - G \left(-G \hat{\omega}_i + \frac{\delta \omega_i}{\delta t} \right) \omega,$$

when use is made of (12), (22) and (23). Or

$$(32) \quad \left[\frac{dv_i}{dt} \right] = (1 + \omega) G^2 \hat{\omega}_i - (2 + \omega) G \frac{\delta \omega}{\delta t} \nu_i.$$

Recalling that the brackets, which are here involved, represent evaluation on the rear of the wave surface, it follows from (31) and (32) that

$$(33) \quad (\lambda + 2\mu)(\hat{\omega}_k \nu_k - 2\omega\Omega) = (1 + \omega) G^2 \bar{\rho} \hat{\omega}_k \nu_k - (2 + \omega) G \bar{\rho} \frac{\delta \omega}{\delta t}.$$

But the product $(1 + \omega)\bar{\rho}$ is equal to the density ρ_0 of the unstrained material from the condition (4) as we observed in §2. Making this substitution and also substituting the value of G^2 given by (9) the relation (33) reduces to

$$(34) \quad \frac{(2 + \omega)\rho_0 G}{1 + \omega} \frac{\delta \omega}{\delta t} = 2(\lambda + 2\mu)\omega\Omega.$$

Let us now denote by $\sigma \geq 0$ the distance measured along the normals to the surface $\Sigma(t_0)$ mentioned in §3. Then

$$\frac{\delta \omega}{\delta t} = G \frac{d\omega}{d\sigma}.$$

When we make this substitution and again avail ourselves of the relation (9) the equation (34) becomes

$$(35) \quad \left(\frac{2 + \omega}{1 + \omega} \right) \frac{d\omega}{d\sigma} = 2\omega\Omega.$$

If we neglect the small dimensionless quantity ω in (35) in comparison with unity and also replace ω by the wave strength w , which is equal to $|\omega|$ for the irrotational wave, we obtain

$$(36) \quad \frac{dw}{d\sigma} = w\Omega.$$

This is the final form of our differential equation for the variation of the strength of an irrotational wave during its propagation.

6. Application to equivoluminal waves. Let us now take

$$(37) \quad \frac{dv_i}{dt} = \frac{\partial^2 u_i}{\partial t^2},$$

which is the usual approximation formula for the acceleration components in this linear theory of elasticity. Then from (1), (28) and (37) we have

$$(38) \quad (\lambda + \mu)[u_{k,ki}] + \mu[u_{i,kk}] = \rho_0 \left[\frac{\partial^2 u_i}{\partial t^2} \right],$$

where use is made of the fact that the density ρ is continuous across the equivoluminal wave (see §2).

If we differentiate the relation $\omega_k \nu_k = 0$, which is valid over the surface of the equivoluminal wave, we find that

$$\omega_{k,\alpha} \nu_k = -\omega_k \nu_{k,\alpha} = \omega_k g^{\sigma\tau} b_{\sigma\alpha} x_{k,\tau}.$$

When this substitution is made in the second term in the right member of (25) it is easily seen that these relations reduce to

$$(39) \quad [u_{k,ki}] = (\hat{\omega}_k \nu_k + g^{\alpha\beta} \omega_{k,\alpha} x_{k,\beta}) \nu_i.$$

Hence from (23), (24), (38) and (39) we have

$$(40) \quad (\lambda + \mu)(\hat{\omega}_k \nu_k + g^{\alpha\beta} \omega_{k,\alpha} x_{k,\beta}) \nu_i + \mu(\hat{\omega}_i - 2\Omega \omega_i) = \rho_0 \left(G^2 \hat{\omega}_i - 2G \frac{\delta \omega_i}{\delta t} \right).$$

When we multiply (40) by ω_i and sum on the repeated index i the resulting relation becomes

$$(41) \quad \mu(\hat{\omega}_k \omega_k - 2\Omega \omega_k \omega_k) = \rho_0 G^2 \hat{\omega}_k \omega_k - 2\rho_0 G \omega_k \frac{\delta \omega_k}{\delta t}.$$

But, making the substitution

$$\frac{\delta \omega_k}{\delta t} = G \frac{d\omega_k}{d\sigma},$$

where σ , which was introduced in §5, is the arc length along the normals to the surface $\Sigma(t_0)$, and also taking account of the relation (10), valid for the equivoluminal wave, we find that (41) reduces to the relation

$$(42) \quad \omega_k \frac{d\omega_k}{d\sigma} = \omega_k \omega_k \Omega.$$

Finally when we introduce the wave strength w , defined by (30), the relation (42) gives

$$(43) \quad \frac{dw}{d\sigma} = w\Omega$$

as the differential equation for the variation in the strength of the equivoluminal wave.

7. Decay of waves. If the wave surfaces $\Sigma(t)$ form a family of parallel planes the mean curvature Ω vanishes at all times t and hence it follows from (36) and (43) that *plane irrotational and equivoluminal waves will show no variations in their strength during propagation.*

If $\Sigma(t)$ represents a family of concentric spherical surfaces propagated outward, *i.e.* such that the radius R increases during the propagation, the mean curvature $\Omega = -1/R$. Evidently σ can be replaced by R in the equations (36) and (43). Hence these equations become⁶

$$\frac{dw}{dR} = -\frac{w}{R}.$$

Integrating we find that the variation in the wave strength w of an irrotational or equivoluminal wave is given by

$$w = w_0 \left(\frac{R_0}{R} \right)$$

where w_0 is the wave strength when the sphere has the radius R_0 .

In the general case we have⁵

$$(44) \quad \Omega = \frac{\Omega_0 - K_0 \sigma}{1 - 2\Omega_0 \sigma + K_0 \sigma^2},$$

where K_0 and Ω_0 denote the Gaussian and mean curvatures of the wave surface $\Sigma(t_0)$ from which the arc length σ is measured. Making the substitution (44) in (36) or (43) and integrating this equation we now find that

$$(45) \quad w = \frac{w_0}{\sqrt{1 - 2\Omega_0 \sigma + K_0 \sigma^2}}$$

where w_0 is the wave strength over the surface $\Sigma(t_0)$.

It is easily seen from (45) that the strength w of an irrotational or an equivoluminal wave will decrease monotonically to zero as $\sigma \rightarrow \infty$, *i.e.* the wave will decay if, and only if, (a) $K_0 = 0$ and $\Omega_0 < 0$ or (b) $K_0 > 0$ and $\Omega_0 < 0$. The other *algebraic* possibility, namely $K_0 > 0$ and $\Omega_0 = 0$, cannot be realized since if Ω_0 vanishes on $\Sigma(t_0)$ this surface will be a minimal surface and for such surfaces the curvature K_0 must be negative. In the first instance $\Sigma(t_0)$ will be a developable surface (not a plane) and in the second it will be a *surface of positive curvature*. Moreover we see from (44) that under condition (a) or (b) the mean curvature Ω

⁶ We have chosen the unit vector ν to have the direction of the propagation of the wave surface. Hence the vector ν is the outwardly directed normal to the above spherical surfaces and in consequence of this fact the mean curvature Ω has the negative value $-1/R$ rather than the usual positive value $1/R$.

will remain negative during the propagation. Similarly it follows from the equation⁵, corresponding to (44), for the variation of the curvature K along the normals to the wave surfaces $\Sigma(t)$, that $K = 0$ under condition (a) and $K > 0$ under condition (b). Hence the wave surface $\Sigma(t)$ will be a developable surface or a surface of positive curvature throughout its propagation if the surface is a developable surface or a surface of positive curvature at the time t_0 .

With the above facts in mind we can now state the following result. *The strength of an irrotational or equivoluminal wave, propagated in an isotropic elastic solid⁷, will approach zero monotonically as the time of propagation increases indefinitely if, and only if, the wave surface is (α) a developable surface (not a plane) or (β) a surface of positive curvature and (γ) the direction of propagation, which is given by the unit normal vector ν to the surface, is such that the mean curvature is negative.*

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⁷ It is implicitly assumed here that the elastic solid extends indefinitely in the directions of the normals to the surface $\Sigma(t_0)$.