*On the Change of Natural Frequencies Induced by Small Constraints**

R. J. DUFFIN & A. SCHILD

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1. Introduction. When a constraint is added to a vibrating linear mechanical system the frequency of each normal mode tends to increase. It is the aim of this paper to show how this qualitative statement can be expressed in a quantitative form. A short outline of the applications of the theory presented here appeared in an earlier paper of the authors [1].

Consider a vibrating linear system with constraints which can be varied continuously. We wish to find the small increase in frequency resulting from a small increase of constraint. The original system will be called the *unconstrained system* S ; the system with the small additional constraints, the *constrained system* S' .

Our problem is resolved by the following simple formula

$$(A) \qquad \delta\lambda = \delta V.$$

Here λ is the square of the circular frequency of a given normal mode of vibration of S , and $\delta\lambda = \lambda' - \lambda$ is the increase in λ resulting from the increase in constraints. To maintain this increase in constraint, external forces of amplitude F must be applied to S ; the resulting vibration of S' is assumed to be suitably normalized. Now let the unconstrained system S be at rest. Let the forces F act to produce a static deflection. Then δV is the potential energy of S in this state of static deflection (Section 4).

In many cases the normalized mode of vibration is known for the constrained system S' . The forces F can then be deduced. Hence $\delta\lambda$ can be obtained without knowing any of the other constrained or unconstrained modes.

Formula (A) holds in the simple form given here when the eigenvalue λ' is non-degenerate. In the case of degenerate states the frequency changes are obtained by solving a secular equation (Section 5). Formula (A) is also valid with a dual interpretation of the potential energy δV in terms of the displace-

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ments of the original system in the region of the additional constraints (Section 6). In this formulation, $\delta\lambda$ can be obtained from the knowledge of a mode of vibration of the unconstrained system.

Formula (A) reduces the dynamic problem to a problem in statics. To facilitate the solution of this static problem, the following formula is employed:

$$(B) \qquad \delta V = \delta V^*.$$

Let S^* be a system which coincides with S in the neighborhood of the applied forces F of additional constraint. Outside of this neighborhood S^* is essentially arbitrary. Formula (B) asserts that, under the same forces F , the potential energies δV and δV^* acquired respectively by S and S^* are equal. An equivalent way of stating (B) is as follows: Close to a rigid constraint the stiffness of an elastic body S is essentially independent of other constraints which are far away. This is a general statement similar in nature to St. Venant's principle. In the applications S^* is taken to be an infinite system because this simplifies the analysis.

In the first part of this paper, Sections 2 to 6, an abstract proof of formula (A) is given, using the methods of Hilbert space. The essential hypothesis needed about S is that the kinetic energy quadratic form T be completely continuous relative to the potential energy quadratic form V .

The second part, Sections 7 to 9, gives some applications of the formulas (A) and (B). Section 7 deals with a vibrating string problem which, though quite trivial in itself, serves to tie down some of the abstract definitions and loose physical ideas to a specific and concrete case.

The last two sections deal with a membrane and a clamped plate problem. No attempt has been made there to discuss the detailed correspondence between the abstract formalism and its concrete realization, to verify the assumptions on which the abstract derivation of Formula (A) is based, or to justify the application of Formula (B). All this can be done without difficulty in the case of the membrane problem. For the plate problem we have not succeeded in obtaining rigorous proofs of all the assumptions made; this is because much less is known of the properties of the biharmonic equation than of the harmonic equation.

An interesting result, which follows from the investigation of the plate problem, is that the lowest mode of vibration of a simply connected clamped plate can have a nodal line. This is contrary to a conjecture of SZEGÖ [2].

The formula (A) has an interesting acoustical application which has not been included in this paper. Consider two rooms separated by a wall. If a small hole is cut in the wall, the frequency changes can be computed. In this case one of the new frequencies is much lower than all of the original frequencies. This vibration can be regarded as a perturbation of a trivial zero-frequency mode of the two uncoupled rooms.

Finally, it should be pointed out that formula (A) has limitations for certain cases of linear vibrating systems. Consider the electromagnetic vibrations of a

cavity with perfectly conducting walls. If the walls are slightly deformed outwards, or if a small hole is opened in the common wall of two such cavities, the resulting system corresponds neither to an increase nor to a decrease in constraints, but to a mixture of both. In the language of the abstract formulation of the following sections, neither of the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}'$ is contained in the other and, corresponding to Eq. (2), we have $\mathfrak{H} = \mathfrak{G} + \mathfrak{F}$, $\mathfrak{H}' = \mathfrak{G} + \mathfrak{F}'$, where $\mathfrak{G}$, $\mathfrak{F}$, $\mathfrak{F}'$ are mutually orthogonal. Further generalization of the methods of this paper would be desirable to include such cases [3, 4].

I. ABSTRACT FORMULATION

2. Potential Energy; Constraints. Consider a continuous mechanical system whose configuration (if bounded and smooth) may be described by a vector v in a suitable infinite dimensional real vector space $\tilde{\mathfrak{C}}$. Let the potential energy of the system be given by a positive definite quadratic form $V(v)$, with the associated symmetric bilinear form $V(v, w)$. Forming the closure of $\tilde{\mathfrak{C}}$ relative to $V(v)$ as a metric, a real Hilbert space $\mathfrak{C}$ is obtained. Of course, $\tilde{\mathfrak{C}}$ is dense in $\mathfrak{C}$. It is assumed that $\mathfrak{C}$ is separable.

Let $\mathfrak{S}$ be a subspace of $\tilde{\mathfrak{C}}$ resulting from a constraint on the system. Let the closure of $\mathfrak{S}$ be $\mathfrak{H}$. It is supposed that the amount of constraint is a function of a positive parameter ϵ . As ϵ decreases, the constraint increases. If $\epsilon_1 > \epsilon_2 > \epsilon_3 > \dots$ is a sequence approaching zero, then the corresponding closed subspaces satisfy $\mathfrak{S}_1 \supset \mathfrak{S}_2 \supset \mathfrak{S}_3 \supset \dots$. Any fixed vector v of $\mathfrak{C}$ can be written in the form $v = v_s + w_s$, where v_s is in $\mathfrak{S}_s$ and w_s is orthogonal to $\mathfrak{S}_s$. As $s \rightarrow \infty$, it is easily shown that v_s converges strongly to a vector v_∞ . The vectors v_∞ define a limit space $\mathfrak{S}_\infty$, which we assume is still infinite dimensional.

In the mechanical system ϵ may be thought of as the length (or area or volume) of a region over which the system is still not constrained. The case of complete constraint corresponds to $\epsilon = 0$. This is defined by a subspace $\mathfrak{S}'$ of $\tilde{\mathfrak{C}}$ with closure $\mathfrak{H}'$. Of course $\mathfrak{H} \supset \mathfrak{H}'$ for every positive ϵ .

On the basis of mechanical continuity it is to be expected that

$$(1) \quad \mathfrak{S}' = \mathfrak{S}_\infty.$$

This is the *first fundamental postulate* which we adopt.

The orthogonal complementary space of $\mathfrak{S}'$ in $\mathfrak{H}$ is denoted by $\mathfrak{F}$, i.e.,

$$(2) \quad \mathfrak{H} = \mathfrak{S}' + \mathfrak{F},$$

$$(3) \quad V(\mathfrak{S}', \mathfrak{F}) = 0.$$

The physical interpretation of the complementary vector space $\mathfrak{F}$ will now be discussed. The mechanical system characterized by $\mathfrak{H}$ will be called the *unconstrained system* S , that characterized by $\mathfrak{S}'$ the *constrained system* S' . The constrained system S' is obtained from the unconstrained system S by applying *additional constraints* over a region of the mechanical system which

depends on ϵ . Consider the static deflections of the unconstrained system S in equilibrium with arbitrary bounded force distributions over the region of additional constraints, with no forces acting outside this region. The vector space $\mathfrak{F}$ consists of the closure of the configuration vectors corresponding to these static deflections.

The following statement is equivalent to the fundamental postulate (1). Let $\mathfrak{F}_1, \mathfrak{F}_2, \dots$ be the sequence of spaces corresponding to the parameter sequence $\epsilon_1 > \epsilon_2 > \dots$ which approaches zero. Let $f_1, f_2, \dots$ be arbitrary unit vectors belonging respectively to $\mathfrak{F}_1, \mathfrak{F}_2, \dots$. Then f_s converges weakly to zero, *i.e.*, for any v in $\mathfrak{E}$,

$$(4) \quad V(f_s, v) \rightarrow 0.$$

The proof follows: Any vector v may be written as $v = v' + u_s + w_s$, where v' is in $\mathfrak{S}'$, u_s is in $\mathfrak{F}_s$, and w_s is orthogonal to $\mathfrak{F}_s$, so that $V(f_s, v) = V(f_s, u_s)$. First assume (1). Then u_s converges strongly to zero while f_s remains bounded. Thus $V(f_s, u_s) \rightarrow 0$, and this establishes (4). Conversely, assume (4), so that $V(f_s, u_s) \rightarrow 0$. Choose the unit vectors $f_s = u_s/[V(u_s)]^{\frac{1}{2}}$. Then $[V(u_s)]^{\frac{1}{2}} \rightarrow 0$. This proves that u_s converges strongly to zero and hence establishes (1).

For the sake of being definite the case of increasing constraint has been considered. The case of decreasing constraint can also be formulated. Here $\mathfrak{S}$ is fixed and $\mathfrak{S}'(\epsilon)$ is a function of ϵ . Equation (1) must be replaced by

$$(1') \quad \mathfrak{S} = \mathfrak{S}'_\infty,$$

and the other equations remain unchanged. Similarly, the case of merely variable constraint can be formulated, where $\mathfrak{S}$ and $\mathfrak{S}'$ are both functions of ϵ , with $\mathfrak{S}(\epsilon) \supset \mathfrak{S}'(\epsilon)$ and with a common limit space as ϵ tends to zero. A further generalization can be made by requiring that the potential energy quadratic form V be only semi-positive definite, but such that the vectors of zero norm span a finite dimensional subspace of $\mathfrak{E}$. The analysis to follow applies to these more general cases with only slight modifications of an obvious kind.

3. Kinetic Energy; Natural Vibrations. The kinetic energy of the system is given by a semi-positive definite quadratic form $T(\dot{v})$, where $\dot{v}$ is the velocity vector. In the applications given in the later sections T is in fact positive definite. However, in the general theory only semi-definiteness is assumed in order to include such systems as the inhomogeneous thin membrane with regions of zero mass density.

It is assumed that in the space $\mathfrak{E}$ the quadratic form $T(v)$ is completely continuous relative to $V(v)$. This is the *second fundamental postulate* which we adopt.

The natural vibrations of the unconstrained system S characterized by $\mathfrak{S}$ will now be considered.

When the system has simple harmonic motion with constant frequency and phase, then the amplitude of the velocity is ωv , where v is the amplitude

of the configuration vector of the system and ω is the circular frequency. The maximum kinetic energy is $\lambda T(v)$, where λ is the square of the circular frequency:

$$(5) \quad \lambda = \omega^2.$$

When the kinetic energy is a maximum the potential energy is zero and *vice versa*. Thus, by the conservation of energy, $V(v) = \lambda T(v)$ for a natural vibration. The permissible frequencies and configurations are given by the eigenvalues λ_n and eigenvectors q_n of a variational principle of mechanics which we now describe.

Let $T(v, w)$ denote the symmetric bilinear form associated with the quadratic form $T(v)$. Let $\lambda(v_1, v_2, \dots, v_{n-1})$ be the greatest lower bound of $V(q)/T(q)$ for all q of $\mathfrak{S}$ such that $T(q, v_1) = T(q, v_2) = \dots = T(q, v_{n-1}) = 0$. Then the eigenvalue λ_n is the least upper bound of $\lambda(v_1, v_2, \dots, v_{n-1})$ as $v_1, v_2, \dots, v_{n-1}$ vary in $\mathfrak{S}$. This is the well known minimax principle for the n^{th} eigenvalue [5]. In this formulation the n^{th} eigenvalue could be infinite.

From the standard theory, which makes use of the complete continuity of T , it follows that the finite eigenvalues form a finite or an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \dots$ with no finite limit point, and that there exists a corresponding sequence of eigenvectors $q_1, q_2, q_3, \dots$ such that

$$(6) \quad V(q_n) = \lambda_n,$$

$$(7) \quad T(q_n, q_m) = \delta_{nm}.$$

The eigenvalues and eigenvectors satisfy

$$(8) \quad V(q_n, v) - \lambda_n T(q_n, v) = 0$$

for any v in $\mathfrak{S}$. The completeness relation is

$$(9) \quad T(v, w) = \sum_{n=1}^{\infty} T(v, q_n) T(w, q_n)$$

for any v and w in $\mathfrak{S}$. In (9) and in what follows it is assumed that there are an infinite number of finite eigenvalues.

The natural vibrations of the constrained system S' characterized by $\mathfrak{S}'$ are given by eigenvalues λ'_n which are determined by a minimax principle analogous to the one above.

It follows directly from the two minimax principles that

$$(10) \quad \lambda_n \leq \lambda'_n.$$

Let us remember that the space $\mathfrak{S}$ and the whole eigenvalue problem of the unconstrained system depend on a parameter ϵ . Consider a sequence $\epsilon_1, \epsilon_2, \dots$ which approaches zero. Since $V(q_n) \leq \lambda'_n$, it may be supposed that the sequence of ϵ 's is so chosen that each q_n converges weakly in the metric V . Let the weak limits be denoted by q'_n . By virtue of the postulate (1) and of its equivalent

formulation (4) it is seen that the vectors q'_n are in the space $\mathfrak{S}'$. From the complete continuity of T it follows that $T(q_n - q'_n) \rightarrow 0$, so that Eq. (7) implies

$$(11) \quad T(q'_n, q'_m) = \delta_{nm}.$$

Moreover, by Eq. (10) and the properties of weak convergence, $V(q'_n) \leq \lambda'_n$. By the minimax definition of λ'_n we have $V(q'_n) \geq \lambda'_n$. Thus

$$(12) \quad V(q'_n) = \lambda'_n,$$

and the q'_n serve as a set of eigenvectors in $\mathfrak{S}'$. It can now be seen, by putting $v = q'_n$ in Eq. (8), that the eigenvalues of the unconstrained problem approach those of the constrained problem as $\epsilon \rightarrow 0$:

$$(13) \quad \lambda_n \rightarrow \lambda'_n,$$

and that the q_n converge strongly to the q'_n as $\epsilon \rightarrow 0$:

$$(14) \quad V(q_n - q'_n) \rightarrow 0.$$

Analogously to Eq. (8), the eigenvalue equation in $\mathfrak{S}'$ is

$$(15) \quad V(q'_n, v') - \lambda'_n T(q'_n, v') = 0$$

for any v' in $\mathfrak{S}'$.

The expression $V(q'_n, v) - \lambda'_n T(q'_n, v)$ defines a bounded linear functional in $\mathfrak{S}$. Thus there exists a vector ϕ_n in $\mathfrak{S}$ such that

$$(16) \quad V(q'_n, v) - \lambda'_n T(q'_n, v) = V(\phi_n, v)$$

for any v in $\mathfrak{S}$. From Eq. (15) it follows that ϕ_n is in $\mathfrak{F}$:

$$(17) \quad V(\phi_n, v') = 0$$

for any v' in $\mathfrak{S}'$. Equation (16) is the key relation for our treatment of the problem.

The physical interpretation of the vector ϕ_n will now be discussed. Consider the n^{th} natural vibration of the constrained system S' , whose configuration has the amplitude q'_n normalized by $T(q'_n) = 1$. This can be regarded as a forced vibration of the unconstrained system S under harmonically varying forces of amplitude F_n acting over the region of additional constraints. Now consider a *static deflection* of the unconstrained system S in equilibrium with the forces F_n . The vector ϕ_n gives the configuration of this static deflection. The potential energy of this equilibrium configuration is $V(\phi_n)$.

The eigenvector q_n of the unconstrained system can be decomposed into a component q''_n in the space $\mathfrak{S}'$ and a component ψ_n in the space $\mathfrak{F}$:

$$(18) \quad q_n = q''_n + \psi_n,$$

$$(19) \quad V(\psi_n, v') = 0$$

for any v' in $\mathfrak{S}'$.

The physical interpretation of the vector ψ_n is as follows. Consider the n^{th} natural vibration of the unconstrained system S , whose configuration has the amplitude q_n normalized by $T(q_n) = 1$. Now consider a *static deflection* of the unconstrained system S under forces F'_n which act over the region of additional constraints and which are such that the static deflection and the configuration q_n are the same in the region of additional constraints. The vector ψ_n gives the configuration of this static deflection. The potential energy of this equilibrium configuration is $V(\psi_n)$.

From the point of view of pure mathematics no physical interpretations need be given. Thus it is not necessary to give a physical interpretation to ϕ_n and ψ_n and the forces F_n and F'_n need not be introduced at all. For example, in the vibrating membrane problem (Section 8) the application of Green's theorem to (16) shows that ϕ_n can be defined as the solution of a certain mixed boundary value problem. Likewise ψ_n can be defined as the solution of a certain Dirichlet problem.

4. The First Fundamental Formula in the Non-degenerate Case. In Eq. (8) replace the suffix n by m and substitute q'_n for v . From this subtract Eq. (16) with q_m for v . This gives

$$(\lambda'_n - \lambda_m)T(q'_n, q_m) = -V(\phi_n, q_m).$$

Eq. (8) with n replaced by m and $v = \phi_n$ becomes

$$V(\phi_n, q_m) = \lambda_m T(\phi_n, q_m).$$

These two results give

$$(20) \quad (\lambda'_n - \lambda_m)T(q'_n, q_m) = -\lambda_m T(\phi_n, q_m).$$

Remembering that $V(q'_n, \phi_n) = 0$ by Eq. (17), the substitution $v = \phi_n$ in Eq. (16) gives

$$V(\phi_n) = -\lambda'_n T(q'_n, \phi_n).$$

By the completeness relation (9) this can be written

$$(21) \quad V(\phi_n) = -\sum_{m=1}^{\infty} \lambda'_n T(q'_n, q_m) T(\phi_n, q_m).$$

In this last relation substitute from Eq. (20) for $T(\phi_n, q_n)$ when $m = n$ and for $T(q'_n, q_m)$ when $m \neq n$. This gives

$$(22) \quad V(\phi_n) = \frac{\lambda'_n}{\lambda_n} (\lambda'_n - \lambda_n) [T(q'_n, q_n)]^2 + \sum'_{m=1}^{\infty} \frac{\lambda'_n \lambda_m}{\lambda'_n - \lambda_m} [T(\phi_n, q_m)]^2,$$

where the prime on Σ' denotes that the summation over m is to be carried out for values $m \neq n$ only.

It will now be shown that the last term on the right side of Eq. (22) is small compared to the left side $V(\phi_n)$. The assumption is made that the eigenvalue

problem of the constrained system is non-degenerate, so that $\lambda'_n \neq \lambda'_m$ whenever $n \neq m$. The degenerate case will be discussed in the following section. Since the λ'_m have no finite limit point and since λ_n approaches λ'_n as $\epsilon \rightarrow 0$, it is clear in the non-degenerate case, and for a sufficiently small ϵ , that $\lambda'_n - \lambda_m$ is bounded away from zero when n is fixed and m varies over all values other than n . Hence there exists a positive bound A_n such that

$$\left| \frac{\lambda'_n \lambda_m}{\lambda'_n - \lambda_m} \right| < A_n$$

for all m different from n . Therefore

$$\begin{aligned} \left| \sum_{m=1}^{\infty} \frac{\lambda'_n \lambda_m}{\lambda'_n - \lambda_m} [T(\phi_n, q_m)]^2 \right| &\leq A_n \sum_{m=1}^{\infty} [T(\phi_n, q_m)]^2 \\ &\leq A_n \sum_{m=1}^{\infty} [T(\phi_n, q_m)]^2 = A_n T(\phi_n). \end{aligned}$$

This can also be written

$$(23) \quad \sum_{m=1}^{\infty} \frac{\lambda'_n \lambda_m}{\lambda'_n - \lambda_m} [T(\phi_n, q_m)]^2 = O[T(\phi_n)].$$

Consider the unit vector $\phi_n/[V(\phi_n)]^{\frac{1}{2}}$ in the space $\mathfrak{F}$. By Eq. (4) this vector converges weakly to zero as $\epsilon \rightarrow 0$. From the complete continuity of T it then follows that

$$(24) \quad T\left(\frac{\phi_n}{[V(\phi_n)]^{\frac{1}{2}}}\right) = \frac{T(\phi_n)}{V(\phi_n)} \rightarrow 0$$

as $\epsilon \rightarrow 0$. This, combined with Eq. (23), gives the desired result:

$$(25) \quad \sum_{m=1}^{\infty} \frac{\lambda'_n \lambda_m}{\lambda'_n - \lambda_m} [T(\phi_n, q_m)]^2 = o[V(\phi_n)].$$

Using this result and the fact that λ'_n/λ_n and $T(q'_n, q_n)$ both tend to unity as $\epsilon \rightarrow 0$, Eq. (22) becomes

$$(26) \quad \lambda'_n - \lambda_n = V(\phi_n) + o[V(\phi_n)].$$

This is the *first fundamental formula* in the non-degenerate case. It is a precise statement of the formula (A) of the introduction.

5. The First Fundamental Formula in the Degenerate Case. For the study of degeneracy in the constrained eigenvalue problem the new double suffix notation $n = N\alpha$ will be introduced. The first suffix $N = 1, 2, \dots, \infty$ enumerates the distinct eigenvalues λ'_N , $0 < \lambda'_1 < \lambda'_2 < \dots$. The second suffix $\alpha = 1, 2, \dots, n_N$ enumerates the independent eigenvectors $q'_{N\alpha} : q'_{N1}, q'_{N2}, \dots, q'_{Nn_N}$, which belong to the degenerate eigenvalue λ'_N .

The eigenvalues of the unconstrained problem which correspond to λ'_N will be denoted by $\lambda_{N\alpha}$. Since, by Eq. (13),

$$(27) \quad \lambda_{N\alpha} \rightarrow \lambda'_N$$

as $\epsilon \rightarrow 0$, it follows for a sufficiently small ϵ that the unconstrained problem cannot have more degeneracy than the constrained problem, *i.e.*, $\lambda_{N\alpha} \neq \lambda_{M\beta}$ if $N \neq M$. However, it can have less degeneracy, so that $\lambda_{N\alpha}$ and $\lambda_{N\beta}$ may be different when $\alpha \neq \beta$. The eigenvectors of the unconstrained problem are denoted by $q_{N\alpha}$.

From the general form of the fundamental formula (26) and from the nature of the vector ϕ_n it is clear that our point of view is essentially as follows: We regard as known the solution of the constrained problem and the quantities λ'_N , $q'_{N\alpha}$. We then ask for the eigenvalues $\lambda_{N\alpha}$ of the unconstrained problem when ϵ is small but non-zero.

We therefore change our notation from that used in Section 3. The eigenvalues λ'_N and eigenvectors $q'_{N\alpha}$ constitute any normalized solution of the constrained problem, so that Eqs. (11), (12), (15), (16), and (17) remain valid. Also Eq. (13) or (27) still holds. But we do not assume that the $q'_{N\alpha}$ are the strong limits of the $q_{N\alpha}$, so that Eq. (14) is not necessarily true. Without loss of generality, the strong limits of the $q_{N\alpha}$ may be assumed to exist and will now be denoted by $\tilde{q}'_{N\alpha}$:

$$(28) \quad V(q_{N\alpha} - \tilde{q}'_{N\alpha}) \rightarrow 0$$

as $\epsilon \rightarrow 0$. The $\tilde{q}'_{N\alpha}$ satisfy the normalization condition

$$(29) \quad T(\tilde{q}'_{N\alpha}, \tilde{q}'_{M\beta}) = \delta_{NM} \delta_{\alpha\beta}.$$

It follows from the general theory that the $\tilde{q}'_{N\gamma}$ must be linear combinations of the $q'_{N\alpha}$:

$$(30) \quad \tilde{q}'_{N\gamma} = \sum_{\alpha=1}^{n_N} A_{N\alpha\gamma} q'_{N\alpha}.$$

As a consequence of the normalizations (11) and (29) the coefficients $A_{N\alpha\gamma}$ are given by

$$(31) \quad A_{N\alpha\gamma} = T(q'_{N\alpha}, \tilde{q}'_{N\gamma})$$

and satisfy the orthogonality relations

$$(32) \quad \sum_{\alpha=1}^{n_N} A_{N\alpha\beta} A_{N\alpha\gamma} = \delta_{\beta\gamma}.$$

The derivations of the fundamental formula in the degenerate case is parallel to that given in the previous section. In the double suffix notation Eq. (20) becomes

$$(33) \quad (\lambda'_N - \lambda_{M\gamma}) T(q'_{N\alpha}, q_{M\gamma}) = -\lambda_{M\gamma} T(\phi_{N\alpha}, q_{M\gamma}).$$

Eq. (21) has to be generalized. Substitute $n = N\alpha$ and $v = \phi_{N\beta}$ in Eq. (16) to obtain

$$(34) \quad \begin{aligned} V(\phi_{N\alpha}, \phi_{N\beta}) &= -\lambda'_N T(q'_{N\alpha}, \phi_{N\beta}) \\ &= -\sum_{M=1}^{\infty} \sum_{\gamma=1}^{nM} \lambda'_N T(q'_{N\alpha}, q_{M\gamma}) T(\phi_{N\beta}, q_{M\gamma}) \end{aligned}$$

by the completeness relation (9). In the right hand side substitute from Eq. (33) for $T(\phi_{N\beta}, q_{N\gamma})$ when $M = N$ and for $T(q'_{N\alpha}, q_{M\gamma})$ when $M \neq N$. This gives

$$(35) \quad \begin{aligned} V(\phi_{N\alpha}, \phi_{N\beta}) &= \sum_{\gamma=1}^{nN} \frac{\lambda'_N}{\lambda_{N\gamma}} (\lambda'_N - \lambda_{N\gamma}) T(q'_{N\alpha}, q_{N\gamma}) T(q'_{N\beta}, q_{N\gamma}) \\ &\quad + \sum_{M=1}^{\infty} \sum_{\gamma=1}^{nM} \frac{\lambda'_N \lambda_{M\gamma}}{\lambda'_N - \lambda_{M\gamma}} T(\phi_{N\alpha}, q_{M\gamma}) T(\phi_{N\beta}, q_{M\gamma}), \end{aligned}$$

where the prime on Σ' denotes that the summation over M is to be carried out for values $M \neq N$ only.

As in the previous section, $\lambda'_N - \lambda_{M\gamma}$ is bounded away from zero for $M \neq N$ and for ϵ sufficiently small. By Schwarz's inequality it follows that the last term of Eq. (35) is $O\{[T(\phi_{N\alpha})]^{\frac{1}{2}}[T(\phi_{N\beta})]^{\frac{1}{2}}\}$. But

$$[T(\phi_{N\alpha})]^{\frac{1}{2}}[T(\phi_{N\beta})]^{\frac{1}{2}} \leq \max_{\alpha} T(\phi_{N\alpha}),$$

and by Eq. (24) it is seen that the last term of Eq. (35) is small compared to $\max_{\alpha} V(\phi_{N\alpha})$:

$$(36) \quad \sum_{M=1}^{\infty} \sum_{\gamma=1}^{nM} \frac{\lambda'_N \lambda_{M\gamma}}{\lambda'_N - \lambda_{M\gamma}} T(\phi_{N\alpha}, q_{M\gamma}) T(\phi_{N\beta}, q_{M\gamma}) = o[\max_{\alpha} V(\phi_{N\alpha})].$$

By Eqs. (28) and (31)

$$(37) \quad T(q'_{N\alpha}, q_{N\gamma}) \rightarrow A_{N\alpha\gamma}$$

as $\epsilon \rightarrow 0$, and $\lambda'_N/\lambda_{N\gamma} \rightarrow 1$.

Equation (35) can therefore be written

$$(38) \quad V(\phi_{N\alpha}, \phi_{N\beta}) = \sum_{\gamma=1}^{nN} \mu_{N\gamma} A_{N\alpha\gamma} A_{N\beta\gamma},$$

where

$$(39) \quad \lambda'_N - \lambda_{N\gamma} = \mu_{N\gamma} + o[\max_{\alpha} V(\phi_{N\alpha})].$$

Equation (38) shows that the orthogonal transformation $A_{N\alpha\gamma}$ relates the matrix $V(\phi_{N\alpha}, \phi_{N\beta})$ to its diagonal form $\delta_{\alpha\beta} \mu_{N\alpha}$. Thus the $\mu_{N\alpha}$ must be the roots of the secular equation

$$(40) \quad \det [V(\phi_{N\alpha}, \phi_{N\beta}) - \mu_N \delta_{\alpha\beta}] = 0.$$

The last two equations constitute the *first fundamental formula* in the degenerate case.

If the limits $\tilde{q}'_{N\alpha}$ of the eigenvectors $q_{N\alpha}$ of the unconstrained problem are known in advance, then it is unnecessary to solve a secular equation such as (40). The frequency changes are given by

$$(41) \quad \lambda'_N - \lambda_{N\alpha} = V(\tilde{\phi}_{N\alpha}) + o[\max_{\beta} V(\tilde{\phi}_{N\beta})],$$

where $\tilde{\phi}_{N\alpha}$ is the ϕ vector associated with the eigenvector $\tilde{q}'_{N\alpha}$ of the constrained problem.

6. The Second Fundamental Formula. In order to derive the second fundamental formula, a *third fundamental postulate* will be adopted. It states that with respect to the kinetic energy quadratic form T the eigenvectors q'_n of the constrained system form a complete set in $\mathfrak{H}$:

$$(42) \quad T(v, w) = \sum_{n=1}^{\infty} T(v, q'_n)T(w, q'_n).$$

The physical content of this postulate is that the kinetic energy of the particles in the region of additional constraints is always zero. This is clearly so in the membrane, plate, and acoustic problems discussed in the Introduction and in the following sections. There the region of additional constraints consists of a gap of zero measure which therefore contains zero mass. On the other hand, in the simple vibrating string problem of Section 7 the region of additional constraints contains a finite amount of mass so that the assumption (42) does not hold. In this case the following derivation of the second fundamental formula is not valid.

As before, Eqs. (8) and (16) give

$$(\lambda'_m - \lambda_n)T(q_n, q'_m) = -V(\phi_m, q_n).$$

By Eq. (18) the right hand side equals $-V(\phi_m, \psi_n)$, since $V(\phi_m, q'_n) = 0$. By Eq. (16) this becomes $\lambda'_m T(q'_m, \psi_n)$, since $V(q'_m, \psi_n) = 0$. Thus

$$(43) \quad (\lambda'_m - \lambda_n)T(q_n, q'_m) = \lambda'_m T(q'_m, \psi_n).$$

When there is no degeneracy, consider

$$V(\psi_n) = V(q_n - q'_n, \psi_n) = V(q_n, \psi_n).$$

By Eq. (8) this becomes

$$(44) \quad V(\psi_n) = \lambda_n T(q_n, \psi_n).$$

By the completeness relation (42) this can be written

$$(45) \quad V(\psi_n) = \sum_{m=1}^{\infty} \lambda_n T(q_n, q'_m)T(\psi_n, q'_m).$$

Substituting from Eq. (43),

$$(46) \quad V(\psi_n) = \frac{\lambda_n}{\lambda'_n} (\lambda'_n - \lambda_n) [T(q_n, q'_n)]^2 + \sum_{m=1}^{\infty} \frac{\lambda_n \lambda'_m}{\lambda'_m - \lambda_n} [T(\psi_n, q'_m)]^2.$$

The argument is now quite analogous to that of Section 4. The last term of Eq. (46) is $O[T(\psi_n)]$ and therefore $o[V(\psi_n)]$. It follows that

$$(47) \quad \lambda'_n - \lambda_n = V(\psi_n) + o[V(\psi_n)].$$

This is the *second fundamental formula* in the non-degenerate case. It is another precise formulation of the formula (A) of the Introduction.

In the case of degeneracy the derivation is quite similar to that of Section 5 and will not be given. Perhaps the one important difference is the following. It is now more useful to think in terms of a system with decreasing constraints. The unique unconstrained problem has distinct eigenvalues λ_N and eigenvectors $q_{N\alpha}$ and its solution is regarded as known. The constrained problem depends on the parameter ϵ and, for small values of this parameter, may have less degeneracy but can have no more than the unconstrained problem. The eigenvalues and eigenvectors are $\lambda'_{N\alpha}$, $q'_{N\alpha}$.

The *second fundamental formula* in the degenerate case is given by the secular equation

$$(48) \quad \det [V(\psi_{N\alpha}, \psi_{N\beta}) - \mu_N \delta_{\alpha\beta}] = 0,$$

$$(49) \quad \lambda'_{N\gamma} - \lambda_N = \mu_{N\gamma} + o[\max_{\alpha} V(\psi_{N\alpha})].$$

If the limits $\tilde{q}_{N\alpha}$ of the eigenvectors $q'_{N\alpha}$ of the constrained problem are known in advance, then the frequency changes are given by the simpler formula

$$(50) \quad \lambda'_{N\alpha} - \lambda_N = V(\tilde{\psi}_{N\alpha}) + o[\max_{\beta} V(\tilde{\psi}_{N\beta})].$$

II. APPLICATIONS

7. The Vibrating String. To illustrate the abstract theory we shall apply it first to the almost trivial problem of a uniform vibrating string. This problem is sufficiently simple so that some details can be given of the correspondence between the abstract formalism and its concrete realization. Also the solutions are known explicitly so that the approximate change in frequency given by the formula can be compared to the rigorous value. It is perhaps worth noting that the methods of this section apply with suitable changes to the less trivial case of a string with a variable mass distribution along its length.

The unconstrained system S consists of a string stretched between the points A at $x = 0$ and B at $x = \pi$, as shown in Fig. 1. In the constrained system S' , shown in Fig. 2, the string is prevented from moving in an interval BB' of length ϵ . Clearly this is a case of decreasing constraints. For a suitable choice of the physical units the linear mass density and the tension in the string will both be unity.

Choose $\tilde{\mathfrak{E}} = \tilde{\mathfrak{F}}$ to be the space of all functions $v(x)$ defined over the closed interval $[0, \pi]$ which vanish at the endpoints, which are continuous, and which have piecewise continuous first derivatives. The potential energy is given by the positive definite quadratic form

Fig. 1**Unconstrained System S.** **Fig. 2****Constrained System S'.**

$$(51) \quad V(v) = \frac{1}{2} \int_0^\pi \left(\frac{dv}{dx} \right)^2 dx.$$

The Hilbert space $\mathfrak{H}$ is the closure of $\tilde{\mathfrak{H}}$ with respect to the metric V . It consists of all functions $v(x)$ which are the integrals

$$(52) \quad v(x) = \int_0^x \bar{v}(x) dx$$

of associated functions $\bar{v}(x)$ in $L_2[0, \pi]$ with zero mean:

$$(53) \quad \int_0^\pi \bar{v} dx = 0.$$

The bilinear form V is given by

$$(54) \quad V(v, w) = \frac{1}{2} \int_0^\pi \bar{v} \bar{w} dx.$$

The functions $v(x)$ are all continuous in the closed interval $[0, \pi]$ and vanish at the endpoints

$$(55) \quad v(0) = v(\pi) = 0.$$

The Hilbert space $\mathfrak{H}'$ consists of those functions $v'(x)$ of $\mathfrak{H}$ which are identically zero in the interval $[\pi - \epsilon, \pi]$. Without loss of generality it can be assumed that they are the integrals of associated functions $\bar{v}'$ in L_2 which are identically zero in the interval of constraint,

$$(56) \quad \bar{v}' = 0 \quad \text{for} \quad \pi - \epsilon < x \leq \pi,$$

and which satisfy the condition of zero mean

$$(57) \quad \int_0^{\pi-\epsilon} \bar{v}' dx = 0.$$

The complementary space $\mathfrak{F}$ consists of functions f of $\mathfrak{S}$, with associated functions $\bar{f}$, such that

$$V(v', f) = \frac{1}{2} \int_0^{\pi-\epsilon} \bar{v}' \bar{f} dx = 0$$

for any $\bar{v}'$ in L_2 which satisfies Eq. (57). It follows that

$$(58) \quad \bar{f} = \bar{f}^{(1)}, \text{ a constant, for } 0 \leq x < \pi - \epsilon.$$

In the interval of constraint $\bar{f}$ can be arbitrary except for the condition of zero mean (53). The continuous functions $f(x)$ therefore start out as straight lines through the origin until $x = \pi - \epsilon$, then assume essentially arbitrary shapes, and finally return to zero when $x = \pi$. A typical function f of $\mathfrak{F}$ is shown in Fig. 3. It clearly represents the static deflection of the string AB under suitable forces which act only in the interval $B'B$ of constraint.

Fig. 3
Function $f(x)$ in $\mathfrak{F}$

The first fundamental postulate in the form of Eq. (4) will now be verified. Consider a fixed function v of $\mathfrak{S}$, with associated function $\bar{v}$, and a function f of $\mathfrak{F}$ which depends on ϵ as a parameter and satisfies the normalizing condition

$$V(f) = \frac{1}{2} \int_0^\pi \bar{f}^2 dx = 1.$$

Then

$$V(v, f) = \frac{1}{2} \int_0^{\pi-\epsilon} \bar{v} \bar{f} dx + \frac{1}{2} \int_{\pi-\epsilon}^\pi \bar{v} \bar{f} dx.$$

By Eqs. (58) and (53) the first term on the right side gives

$$\frac{1}{2} \int_0^{\pi-\epsilon} \bar{v} \bar{f} dx = \frac{1}{2} \bar{f}^{(1)} \int_0^{\pi-\epsilon} \bar{v} dx = -\frac{1}{2} \bar{f}^{(1)} \int_{\pi-\epsilon}^\pi \bar{v} dx \rightarrow 0$$

as $\epsilon \rightarrow 0$, since $\bar{v}$ is independent of the interval of integration ϵ and $\bar{f}^{(1)}$ is bounded. Applying Schwarz's inequality to the second term on the right side and using the normalizing condition gives

$$\left| \frac{1}{2} \int_{\pi-\epsilon}^\pi \bar{v} \bar{f} dx \right|^2 \leq \left(\frac{1}{2} \int_{\pi-\epsilon}^\pi \bar{v}^2 dx \right) \left(\frac{1}{2} \int_{\pi-\epsilon}^\pi \bar{f}^2 dx \right) \leq \frac{1}{2} \int_{\pi-\epsilon}^\pi \bar{v}^2 dx \rightarrow 0$$

as $\epsilon \rightarrow 0$. Thus $V(v, f) \rightarrow 0$, and this establishes the first fundamental postulate.

The kinetic energy is given by the positive definite form

$$(59) \quad T(v, w) = \frac{1}{2} \int_0^\pi vw \, dx.$$

It is well known that T is completely continuous with respect to V , so that the second fundamental postulate holds. The details of the proof will not be given. It can be obtained easily from the fact that the functions $v(x)$ of $\mathfrak{S}$ have Fourier series of the form

$$(60) \quad v = \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx, \quad \sum_{n=1}^{\infty} a_n^2 < \infty,$$

and that

$$(61) \quad V(v) = \frac{\pi}{4} \sum_{n=1}^{\infty} a_n^2, \quad T(v) = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{a_n^2}{n^2}.$$

We shall also accept as known that the eigenvalue problem (8) has eigenfunctions $q_n(x)$ which have second derivatives throughout the interval $(0, \pi)$. Then Eq. (8) becomes

$$(62) \quad \int_0^\pi \frac{dq_n}{dx} \bar{v} \, dx - \lambda_n \int_0^\pi q_n v \, dx = 0$$

for any v in $\mathfrak{S}$. Integrating by parts, this becomes

$$(63) \quad \left[\frac{dq_n}{dx} v \right]_0^\pi - \int_0^\pi \left[\frac{d^2 q_n}{dx^2} + \lambda_n q_n \right] v \, dx = 0.$$

By Eq. (55) the first term drops out, and since v is an arbitrary function of $\mathfrak{S}$ the vanishing of the integral leads to the differential equation

$$(64) \quad \frac{d^2 q_n}{dx^2} + \lambda_n q_n = 0 \quad \text{for } 0 < x < \pi,$$

with the boundary conditions

$$(65) \quad q_n(0) = q_n(\pi) = 0.$$

This is the standard formulation of the eigenvalue problem for the vibrating string.

Similarly, in the case of the constrained string, the eigenvalue problem (15) leads to the standard formulation

$$(66) \quad \frac{d^2 q'_n}{dx^2} + \lambda'_n q'_n = 0 \quad \text{for } 0 < x < \pi - \epsilon,$$

$$(67) \quad q'_n(0) = q'_n(\pi - \epsilon) = 0,$$

and where q'_n vanishes identically in the interval $[\pi - \epsilon, \pi]$. The solutions of this simple eigenvalue problem are

$$(68) \quad \lambda'_n = \left(\frac{n\pi}{\pi - \epsilon} \right)^2,$$

$$(69) \quad q'_n = \frac{2}{(\pi - \epsilon)^{\frac{1}{2}}} \sin \frac{n\pi}{\pi - \epsilon} x, \quad T(q'_n) = \frac{1}{2} \int_0^{\pi - \epsilon} q_n'^2 dx = 1.$$

The function ϕ_n is in the complementary space $\mathfrak{F}$ and is determined by Eq. (16) which gives

$$(70) \quad \int_0^{\pi - \epsilon} \frac{dq'_n}{dx} \bar{v} dx - \lambda'_n \int_0^{\pi - \epsilon} q'_n v dx = \int_0^{\pi} \bar{\phi}_n \bar{v} dx,$$

since q'_n is differentiable and vanishes in $[\pi - \epsilon, \pi]$. Integrating the first term by parts and using Eqs. (66), (52), (55), and (58), this becomes

$$\left(\frac{dq'_n}{dx} v \right)_{\pi - \epsilon} = \bar{\phi}_n^{(1)} v_{\pi - \epsilon} + \int_{\pi - \epsilon}^{\pi} \bar{\phi}_n \bar{v} dx$$

for any $\bar{v}$ in L_2 which, by Eqs. (52) and (53), satisfies

$$\int_{\pi - \epsilon}^{\pi} \bar{v} dx = -v_{\pi - \epsilon}.$$

It follows that ϕ_n must also be a constant $\bar{\phi}_n^{(2)}$ in the interval $(\pi - \epsilon, \pi]$ and that the discontinuity at $x = \pi - \epsilon$ is given by

$$(71) \quad \bar{\phi}_n^{(1)} - \bar{\phi}_n^{(2)} = \left(\frac{dq'_n}{dx} \right)_{\pi - \epsilon}.$$

Also the condition (53) must be satisfied, so that

$$(72) \quad (\pi - \epsilon) \bar{\phi}_n^{(1)} + \epsilon \bar{\phi}_n^{(2)} = 0.$$

Therefore the function ϕ_n consists of two straight line segments, respectively passing through $x = 0$ and $x = \pi$, and has a discontinuity of the slope at $x = \pi - \epsilon$ given by Eq. (71). The function is shown in Fig. 4. Its value at $x = \pi - \epsilon$ is

$$(73) \quad (\phi_n)_{\pi - \epsilon} = \frac{\epsilon(\pi - \epsilon)}{\pi} \left(\frac{dq'_n}{dx} \right)_{\pi - \epsilon} = (-1)^n \frac{2\epsilon n}{(\pi - \epsilon)^{\frac{1}{2}}}.$$

If the vibration q'_n (*cf.* Fig. 2) is regarded as a forced vibration of the unconstrained string AB , then a force

$$(74) \quad F_n = \left(\frac{dq'_n}{dx} \right)_{\pi - \epsilon} = (-1)^n \frac{2n\pi}{(\pi - \epsilon)^{\frac{1}{2}}}$$

must act at B' . The unconstrained string AB in static deflection under such a load F_n has just the shape given by $\phi_n(x)$ and shown in Fig. 4.

The approximate change in frequency, as given by the first fundamental formula (26), is

$$(75) \quad \lambda'_n - \lambda_n = V(\phi_n) = \frac{1}{2} \int_0^{\pi} \left(\frac{d\phi_n}{dx} \right)^2 dx.$$

Fig. 4**The Function $\phi_n(x)$.**

The right side can easily be computed directly, but it is instructive to perform an integration by parts which gives

$$\lambda'_n - \lambda_n = \frac{1}{2}(\phi_n)_{\pi-\epsilon} (\bar{\phi}_n^{(1)} - \bar{\phi}_n^{(2)}).$$

By Eqs. (71) and (74) this can be written

$$(76) \quad \lambda'_n - \lambda_n = \frac{1}{2}(\phi_n)_{\pi-\epsilon} F_n.$$

This is the expression for the potential energy of an elastic system deformed by a single point force which increases linearly with displacement to a maximum value F_n .

Substituting from Eqs. (73) and (74), the following explicit expression is obtained for the change in frequency:

$$(77) \quad \lambda'_n - \lambda_n = \frac{2\epsilon\pi n^2}{(\pi - \epsilon)^2} + o(\epsilon) = \frac{2n^2}{\pi} \epsilon + o(\epsilon).$$

The eigenvalues $\lambda_n = n^2$ of the unconstrained problem S (Eqs. (64) and (65)) are of course well known, and the result (77) agrees with the rigorous expression

$$(78) \quad \lambda'_n - \lambda_n = n^2 \left(1 - \frac{\epsilon}{\pi}\right)^{-2} - n^2 = \frac{2n^2}{\pi} \epsilon + O(\epsilon^2).$$

To show how the formula (B) can be applied in this simple case, consider the system S^* in which the point A has been allowed to recede to $x = -\infty$. Under the force F_n at B' the static deflection ϕ_n^* is as shown in Fig. 5, and

$$(79) \quad (\phi_n^*)_{\pi-\epsilon} = \epsilon \left(\frac{dq'_n}{dx} \right)_{\pi-\epsilon} = (-1)^n \frac{2n\epsilon\pi}{(\pi - \epsilon)^{\frac{3}{2}}}.$$

Fig. 5

**The Function $\phi_n^*(x)$
Distorted System S^* .**

This gives the frequency change

$$\lambda'_n - \lambda_n = V(\phi_n^*) + o(\epsilon) = \frac{1}{2}(\phi_n^*)_{\pi-\epsilon} F_n + o(\epsilon) = \frac{2\epsilon\pi^2 n^2}{(\pi - \epsilon)^3} + o(\epsilon),$$

in agreement with the expressions (77) and (78).

8. The Vibrating Membrane. The unconstrained system S consists of the uniform, thin, elastic membrane, shown in Fig. 6, in the shape of two contiguous

Fig. 6
Unconstrained System S.

plane regions $R^{(1)}$ and $R^{(2)}$ with a common straight boundary l and joined by a small gap in l of length 2ϵ . $B^{(1)}$ denotes the portion of the boundary curve of $R^{(1)}$ which does not include the gap 2ϵ ; $B^{(2)}$ is similarly defined. For a suitable choice of the physical units the areal mass density and the tension will both be unity.

The corresponding constrained system S' , shown in Fig. 7, is obtained by closing the gap 2ϵ , so that S' consists of two disconnected membranes vibrating independently, one in the region $R^{(1)}$ and the other in $R^{(2)}$. For simplicity it is assumed that the individual regions $R^{(1)}$ and $R^{(2)}$ are such that their normal vibrations are non-degenerate and their frequencies are distinct. Then the constrained eigenvalue problem will be non-degenerate.

The quadratic forms can be taken to be

$$V(v) = \frac{1}{2} \int_{R^{(1)} + R^{(2)}} (\nabla v)^2 dx dy$$

and

$$T(v) = \frac{1}{2} \int_{R^{(1)} + R^{(2)}} v^2 dx dy.$$

Fig. 7
Constrained System S'

The unconstrained eigenvalue problem is given by the differential equation

$$(80) \quad \nabla^2 q_n + \lambda_n q_n = 0 \quad \text{inside } R^{(1)}, R^{(2)} \text{ and on gap } 2\epsilon$$

and the boundary conditions

$$(81) \quad q_n = 0 \quad \text{on } B^{(1)}, B^{(2)}.$$

The constrained eigenvalue problem is given by the differential equation

$$(82) \quad \nabla^2 q'_n + \lambda'_n q'_n = 0 \quad \text{inside } R^{(1)}, R^{(2)}$$

and the boundary conditions

$$(83) \quad q'_n = 0 \quad \text{on } B^{(1)}, B^{(2)} \text{ and on gap } 2\epsilon.$$

Because of the non-degeneracy of the constrained problem, the eigenfunction q'_n will be non-zero in only one of the two regions $R^{(1)}$ and $R^{(2)}$, say in $R^{(1)}$, so that

$$(84) \quad \begin{aligned} q'_n &= q_n^{(1)'} \quad \text{inside } R^{(1)} & (y > 0) \\ &= 0 \quad \text{inside } R^{(2)} & (y < 0). \end{aligned}$$

The normalization (11) gives

$$(85) \quad T(q'_n) = \frac{1}{2} \int_{R^{(1)}} [q_n^{(1)'}]^2 dx dy = 1.$$

At the maximum deflection q'_n the additional constraints consist of a linear force density along the gap given by

$$(86) \quad F_n = -\left(\frac{\partial q_n^{(1)'}}{\partial y}\right)_{y=0}, \quad -\epsilon \leq x \leq \epsilon.$$

Under such a force distribution along the gap the static deflection ϕ_n of the unconstrained system S is determined by the differential equation

$$(87) \quad \nabla^2 \phi_n = 0 \quad \text{inside } R^{(1)}, R^{(2)},$$

where

$$(88) \quad \phi_n = 0 \quad \text{on } B^{(1)}, B^{(2)}$$

and where ϕ_n is continuous along the gap but has a discontinuity in its normal derivative given by

$$(89) \quad \left(\frac{\partial \phi_n}{\partial y} \right)_{y \rightarrow +0} - \left(\frac{\partial \phi_n}{\partial y} \right)_{y \rightarrow -0} = -F_n \quad \text{for } -\epsilon < x < \epsilon.$$

Green's theorem shows that ϕ_n so defined satisfies the key relation (16). Once this static problem is solved, the approximate change in frequency is given by the first fundamental formula

$$(90) \quad \lambda'_n - \lambda_n = V(\phi_n) = \frac{1}{2} \int_{-\epsilon}^{\epsilon} (\phi_n)_{y=0} F_n dx.$$

The function $q_n^{(1) '}$ describes the vibration of the single region $R^{(1)}$ when the gap is closed. Therefore $q_n^{(1) '}$ and its derivatives are continuous along the gap and, since ϵ is small, the force density (86) can be replaced by its constant value at the center O ($x = y = 0$) of the gap:

$$(91) \quad F_n^* = - \left(\frac{\partial q_n^{(1) '}}{\partial y} \right)_O.$$

For small values of ϵ the static deflection ϕ_n is limited to the immediate neighborhood of the gap. It is determined by the part of the boundary at distances of order ϵ from the gap and is independent of the shape of the boundary at finite distances from the gap. Therefore, to the lowest order in ϵ , ϕ_n can be replaced by ϕ_n^* which gives the static deflection of the membrane S^* , shown in

Fig. 8
Distorted System S^*

Fig. 8, under the uniform force distribution F_n^* along the gap. The function ϕ_n^* is determined by the differential equation

$$(92) \quad \nabla^2 \phi_n^* = 0 \quad \text{for } y > 0 \quad \text{and } y < 0,$$

where

$$(93) \quad \begin{aligned} \phi_n^* &= 0 \quad \text{for } y = 0, \quad |x| > \epsilon, \\ \phi_n^* &\rightarrow 0 \quad \text{as } x^2 + y^2 \rightarrow \infty, \end{aligned}$$

and where ϕ_n^* is continuous along the gap but has a discontinuity in its normal derivative given by

$$(94) \quad \left(\frac{\partial \phi_n^*}{\partial y} \right)_{y \rightarrow +0} - \left(\frac{\partial \phi_n^*}{\partial y} \right)_{y \rightarrow -0} = -F_n^* \quad \text{for } -\epsilon < x < \epsilon.$$

To the lowest order in ϵ the approximate change in frequency between the vibrations of S and S' is given by

$$(95) \quad \lambda'_n - \lambda_n = V(\phi_n^*) = \frac{1}{2} F_n^* \int_{-\epsilon}^{\epsilon} (\phi_n^*)_{y=0} dx.$$

Consider now the standardized problem of an infinite membrane Σ constrained along two half lines which are separated by a gap of length 2 as shown in Fig. 9.

Fig. 9

Standardized System Σ .

Let Φ be the static deflection of this membrane produced by a unit force density acting along the gap, so that

$$(96) \quad \nabla^2 \Phi = 0 \quad \text{for } y > 0 \quad \text{and } y < 0,$$

where

$$(97) \quad \begin{aligned} \Phi &= 0 \quad \text{for } y = 0, \quad |x| > 1, \\ \Phi &\rightarrow 0 \quad \text{as } x^2 + y^2 \rightarrow \infty, \end{aligned}$$

and where Φ is continuous along the gap but has a discontinuity in its normal derivative given by

$$(98) \quad \left(\frac{\partial \Phi}{\partial y} \right)_{y \rightarrow +0} - \left(\frac{\partial \Phi}{\partial y} \right)_{y \rightarrow -0} = -1 \quad \text{for} \quad -1 < x < 1.$$

The potential energy of this standardized deflection is given by the constant

$$(99) \quad c = \frac{1}{2} \int_{-1}^1 (\Phi)_{y=0} dx.$$

By means of the standardized deflection Φ the function ϕ_n^* can be expressed in the form

$$(100) \quad \phi_n^*(x, y) = -\epsilon \left(\frac{\partial q_n^{(1)'}}{\partial y} \right)_o \Phi \left(\frac{x}{\epsilon}, \frac{y}{\epsilon} \right).$$

This is easily seen to satisfy Eqs. (92), (93), and (94). Substitution into Eq. (95) and use of Eq. (99) give

$$(101) \quad \lambda'_n - \lambda_n = \epsilon^2 \left(\frac{\partial q_n^{(1)'}}{\partial y} \right)_o^2 c + o(\epsilon^2).$$

This is the main result of the present section. The formula (101) gives, to the lowest order in ϵ , the change in the frequency of any mode in terms of the known vibrations of the constrained system and in terms of a single constant c .

A similar result will be obtained in the following section for the clamped plate. Here, in the simpler case of the membrane, the constant c can in fact be obtained explicitly in terms of integrals.

The Green's function for the standardized membrane of Fig. 9 can be obtained by a conformal mapping and is given by

$$(102) \quad G(x, y; x_0, y_0) = -\frac{1}{2\pi} \operatorname{Re} \left[\log \frac{(z + i\sqrt{1-z^2}) - (z_0 + i\sqrt{1-z_0^2})}{(z + i\sqrt{1-z^2}) - (\bar{z}_0 - i\sqrt{1-\bar{z}_0^2})} \right],$$

where $z = x + iy$, $z_0 = x_0 + iy_0$, $\bar{z}_0 = x_0 - iy_0$. The Green's function G gives the static deflection of the standardized membrane Σ at the point x, y if a unit concentrated load acts at the point x_0, y_0 . The deflection Φ , under a uniform linear force density acting along the gap, is given by

$$(103) \quad \Phi(x, y) = \int_{-1}^1 G(x, y; x_0, 0) dx_0.$$

Finally, Eq. (99) determines the constant c :

$$(104) \quad c = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 G(x, 0; x_0, 0) dx dx_0.$$

9. The Vibrating Clamped Plate. The last application will be to a clamped plate problem which is similar to the membrane problem of the preceding section. However, in order to give an example of a degenerate eigenvalue problem and

also to establish a special result about nodal lines, the regions $R^{(1)}$ and $R^{(2)}$ of Figs. 6 and 7 will now be assumed to be congruent and reflections of one another in the line l .

The unconstrained system S consists of the uniform, thin, elastic plate shown in Fig. 10 which is clamped along the boundary $B^{(1)}$ and $B^{(2)}$.

Fig. 10
Unconstrained System S.

The corresponding constrained system S' , shown in Fig. 11, is obtained by additional clamping along the gap 2ϵ , so that S' consists of two identical disconnected clamped plates vibrating independently, one in the region $R^{(1)}$ and the other in $R^{(2)}$. For simplicity it is assumed that the individual region $R^{(1)}$ is such that its normal vibrations are non-degenerate. Then the constrained eigenvalue problem S' of the combined region $R^{(1)} + R^{(2)}$ is exactly twofold degenerate. Corresponding to each eigenvalue λ'_N there are two eigenfunctions. One eigenfunction ($q_N'^{+}$) may be taken to be even with respect to the line l , the other ($q_N'^{-}$) odd:

$$(105) \quad \begin{aligned} q_N'^{+}(x, y) &= q_N'^{+}(x, -y), \\ q_N'^{-}(x, y) &= -q_N'^{-}(x, -y). \end{aligned}$$

Fig. 11
Constrained System S'

The unconstrained system S also has symmetry of reflection in the line l . Therefore the eigenfunctions will again be even (q_N^+) or odd (q_N^-). But in general these will now have different eigenvalues λ_N^+ and λ_N^- . As $\epsilon \rightarrow 0$, it is clear that the even functions q_N^+ of the unconstrained problem tend to the even functions $q_N^{'+}$ of the constrained problem, and similarly for the odd functions. We thus have a degenerate case where it is unnecessary to solve a secular equation and where the first fundamental formula (41) applies.

For a suitable choice of the physical units the unconstrained eigenvalue problem is given by the differential equation

$$(106) \quad \nabla^4 q_N^* - \lambda_N^* q_N^* = 0 \quad \text{inside } R^{(1)}, R^{(2)} \text{ and on gap } 2\epsilon$$

and the boundary conditions

$$(107) \quad q_N^* = \frac{\partial q_N^*}{\partial n} = 0 \quad \text{on } B^{(1)}, B^{(2)},$$

where $\partial/\partial n$ denotes the normal derivative.

For the clamped plate the quadratic forms can be taken as (see [5; Vol. 1, p. 349; Vol. 2, p. 528])

$$V(v) = \frac{1}{2} \int_{R^{(1)} + R^{(2)}} (\nabla^2 v)^2 dx dy$$

and

$$T(v) = \frac{1}{2} \int_{R^{(1)} + R^{(2)}} v^2 dx dy.$$

The constrained eigenvalue problem is given by the differential equation

$$(108) \quad \nabla^4 q_N'' - \lambda_N' q_N'' = 0 \quad \text{inside } R^{(1)}, R^{(2)}$$

and the boundary conditions

$$(109) \quad q_N'' = \frac{\partial q_N''}{\partial n} = 0 \quad \text{on } B^{(1)}, B^{(2)} \text{ and on gap } 2\epsilon.$$

The normalization (11) gives

$$(110) \quad T(q_N'') = \frac{1}{2} \int_{R^{(1)} + R^{(2)}} [q_N'']^2 dx dy = \int_{R^{(1)}} [q_N'']^2 dx dy = 1.$$

In the N^{th} even vibration of the constrained plate the amplitude of the constraining forces along the gap consists of a linear force density

$$(111) \quad F_N = 2 \left(\frac{\partial^3 q_N''}{\partial y^3} \right)_{y \rightarrow +0}, \quad -\epsilon < x < \epsilon.$$

The x derivative term vanishes by (109). In the N^{th} odd vibration the amplitude of the constraining forces along the gap consists of a linear density of force moments

$$(112) \quad M_N = -2 \left(\frac{\partial^2 q_N''}{\partial y^2} \right)_{y \rightarrow +0}, \quad -\epsilon < x < \epsilon.$$

Again the x derivative term vanishes by (109).

Under such force distributions along the gap, the static deflections ϕ_N'' of the unconstrained system S are determined by the differential equation

$$(113) \quad \nabla^4 \phi_N'' = 0 \quad \text{inside } R^{(1)}, R^{(2)},$$

where

$$(114) \quad \phi_N'' = \frac{\partial \phi_N''}{\partial n} = 0 \quad \text{on } B^{(1)}, B^{(2)}.$$

In the even case ϕ_N'' is even so that ϕ_N'' and $\partial^2 \phi_N'' / \partial y^2$ are continuous along the gap 2ϵ , and

$$(115) \quad \left(\frac{\partial \phi_N''}{\partial y} \right)_{y=0} = 0 \quad \text{for } -\epsilon < x < \epsilon,$$

$$(116) \quad \left(\frac{\partial^3 \phi_N''}{\partial y^3} \right)_{y \rightarrow +0} = - \left(\frac{\partial^3 \phi_N''}{\partial y^3} \right)_{y \rightarrow -0} = \frac{1}{2} F_N \quad \text{for } -\epsilon < x < \epsilon.$$

In the odd case ϕ_N^- is odd so that $\partial\phi_N^-/\partial y$ and $\partial^3\phi_N^-/\partial y^3$ are continuous along the gap 2ϵ , and

$$(117) \quad (\phi_N^-)_{y=0} = 0 \quad \text{for} \quad -\epsilon < x < \epsilon,$$

$$(118) \quad \left(\frac{\partial^2\phi_N^-}{\partial y^2}\right)_{y \rightarrow +0} = -\left(\frac{\partial^2\phi_N^-}{\partial y^2}\right)_{y \rightarrow -0} = -\frac{1}{2}M_N \quad \text{for} \quad -\epsilon < x < \epsilon.$$

Green's theorem shows that ϕ_N^* so defined satisfy the key relation (16). Once these static problems are solved, the approximate changes in frequency are given by the first fundamental formula

$$(119) \quad \lambda_N' - \lambda_N^+ = V(\phi_N^+) = \frac{1}{2} \int_{-\epsilon}^{\epsilon} (\phi_N^+)_{y=0} F_N dx,$$

$$(120) \quad \lambda_N' - \lambda_N^- = V(\phi_N^-) = \frac{1}{2} \int_{-\epsilon}^{\epsilon} \left(\frac{\partial\phi_N^-}{\partial y}\right)_{y=0} M_N dx.$$

As in the previous case of the membrane, the force and moment densities (111) and (112) can be replaced by their constant values at the center of the gap:

$$(121) \quad F_N^* = 2 \left(\frac{\partial^3 q_N'}{\partial y^3} \right)_{O+},$$

$$(122) \quad M_N^* = -2 \left(\frac{\partial^2 q_N'}{\partial y^2} \right)_{O+},$$

where the subscript $O+$ indicates the limit of the expression as $x = 0$ and as $y \rightarrow 0$ through positive values

To the lowest order in ϵ the potential energies $V(\phi_N^*)$ are respectively equal to the potential energies $V(\phi_N^{*\pm})$ of the static deflections of an infinite plate S^* , clamped along two half lines as shown in Fig. 8, and under the uniform force distributions (121) or (122) along the gap. These even and odd static deflections $\phi_N^{*\pm}$ are determined by differential equations and boundary conditions similar to Eqs. (113) to (118):

$$(123) \quad \begin{aligned} \nabla^4 \phi_N^{*\pm} &= 0 \quad \text{for} \quad y > 0 \quad \text{and} \quad y < 0, \\ \phi_N^{*\pm} &= \frac{\partial\phi_N^{*\pm}}{\partial y} = 0 \quad \text{for} \quad y = 0, \quad |x| > \epsilon, \\ \nabla\phi_N^{*\pm} &\rightarrow 0 \quad \text{as} \quad x^2 + y^2 \rightarrow \infty, \\ \frac{\partial\phi_N^{*+}}{\partial y} &= 0 \quad \text{for} \quad y = 0, \quad -\epsilon < x < \epsilon, \\ \left(\frac{\partial^3\phi_N^{*+}}{\partial y^3}\right)_{y \rightarrow +0} &= \frac{1}{2}F_N^* \quad \text{for} \quad -\epsilon < x < \epsilon, \\ \phi_N^{*-} &= 0 \quad \text{for} \quad y = 0, \quad -\epsilon < x < \epsilon, \\ \left(\frac{\partial^2\phi_N^{*-}}{\partial y^2}\right)_{y \rightarrow +0} &= -\frac{1}{2}M_N^* \quad \text{for} \quad -\epsilon < x < \epsilon. \end{aligned}$$

To the lowest order in ϵ the change in frequencies between the vibrations of S and S' are given by

$$(124) \quad \lambda'_N - \lambda_N^+ = V(\phi_N^{*+}) = \frac{1}{2} F_N^* \int_{-\epsilon}^{\epsilon} (\phi_N^{*+})_{y=0} dx,$$

$$(125) \quad \lambda'_N - \lambda_N^- = V(\phi_N^{*-}) = \frac{1}{2} M_N^* \int_{-\epsilon}^{\epsilon} \left(\frac{\partial \phi_N^{*-}}{\partial y} \right)_{y=0} dx.$$

Consider now the standardized problem of an infinite plate Σ clamped along two half lines which are separated by a gap of length 2 as shown in Fig. 9. The even static deflection Φ^+ produced by a unit force density acting along the gap and the odd static deflection Φ^- produced by a unit moment density acting along the gap are determined by

$$(126) \quad \begin{aligned} \nabla^4 \Phi^* &= 0 \quad \text{for } y > 0 \quad \text{and } y < 0, \\ \Phi^* = \frac{\partial \Phi^*}{\partial y} &= 0 \quad \text{for } y = 0, \quad |x| > 1, \\ \nabla \Phi^* &\rightarrow 0 \quad \text{as } x^2 + y^2 \rightarrow \infty, \\ \frac{\partial \Phi^+}{\partial y} &= 0 \quad \text{for } y = 0, \quad -1 < x < 1, \\ \left(\frac{\partial^3 \Phi^+}{\partial y^3} \right)_{y \rightarrow +0} &= \frac{1}{2} \quad \text{for } -1 < x < 1, \\ \Phi^- &= 0 \quad \text{for } y = 0, \quad -1 < x < 1, \\ \left(\frac{\partial^2 \Phi^-}{\partial y^2} \right)_{y \rightarrow +0} &= -\frac{1}{2} \quad \text{for } -1 < x < 1. \end{aligned}$$

The potential energies of these standardized deflections are given by the two constants

$$(127) \quad c^+ = \frac{1}{2} \int_{-1}^1 (\Phi^+)_{y=0} dx,$$

$$(128) \quad c^- = \frac{1}{2} \int_{-1}^1 \left(\frac{\partial \Phi^-}{\partial y} \right)_{y=0} dx.$$

Comparing the systems (123) and (126), it is clear that

$$(129) \quad \phi_N^{*+}(x, y) = 2\epsilon^3 \left(\frac{\partial^3 q_N^+}{\partial y^3} \right)_{0+} \Phi^+ \left(\frac{x}{\epsilon}, \frac{y}{\epsilon} \right),$$

$$(130) \quad \phi_N^{*-}(x, y) = -2\epsilon^2 \left(\frac{\partial^2 q_N^-}{\partial y^2} \right)_{0+} \Phi^- \left(\frac{x}{\epsilon}, \frac{y}{\epsilon} \right).$$

Substitution into Eqs. (124), (125) and use of Eqs. (121), (122) give

$$(131) \quad \lambda'_N - \lambda_N^+ = 4\epsilon^4 \left(\frac{\partial^3 q_N^+}{\partial y^3} \right)_{o+} c^+ + o(\epsilon^4),$$

$$(132) \quad \lambda'_N - \lambda_N^- = 4\epsilon^2 \left(\frac{\partial^2 q_N^-}{\partial y^2} \right)_{o+} c^- + o(\epsilon^2).$$

These are the main results of the present section. They give the changes in frequency of any mode in terms of the known vibrations of the constrained system and in terms of two constants c^* , which are independent of the particular mode of vibration and independent of the geometry of the region $R^{(1)}$.

We can now use these results to deal with a special problem in the theory of vibrating plates. In a recent paper [2] SZEGÖ advanced the hypothesis that there are no nodal lines in the lowest mode of vibration of a plane clamped plate. Using this hypothesis, he showed that for clamped plates of a given area the circular plate has the lowest frequency of vibration.

Subtracting Eq. (131) from Eq. (132), with $N = 1$, gives

$$(133) \quad \lambda_1^+ - \lambda_1^- = 4\epsilon^2 \left(\frac{\partial^2 q_1^-}{\partial y^2} \right)_{o+} c^- + o(\epsilon^2).$$

Apart from the exceptional case when the second normal derivative $\partial^2 q_1^- / \partial y^2$ vanishes at the point of the boundary of $R^{(1)}$ where the gap is located, the right hand side of Eq. (133) is positive when ϵ is sufficiently small. Thus

$$(134) \quad \lambda_1^- < \lambda_1^+,$$

and the lowest mode of vibration of a clamped plate in the shape of the simply connected region of Fig. 10 is the odd mode q_1^- . This, like all odd modes, has a nodal line along the gap $y = 0$, $-\epsilon < x < \epsilon$. Thus SZEGÖ's hypothesis cannot be generally true. A specific counter example has also been found [6].

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Westinghouse Research Laboratories
Carnegie Institute of Technology
Pittsburgh, Pennsylvania
and
The University of Texas
Austin, Texas