A Mathematical Foundation for Radiative Transfer Theory

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Communicated by E. HOPF

I. INTRODUCTION

Radiative transfer theory is the quantitative study, on a phenomenological level, of the transfer of radiant energy through media that absorb, scatter, or emit radiant energy. This paper* presents a set of physically motivated axioms phrased in the language of measure theory from which, as a special case, the salient features of radiative transfer theory may be rigorously deduced. The manifold problems of the theory are then subject to rigorous scrutiny by means of the powerful tools of modern analysis, notably measure theory and functional analysis, and thereby the gap between the pure and applied aspects of the mathematics of radiative transfer theory may be closed.

The need for a precise formulation of the notions of radiative transfer has become acute in the past several years. The ramification of the tenets of radiative transfer in particular to the problems of neutron transport theory has given rise to a flurry of mathematical work on the existence and structure of the solutions of the equation of transfer and notions such as group theory [19] and Hilbert space [13] have been pressed into use, all of which are directly transferable to the radiative transfer context. However, the foundations have remained untouched and the problem formulated by HOPF ([9], p. 101) over twenty years ago still stands: "The entire subject ought to be treated anew, and the fundamental equations be derived in a rigorous way, the main tool being the general theory of measure (in the sense of Lebesgue-Radon)."

A perusal of the long history of radiative transfer theory would reveal the enormous number of ramifications of the tenets of the theory and their connections with the foundations of physics from the early classical dynamics of gases to recent quantum theory, neutron diffusion theory, and probability theory. A sketch of the history and an extended bibliography may be found in [17]

* Scripps Contribution No. 953. This work has been supported by the Bureau of Ships of the U.S. Navy under contracts NObsr-43356 and NObs-72092.

along with detailed heuristic discussions leading to the present axiomatic formulations.

In the interests of conceptual simplicity the present formulation will consider only the abstract counterpart to unpolarized radiant energy. It is to be noted, however, that the formal inclusion of the phenomenon of polarization introduces no essential modifications in the mathematical topography of the results (in the classical case, see, *e.g.*, [2], pp. 37–45). We will merely note here that when the notion of the Stokes vector is abstracted, and included in the present formalism, the discussions are elevated from the scalar to the vector level. As a consequence, certain close conceptual relationships may be shown to exist between $\mathcal{R}_*$ (the radiative process) and the Mueller phenomenological algebra for optics on both the *set* and *point* levels in the carrier space.

The mathematical core of present day radiative transfer theory rests in the integro-differential equation:

$$\frac{1}{v} \frac{DN}{Dt} = -\alpha N + \int_{\Xi} N \sigma \, d\Omega + N_{\eta}, \quad (1)$$

which is the *classical equation of transfer* for N . Values of the function N are called the *specific intensity*, or *radiance* of radiation: the time rate of change (the flow) of radiant energy across a unit area in a unit solid angle in a given direction normal to the area. It is customarily assumed that the basic carrier of the radiant energy is a photon. Nuclear physicists concerned with neutron transport theory view (1) as the linearized form of the Boltzmann equation [3], and the role of the photon is replaced by that of the neutron. Pure mathematicians concerned with the probability theory view (1) and related variants of (1) as a representation of certain Markoff processes [5]–[7], [10]. The domain of definition of N is some well defined subset of Φ of Euclidean 5-space E_5 , where Φ is the Cartesian product $X \times \Xi$ of a given subset X of E_3 and the unit sphere Ξ in E_3 . The functions α and σ are the *volume attenuation* and *volume scattering functions* respectively with domains on X and $X \times \Xi \times \Xi$ respectively. v is the *velocity function* with domain X . N_{η} is the *emission function* defined on Φ . All of these functions have as their range the subset of non-negative real numbers; it is implicitly understood that they are in addition dependent on wavelength and time. D/Dt is the *Lagrangian* (or *mobile*) *derivative operator* associated with Φ .

From the point of view of the mathematician radiative transfer takes place in the measure space $(\Phi, \mathfrak{P}, \nu)$, where $\mathfrak{P}$ is the family of Lebesgue measurable subsets of Φ and ν is the *classical optical volume measure* on $\mathfrak{P}$ defined by $\nu = \int n^3 \, dp$, where p is the usual Lebesgue measure on $\mathfrak{P}$ and n is the (non-negative real-valued bounded) *index of refraction function* on X . It should be observed that p is the completed product measure of the *volume measure* V on $\mathbf{X}$ and the *solid angle measure* Ω on Ξ , where $\mathbf{X}$ and Ξ are the families of Lebesgue measurable subsets of X and Ξ respectively. $(\Phi, \mathfrak{P}, \nu)$ is the *classical carrier space*. The *classical radiative measure* U on $\mathfrak{P}$ is defined by $U = \int (N/v) \, dp$, so that

$$U(X_0 \times \Xi) \equiv U(X_0) = \int_{x_0 \times \Xi} \left(\frac{N}{v} \right) dp$$

is the radiant energy content customarily associated with the subset X_0 of X . In measure-theoretic parlance, N/v is the Radon-Nikodym derivative of U with respect to p . The Radon-Nikodym derivative of U with respect to ν is denoted by G and is clearly related to N by the expression $G = N/n^2 c [p]$, where c is the universal constant associated with the speed of light in a vacuum. G has the advantage of being invariant along the natural path of a stream of photons in a transparent isotropic optical medium and is related to the fundamental Straubel invariant [18].

The principal results of the paper may now be stated as follows:

- (i) An explication, by means of measure-theoretic concepts, of the notions of *optical medium* (the carrier space axiom C_1 , §1), *radiant energy* (the radiative measure axiom M_1 , §2), the *evolution* of radiant energy in an optical medium (radiative process axiom R_1 , §3), and the *transfer* of radiant energy in an optical medium (transfer process axiom T_1 , §18).
- (ii) The formal deduction from C_1 , M_1 , R_1 , and T_1 of the existence and salient properties of G (and hence N) (§2), α and σ (§§13, 14), and ν (§21).
- (iii) A rigorous derivation of the most general homogeneous form of (1) in an arbitrary carrier space (§§15, 17) and necessary and sufficient conditions which yield as a special case the correct classical homogeneous form of (1) (§21).

Several intermediate results, mainly of mathematical interest, are obtained as the discussion is carried toward the principal goals, such as:

- (iv) The *static transition relation* for relative radiative measures (§4) which is the most primitive form of the equation of transfer and which describes the radiative process in an arbitrary carrier space. It is the radiative transfer counterpart to the Chapman-Kolmogoroff transition relation for transition probabilities [10].
- (v) The *dynamic transition relation* for relative measures (§12) which in the present formulations is the first recognizable counterpart to the classical equation of transfer on a set level.
- (vi) The *static* (§9) and *dynamic* (§15) transition relations for certain densities g and γ which complement the general density G .

The measure-theoretic terminology and notation used in the sequel follows, wherever possible, that given in the standard treatise [8].

II. RADIATIVE PROCESSES

1. Carrier Space Axiom. Let X be a non-empty set and $\mathbf{S}$ a σ -algebra of subsets of X . Let ν be a totally σ -finite measure on $\mathbf{S}$ such that $\nu(X) > 0$. The measure space $(X, \mathbf{S}, \nu)$ will be referred to as the *carrier space*, and in the interests

of brevity we will occasionally indulge in the customary symbolic confusion of X and the triple $(X, \mathbf{S}, \nu)$. The measure ν will be called the *carrier measure*. The elements of $\mathbf{S}$ are the *measurable sets* of X . The elements of X are the *points* of X .

C_1 : *There exists a carrier space $(X, \mathbf{S}, \nu)$.*

2. Radiative Measure Axiom. Let $(\Lambda, \mathbf{L}, l)$ denote the measure space consisting of the subset Λ of non-negative real numbers, the usual σ -algebra $\mathbf{L}$ of Lebesgue measurable subsets of Λ , and Lebesgue measure l . A point $\lambda \in \Lambda$ will be called the *wavelength* λ .

Let $(T, +, <)$ denote the additive semigroup consisting of the subset T of non-negative real numbers with the customary binary operation $+$ of addition and linear ordering $<$. Occasionally limit arguments involving elements of T will be required. These will be with respect to the order topology induced by $<$. A point $t \in T$ will be called the *time* t .

Suppose $\mathfrak{M}$ is the set of all measures on $\mathbf{S}$. $\mathfrak{M}$ is not empty since $\nu \in \mathfrak{M}$. Let $\mathfrak{M}_\nu$ denote the subset of $\mathfrak{M}$ consisting of all totally finite ν -continuous (*i.e.*, absolutely continuous with respect to ν) measures. $\mathfrak{M}_\nu$ is not empty since the zero measure is in $\mathfrak{M}_\nu$. The elements of $\mathfrak{M}$ will be referred to as the *radiative measures* on $\mathbf{S}$. We will say that a collection M of measures on $\mathbf{T}$ of a measurable space $(Y, \mathbf{T})$ forms a *non-negative cone* iff (if and only if) for any two $m_1, m_2 \in M$ and for any two non-negative real numbers r_1, r_2 , $r_1 m_1 + r_2 m_2 \in M$, where scalar multiplication and addition of elements of M are defined in the obvious way. $\mathfrak{M}_\nu$ forms a non-negative cone.

If m is a measure on $\mathbf{T}$ of a measurable space $(Y, \mathbf{T})$, we define for each $E \in \mathbf{T}$ the set function $\bar{m}_E$ on $\mathbf{T}$ by the relation $\bar{m}_E(F) = m(E \cap F)$ for $F \in \mathbf{T}$. Clearly $\bar{m}_E$ is a measure on $\mathbf{T}$. Finally, let $f(\cdot, \cdot)$ be a function defined on the product $X \times Y$ of two sets X and Y and with range in some set Z . If $x \in X$, the function $f(x, \cdot)$ defined on Y is an *X-section* of f determined by $x \in X$; similarly, if $y \in Y$, the function $f(\cdot, y)$ defined on X is a *Y-section* of f determined by $y \in Y$. For arbitrary but given $(x, y) \in X \times Y$ the two sections $f(x, \cdot)$, $f(\cdot, y)$ of f will be called *corresponding functions*.

- M_1 : (i) *For every $(t, \lambda) \in T \times \Lambda$, there is a non-empty subset $\mathfrak{M}_{\nu, t, \lambda} \subset \mathfrak{M}_\nu$, which forms a non-negative cone.*
 (ii) *If $\mu \in \mathfrak{M}_\nu$ and if for every $E \in \mathbf{S}$ there is a $\mu_1 \in \mathfrak{M}_{\nu, t, \lambda}$ and an $E_1 \in \mathbf{S}$ such that $\mu(E) = \mu_1(E_1)$, then $\mu \in \mathfrak{M}_{\nu, t, \lambda}$.*
 (iii) *If $\mu(t, \lambda, \cdot) \in \mathfrak{M}_{\nu, t, \lambda}$ and t is fixed, the corresponding function $\mu(t, \cdot, E)$, $E \in \mathbf{S}$, is Lebesgue integrable on Λ .*

As an immediate consequence of (ii) M_1 , if $\mu \in \mathfrak{M}_{\nu, t, \lambda}$ and $F \in \mathbf{S}$, then $\bar{\mu}_F \in \mathfrak{M}_{\nu, t, \lambda}$. For $\bar{\mu}_F \in \mathfrak{M}_\nu$, and for every $E \in \mathbf{S}$, there is a $\mu_1 \in \mathfrak{M}_{\nu, t, \lambda}$, namely μ , and an $E_1 \in \mathbf{S}$, namely $E \cap F$, such that $\bar{\mu}_F(E) = \mu(E \cap F)$.

λ -convention: In the sequel, the wavelength $\lambda \in \Lambda$ will be considered arbitrary but fixed; the λ will be dropped from the notation.

Let $\mu \in \mathfrak{M}$, $E \in \mathbf{S}$. E is said to *support* μ (or is a *support* for μ) iff $\mu(E) > 0$. E is said to *fully support* μ (or is a *full support* for μ) iff $F \in \{G: G \in \mathbf{S}, \nu(E \cap G) > 0\}$ implies $\mu(E \cap F) > 0$. E is said to *maximally fully support* μ (or is a *maximal full support* for μ) iff (a) E is a full support for μ and (b) $F \in \mathbf{S}$ implies $\mu(F) \leq \mu(E)$.

If $E \in \mathbf{S}$ supports $\mu \in \mathfrak{M}_\nu$, the set function μ_E on $\mathbf{S}$ defined by $\mu_E(F) = \mu(E \cap F)/\mu(E)$ is an element of $\mathfrak{M}_\nu$. We use the convention that $\mu_E \equiv 0$ on $\mathbf{S}$ if E does not support μ . μ_E will be referred to as a *relative (conditional) radiative measure*. Similarly, for $E \in \mathbf{S}$, $\mu \in \mathfrak{M}_\nu$, the set function $\bar{\mu}_E \in \mathfrak{M}_\nu$ will be called an *absolute (conditional) radiative measure*.

Theorem A. *Let $\mu \in \mathfrak{M}_\nu$. If X supports μ , then there exists a unique $[\nu]$ maximal full support for μ .*

Proof: Since $\mu \in \mathfrak{M}_\nu$, by the Radon-Nikodym theorem ([8], p. 128), there exists a finite-valued non-negative measurable function f on X unique $[\nu]$ such that $\mu(E) = \int_E f d\nu$ for every $E \in \mathbf{S}$. Define $F = \{x: f(x) > 0\}$. Since $\mu(X) > 0$, F is a set of positive ν -measure, is unique $[\nu]$, and $\int_X f d\nu = \int_F f d\nu$. If $E \in \mathbf{S}$ and $\nu(F \cap E) > 0$, then $\mu(F \cap E) = \int_{F \cap E} f d\nu > 0$, so that F fully supports μ . Finally, if $E \in \mathbf{S}$, then

$$\mu(E) = \int_E f d\nu \leq \int_X f d\nu = \int_F f d\nu = \mu(F),$$

so that F is a maximal full support for μ . This completes the proof.

For any $\mu \in \mathfrak{M}_{\nu,t}$, the value $\mu(E)$ of μ at $E \in \mathbf{S}$ will be called the *radiant energy in E* (at $t \in T$ and at $\lambda \in \Lambda$). If the carrier space supports μ , the preceding theorem asserts that the radiant energy in X may be associated with a unique $[\nu]$ measurable set which maximally fully supports μ . The R-N (Radon-Nikodym) derivative of μ will be called the *general radiant density function on X associated with μ* , and its value at $x \in X$ the *general radiant density at x* (at $t \in T$ and at $\lambda \in \Lambda$). Paraphrasing the proof of the theorem, we may say that a radiative measure supported by the carrier space determines a unique $[\nu]$ general radiant density function on X . If the radiative measure is not supported by X , it is clear that the general radiant density function vanishes a.e. $[\nu]$ on X . We collect these observations in the following

Corollary. *Every radiative measure uniquely $[\nu]$ determines a general radiant density function on X .*

3. Radiative Process Axiom. If $\{\mu_n\}$ is a sequence of measures in $\mathfrak{M}$, the sequence will be said to *converge to the measure $\mu \in \mathfrak{M}$* , in symbols $\mu_n \rightarrow \mu \in \mathfrak{M}$, iff $\lim_n \mu_n(E) = \mu(E)$ for every $E \in \mathbf{S}$. If $\{\mu_n\}$ is a sequence of measures in $\mathfrak{M}$, which converges to $\mu \in \mathfrak{M}$, then $\mu \ll \nu$; for if $F \in \mathbf{S}$ and $\nu(F) = 0$, $0 = \lim_n \mu_n(F) = \mu(F)$. Let $\mathcal{R}$ be the set of all linear transformations from the non-negative cone $\mathfrak{M}_+$ into $\mathfrak{M}_+$. $\mathcal{R}$ is not empty: the identity transformation I_ϕ is

an element of $\mathfrak{R}$. The image of μ under $R \in \mathfrak{R}$ will be written as μR . Let

$$D = \{(t_1, t_2) : (t_1, t_2) \in T \times T, t_2 \geq t_1\}.$$

R_1 : There exists a transformation ϕ from D into $\mathfrak{R}$ such that $\phi(D) \equiv \mathfrak{R}_\phi$ has the following properties. Let $\phi[(t_1, t_2)] \equiv R_{t_1, t_2}$, $(t_1, t_2) \in D$, then:

- (i) $\mu \in \mathfrak{N}_{\nu, t_1}$ implies $\mu R_{t_1, t_2} \in \mathfrak{N}_{\nu, t_2}$.
- (ii) If t_1, t_2, t_3 are elements of T and if $t_1 \leq t_2 \leq t_3$, then $R_{t_1, t_2} = R_{t_1, t_2} R_{t_2, t_3}$.
- (iii) For every $t \in T$, $R_{t, t} = I_\phi$.
- (iv) If $\{\mu_n\}$ is a sequence of measures in $\mathfrak{N}_{\nu, t}$ and $\mu_n \rightarrow \mu \in \mathfrak{N}_{\nu, t}$, then for every $(t, t') \in D$, $\mu_n R_{t, t'} \rightarrow \mu R_{t, t'} \in \mathfrak{N}_{\nu, t'}$.
- (v) If $\mu \in \mathfrak{N}_{\nu, t_1}$, $\mu R_{t_1, t_2}(X) \leq \mu(X)$ for every $(t_1, t_2) \in D$.

The set $\mathfrak{R}_\phi$ of linear transformations on $\mathfrak{N}_\nu$ will be called the *radiative process on the carrier space* (X, S, ν) . The axioms C_1 , M_1 , and R_1 are the *structure axioms* for $\mathfrak{R}_\phi$. Property (i) asserts the *radiative transition property* of $\mathfrak{R}_\phi$ and reflects the classical radiative transfer of radiant energy under linear (integral) operators. Property (ii) asserts that the radiative transitions are uniquely determined by every given $(t_1, t_3) \in D$ and reflects the uniqueness of solutions of the classical equation of transfer which can be demonstrated under a rather wide set of conditions using the results of [12]. Property (ii) is the *unicity property* of $\mathfrak{R}_\phi$. Property (iii) is the *identity property* of $\mathfrak{R}_\phi$. Property (iv) is the *convergence property* of $\mathfrak{R}_\phi$. Finally, (v) asserts the *contraction property* of $\mathfrak{R}_\phi$ and reflects the physical situation on X when X is source-free, but by itself is not sufficient to insure that X is source-free. All the salient mechanisms of $\mathfrak{R}_\phi$ can be studied using the contraction property (*e.g.*, the scattering and absorption mechanisms). The introduction of sources into X can be represented by the adjunction of an arbitrary general radiant density function onto the equation of transfer (Source Axiom S_1). The contraction property will not be used until the discussion of the measure attenuating functions (see lemma, §8; and (B, §11)).

4. Transition Relation for Radiative Measures. A *partition* of a measurable set E is a finite disjoint class of measurable sets whose union is E . If $\mu \in \mathfrak{N}$ and $\{F_i\}$ is a partition of X , then, for every $F \in S$, $\bigcup_{i=1}^n (F \cap F_i) = F$, and

$$\mu(F) = \sum_{i=1}^n \mu(F \cap F_i) = \sum_{i=1}^n \mu(F_i) \mu_{F_i}(F),$$

that is, $\mu = \sum_{i=1}^n \mu(F_i) \mu_{F_i}$.

Let $\mu \in \mathfrak{N}_{\nu, t_1}$, $E \in S$, and $R_{t_1, t_2} \in \mathfrak{R}_\phi$. By (i) M_1 and (ii) M_1 we have $\mu_E \in \mathfrak{N}_{\nu, t_1}$, and, by (i) R_1 , $\mu_E R_{t_1, t_2} \in \mathfrak{N}_{\nu, t_2}$. By the preceding discussion, if $\{F_i\}$ is a partition of X , then

$$\mu_E R_{t_1, t_2} = \sum_{i=1}^n \mu_E R_{t_1, t_2}(F_i) \cdot (\mu_E R_{t_1, t_2})_{F_i}.$$

For any $R_{t_1, t_3} \in \mathcal{R}_\phi$, by (ii) R_1 ,

$$\mu_E R_{t_1, t_3} = (\mu_E R_{t_1, t_2}) R_{t_2, t_3} = \sum_{i=1}^n \mu_E R_{t_1, t_2}(F_i) \cdot [(\mu_E R_{t_1, t_2})_{F_i}] R_{t_2, t_3}.$$

For simplicity we adopt the notation $\mu_E R_{t_1, t_3} \equiv \mu(t_1, E; t_2, \cdot)$, so that, for example, $\mu_E R_{t_1, t_3}(F) = \mu(t_1, E; t_2, F)$. We shall write $\bar{\mu}_E R_{t_1, t_3} \equiv \bar{\mu}(t_1, E; t_2, \cdot)$, and frequently the further abbreviation $\mu' \equiv \mu_E R_{t_1, t_3}$ will be used; it will always be clear from the context which triple $((t_1, t_2) \in D, \mu \in \mathfrak{M}_{\nu, t_1}, E \in \mathbf{S})$ is being considered. With these notational devices the above relation may be written as

$$\mu(t_1, E; t_3, \cdot) = \sum_{i=1}^n \mu(t_1, E; t_2, F_i) \mu'(t_2, F_i; t_3, \cdot),$$

where $t_1 \leq t_2 \leq t_3$, and $\{F_i\}$ is a partition of X . This is the *transition relation for relative radiative measures* and will be the starting point of the chain of arguments leading through the main features of the abstract radiative transfer theory.

There exists an alternative *transition relation for absolute radiative measures*. It may be obtained from the former by writing

$$\mu_E R_{t_1, t_2} = (1/\mu(E)) \bar{\mu}_E R_{t_1, t_2}$$

and

$$\mu'(t_2, F_i; t_3, \cdot) = [\overline{(\bar{\mu}_E R_{t_1, t_2})_{F_i} R_{t_2, t_3}}] / \bar{\mu}(t_1, E; t_2, F_i),$$

so that

$$\bar{\mu}_E R_{t_1, t_3} = \sum_{i=1}^n \overline{(\bar{\mu}_E R_{t_1, t_2})_{F_i} R_{t_2, t_3}},$$

where $t_1 \leq t_2 \leq t_3$ and $\{F_i\}$ is a partition of X .

5. G- and g-densities. The $\mathcal{R}_\phi$ -transforms of the relative radiative measures were used to obtain the fundamental transition relation for radiative measures. This relation may be considered as the necessary representation of a general radiative process $\mathcal{R}_\phi$ on a carrier space X . Its order of generality is such that it conveys little useful information except the single pleasant fact that the idea of a classical process can be extended to a carrier space which has a minimum of requisite structure, namely an arbitrary basic space X , a σ -algebra $\mathbf{S}$ of subsets of X , and a totally σ -finite measure on $\mathbf{S}$. In fact, an examination of the derivation of the transition relations will show that it would indeed be permissible to employ an arbitrary measurable space and a “radiative process” which simply used some cone (not necessarily non-negative) of functions on $\mathbf{S}$ for its “radiative measures.”

These observations show that the fundamental concepts of a classical radiative process can be pushed so far into the backwoods of abstraction that they become almost lost in the nebulous generalities of measure theory; and yet, despite this

abstract character, the "radiative process" may still retain enough structure to yield a recognizable transition relation for the correspondingly abstract "radiative measures." As they stand now, the structure axioms are sufficiently detailed to allow a running start towards our goal of obtaining necessary characterizations of the radiative process in terms of abstract counterparts of the absorption and scattering functions in an integro-differential equation. The transition relation for radiative measures describes the radiative process on a set level (with respect to X). The purpose of this section is to prepare the ground for the description of the radiative process on a point level in X . This will require the consideration of various densities of elements of $\mathfrak{M}_\nu$.

Theorem A. *Let $(t_1, t_2) \in D$ be arbitrary but fixed. Let $\mu \in \mathfrak{M}_{\nu, t_1}$ and $E \in \mathbf{S}$; then there exists a finite-valued non-negative measurable function $G(t_1, E; t_2, \cdot)$ on X , unique $[\nu]$, such that for every $F \in \mathbf{S}$*

$$\bar{\mu}(t_1, E; t_2, F) = \int_F G(t_1, E; t_2, \cdot) d\nu.$$

Proof: By hypothesis $\mu \in \mathfrak{M}_{\nu, t_1}$, so that, by (ii)M₁, $\bar{\mu}_E \in \mathfrak{M}_{\nu, t_1}$, and, by (ii)R₁ and (i)M₁, $\bar{\mu}_E R_{t_1, t_2} \in \mathfrak{M}_{\nu, t_2}$. Since $\bar{\mu}(t_1, E; t_2, \cdot) \ll \nu$, use of C₁ and a Radon-Nikodym argument completes the proof.

The elements of the collection of functions on X whose existence was demonstrated in the preceding theorem will be referred to as the G -densities (with respect to ν) of radiative measures. If X does not support $\bar{\mu}(t_1, E; t_2, \cdot)$, the ν -uniqueness of the associated G -density will permit the convention of defining $G(t_1, E; t_2, \cdot)$ as the zero function on X . On the other hand, if X supports $\bar{\mu}(t_1, E; t_2, \cdot)$, then by (A, §2) there is a unique $[\nu]$ maximal full support F for this radiative measure and we can establish the convention that $G(t_1, E; t_2, x) = 0$ for every $x \in F'$ (the complement of F).

Theorem B. *Let $(t_1, t_2) \in D$ be arbitrary but fixed. Let $\mu \in \mathfrak{M}_{\nu, t_1}$ and $F \in \mathbf{S}$; then there exists a finite-valued non-negative measurable function $g(t_1, \cdot; t_2, F)$ on X , unique $[\nu]$, such that, for every $E \in \mathbf{S}$,*

$$\bar{\mu}(t_1, E; t_2, F) = \int_E g(t_1, \cdot; t_2, F) d\nu.$$

Proof: The set function $\bar{\mu}(t_1, \cdot; t_2, F)$ is finite-valued and non-negative on $\mathbf{S}$. If $E \in \mathbf{S}$ and $\nu(E) = 0$, then $\bar{\mu}_E \equiv 0$, so that, by the linear properties of the elements of $\mathfrak{M}_\nu$, $\bar{\mu}(t_1, E; t_2, F) = 0$. Hence $\bar{\mu}(t_1, \cdot; t_2, F) \ll \nu$. It remains to show that $\bar{\mu}(t_1, \cdot; t_2, F)$ is countably additive on $\mathbf{S}$. Let $\{E_j\}$ be any disjoint sequence of measurable sets in $\mathbf{S}$. Let $E^{(n)} = \bigcup_{j=1}^n E_j$. Then $\{E^{(n)}\}$ is an increasing sequence of measurable sets such that $\lim_n E^{(n)} = E = \bigcup_{j=1}^\infty E_j \in \mathbf{S}$. The sequence $\{\bar{\mu}_{E^{(n)}}\}$ of elements in $\mathfrak{M}_{\nu, t_1}$ has the limit $\bar{\mu}_E \in \mathfrak{M}_{\nu, t_1}$; for

$$\lim_n \bar{\mu}_{E^{(n)}}(G) = \lim_n \sum_{j=1}^n \mu(E_j \cap G) = \sum_{j=1}^\infty \mu(E_j \cap G) = \mu(E \cap G) = \bar{\mu}_E(G)$$

for every $G \in \mathbf{S}$. By (iv) R_1 it follows that $\mu_{E(n)}R_{t_1, t_2} \rightarrow \mu_E R_{t_1, t_2}$. For every positive integer n , $\mu_{E(n)}R_{t_1, t_2} = \sum_{i=1}^n \mu_{E_i} R_{t_1, t_2} \in \mathfrak{M}_{\nu, t_2}$, by (i) M_1 and (i) R_1 , hence

$$\lim_n \left(\sum_{i=1}^n \mu_{E_i} R_{t_1, t_2} \right) = \sum_{i=1}^{\infty} \mu_{E_i} R_{t_1, t_2} = \mu_E R_{t_1, t_2}$$

implies that, for $F \in \mathbf{S}$, $\sum_{i=1}^{\infty} \mu(t_1, E_i; t_2, F) = \mu(t_1, E; t_2, F)$. It follows that $\mu(t_1, \cdot; t_2, F)$ is a totally finite ν -continuous measure on $\mathbf{S}$. An R-N argument yields the function $g(t_1, \cdot; t_2, F)$ on X satisfying all the requirements of the theorem.

The elements of the collection of functions on X whose existence was demonstrated in the preceding theorem will be referred to as the *g-densities* (with respect to ν) of *radiative measures*. We observe that the ν -continuous measure $\mu(t_1, \cdot; t_2, F)$ is, by (ii) M_1 , an element of $\mathfrak{M}_{\nu, t_2}$.

The set function $g(t_1, x; t_2, \cdot)$ which is the correspondent to $g(t_1, \cdot; t_2, F)$ is clearly non-negative and finite-valued on $\mathbf{S}$. We will show that $g(t_1, x; t_2, \cdot)$ is almost a ν -continuous measure on $\mathbf{S}$. The sense of "almost" will be made precise in the theorem which follows. The theorem and the subsequent discussion are designed to show certain limitations of the axioms C_1 , M_1 , and R_1 and to motivate the axiom R_2 . The key to the following discussion lies in the observation that each $g(t_1, \cdot; t_2, F)$ is dependent upon $(t_1, t_2) \in D$ and $F \in \mathbf{S}$ so that the exceptional ν -null sets on which $g(t_1, \cdot; t_2, F)$ is perhaps undetermined depend upon $F \in \mathbf{S}$ for fixed $(t_1, t_2) \in D$. This is the reason which generally bars $g(t_1, x; t_2, \cdot)$ from being a measure for arbitrary $x \in X$. The removal of this restriction can be accomplished by considering special classes of radiative measures at the outset, or special classes of carrier spaces ([8], p 210).

Theorem C. Let $(t_1, t_2) \in D$ be arbitrary but fixed. Let $\mu \in \mathfrak{M}_{\nu, t_1}$; then:

(a) For each fixed disjoint sequence $\{F_i\}$ of measurable sets

$$g(t_1, x; t_2, \bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} g(t_1, x; t_2, F_i) [\nu].$$

(b) If $F_1, F_2 \in \mathbf{S}$, and $F_1 \subset F_2$, then $g(t_1, x; t_2, F_1) \leq g(t_1, x; t_2, F_2) [\nu]$.

(c) For each fixed decreasing sequence $\{F_i\}$ of measurable sets

$$\lim_i g(t_1, x; t_2, F_i) = g(t_1, x; t_2, \bigcap_{i=1}^{\infty} F_i) [\nu].$$

(d) If, for $F \in \mathbf{S}$, $\nu(F) = 0$, then $g(t_1, x; t_2, F) = 0 [\nu]$.

Proof: (a) Let $F = \bigcup_{i=1}^{\infty} F_i$. Let the *g-densities* associated with $\mu(t_1, \cdot; t_2, F_i)$ be designated by $g_i = g(t_1, \cdot; t_2, F_i)$, $i = 1, 2, 3, \dots$, each being unique $[\nu]$. Then $\mu(t_1, E; t_2, F_i) = \int_E g_i d\nu$ for every $E \in \mathbf{S}$. Since $\mu(t_1, E; t_2, \cdot) \in \mathfrak{M}_{\nu, t_2}$,

$$\sum_{i=1}^{\infty} \mu(t_1, E; t_2, F_i) = \mu(t_1, E; t_2, F) < \infty,$$

and the series $\sum_{i=1}^{\infty} g_i(x)$ converges a.e. $[\nu]$ to a ν -integrable function g ([8], p. 114) such that

$$\int_E g \, d\nu = \sum_{i=1}^{\infty} \int_E g_i \, d\nu = \bar{\mu}(t_1, E; t_2, F)$$

for every $E \in \mathbf{S}$. Hence

$$g(t_1, x; t_2, \bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} g(t_1, x; t_2, F_i) \quad [\nu]$$

by the uniqueness of R-N derivatives.

(b) If $F_1, F_2 \in \mathbf{S}$, and $F_1 \subset F_2$, then, since $\bar{\mu}(t_1, E; t_2, \cdot) \in \mathfrak{M}_{\nu, t_2}$ for every $E \in \mathbf{S}$, $\bar{\mu}(t_1, E; t_2, F_1) \leq \bar{\mu}(t_1, E; t_2, F_2)$ for every $E \in \mathbf{S}$, so that

$$\int_E g(t_1, \cdot; t_2, F_1) \, d\nu \leq \int_E g(t_1, \cdot; t_2, F_2) \, d\nu$$

for every $E \in \mathbf{S}$, and the conclusion follows.

(c) If $\{F_i\}$ is a decreasing sequence of measurable sets, then, since $\bar{\mu}(t_1, E; t_2, \cdot) \in \mathfrak{M}_{\nu, t_2}$, we have for every $E \in \mathbf{S}$

$$\begin{aligned} \int_E \lim_n g(t_1, \cdot; t_2, F_n) \, d\nu &= \lim_n \int_E g(t_1, \cdot; t_2, F_n) \, d\nu \\ &= \int_E g\left(t_1, \cdot; t_2, \bigcap_{n=1}^{\infty} F_n\right) \, d\nu \end{aligned}$$

and the conclusion follows.

(d) If F is a ν -null set, then, since $\bar{\mu}(t_1, E; t_2, \cdot) \in \mathfrak{M}_{\nu, t_2}$, $\int_E g(t_1, \cdot; t_2, F) \, d\nu = 0$ for every $E \in \mathbf{S}$, and the conclusion follows. This completes the proof of the theorem.

Part (a) of the preceding theorem shows a restricted countable additivity property for $g(t_1, x; t_2, \cdot)$ and clearly limits the points $x \in X$ for which it holds to sets depending upon the given fixed sequence of measurable sets. Similarly, parts (b) and (c) in turn gather evidence for restricted types of monotonicity and continuity from above; and, finally, (d) asserts a restricted type of ν -continuity for $g(t_1, x; t_2, \cdot)$. These properties are reminiscent of the corresponding properties of densities of conditional measures on probability spaces ([8], pp. 206–211).

If the σ -ring $\mathbf{S}$ has at most an enumerable number of elements, the union of the exceptional ν -null sets of the elements of a chosen collection $\{g(t_1, \cdot; t_2, F) : F \in \mathbf{S}\}$ of g -densities would be a ν -null set, thereby allowing the definition of a set function $g(t_1, x; t_2, \cdot)$ for almost every $[\nu] \, x \in X$ such that it is an element of $\mathfrak{M}_{\nu, t_2}$. Since a σ -ring cannot have an enumerably infinite number of distinct elements, this ruse will work only in the more trivial σ -rings.

6. Measurable Process Axiom. The corresponding set function $g(t_1, x; t_2, \cdot)$ to the g -density just misses being a radiative measure a.e. $[\nu]$ on X . Some ways

of removing this restriction have been discussed or implied in the preceding section. Since ways and means do exist which will allow the g -density's correspondent to be a radiative measure a.e. $[\nu]$ we will postulate that this property is actually possessed by the correspondent without, however, making any commitment as to what route was taken. This is done by introducing the following measurable process axiom.

R_2 : For each $(t_1, t_2) \in D$, $\mu \in \mathfrak{M}_{\nu, t_1}$, and ν -continuous measure m on $\mathbf{S}$ such that $\mu \ll m$, there exists a set function $g(t_1, x; t_2, \cdot)$ on $\mathbf{S}$ with the following properties:

- (i) $g(t_1, x; t_2, \cdot) \in \mathfrak{M}_{\nu, t_2}$ for almost every $[m] x \in X$ and has an R - N derivative with respect to ν which is the X -section, determined by $x \in X$, of a measurable function on the measurable space $(X \times X, \mathbf{S} \times \mathbf{S})$.
- (ii) For each $F \in \mathbf{S}$, the corresponding function $g(t_1, \cdot; t_2, F)$ on X is a g -density with respect to m of $\bar{\mu}(t_1, \cdot; t_2, F) \in \mathfrak{M}_{\nu, t_2}$.

This axiom will always allow the choice of a g -density whose correspondent is a radiative measure a.e. $[\nu]$. Henceforth the radiative process will be considered to obey R_2 .

7. γ -density.

Theorem A. For each $(t_1, t_2) \in D$, $\mu \in \mathfrak{M}_{\nu, t_1}$, there exists a finite-valued non-negative measurable function $\gamma(t_1, \cdot; t_2, \cdot)$ on $(X \times X, \mathbf{S} \times \mathbf{S}, \nu \times \nu)$ unique $[\nu \times \nu]$ such that:

$$(a) \quad \bar{\mu}(t_1, E_1; t_2, E_2) = \int_{E_1} \int_{E_2} \gamma(t_1, x_1, t_2, x_2) d\nu(x_2) d\nu(x_1)$$

for every pair (E_1, E_2) of measurable sets.

$$(b) \quad g(t_1, x_1; t_2, E_2) = \int_{E_2} \gamma(t_1, x_1; t_2, x_2) d\nu(x_2)$$

for almost every $[\nu] x_1 \in X$ and all measurable sets E_2 , and where $g(t_1, \cdot; t_2, E_2)$ is a g -density of $\bar{\mu}(t_1, \cdot; t_2, E_2) \in \mathfrak{M}_{\nu, t_2}$.

$$(c) \quad G(t_1, E_1; t_2, x_2) = \int_{E_1} \gamma(t_1, x_1; t_2, x_2) d\nu(x_1)$$

for almost every $[\nu] x_2 \in X$ and all measurable sets E_1 , and where $G(t_1, E_1; t_2, \cdot)$ is a G -density of $\bar{\mu}(t_1, E_1; t_2, \cdot) \in \mathfrak{M}_{\nu, t_2}$.

Proof: In R_2 let m be the carrier measure. (i) R_2 then supplies a measurable function $\gamma(t_1, \cdot; t_2, \cdot)$ on $X \times X$ for each $(t_1, t_2) \in D$ such that it is non-negative, finite-valued and unique $[\nu \times \nu]$, and

$$g(t_1, x_1; t_2, E_2) = \int_{E_2} \gamma(t_1, x_1; t_2, x_2) d\nu(x_2)$$

for every $E_2 \in \mathbf{S}$ and almost every $[\nu] x_1 \in X$. By (ii) R_2 , for each fixed $E_2 \in \mathbf{S}$, $g(t_1, \cdot; t_2, E_2)$ is a g -density of $\bar{\mu}(t_1, \cdot; t_2, E_2) \in \mathfrak{M}_{\nu, t_2}$, i.e.,

$$\begin{aligned} \bar{\mu}(t_1, E_1; t_2, E_2) &= \int_{E_1} g(t_1, x_1; t_2, E_2) d\nu(x_1) \\ &= \int_{E_1} \left\{ \int_{E_2} \gamma(t_1, x_1; t_2, x_2) d\nu(x_2) \right\} d\nu(x_1) < \infty \end{aligned}$$

for each $E_1, E_2 \in \mathbf{S}$. It follows ([8], p. 147) that $\gamma(t_1, \cdot; t_2, \cdot)$ is in fact integrable on $X \times X$ for each $(t_1, t_2) \in D$ and that

$$\bar{\mu}(t_1, E_1; t_2, E_2) = \int_{E_2} \int_{E_1} \gamma(t_1, x_1; t_2, x_2) d\nu(x_1) d\nu(x_2)$$

for every pair (E_2, E_1) of measurable sets. Finally, by (A, §5) and the uniqueness of R-N derivatives,

$$G(t_1, E_1; t_2, x_2) = \int_{E_1} \gamma(t_1, x_1; t_2, x_2) d\nu(x_1)$$

for almost every $[\nu] x_2 \in X$ and all measurable sets E_1 . This completes the proof.

The functions $\gamma(t_1, \cdot; t_2, \cdot)$ discussed in the preceding theorem will be referred to as the γ -densities (with respect to ν) of radiative measures.

8. g' -density. We continue with the ground work which will lead to the transition relations for the G -, g -, and γ -densities. In this section we consider radiative measures which are specific images, under two successive applications of elements of $\mathcal{R}_\phi$, of given radiative measures. More precisely, let $(t_1, t_2), (t_2, t_3) \in D$ be arbitrary but fixed, and let $E_1, E_2 \in \mathbf{S}$, and finally let $\mu \in \mathfrak{M}_{\nu, t_1}$. Then $\mu_{E_1} \in \mathfrak{M}_{\nu, t_1}$ and $\mu_{E_1} R_{t_1, t_2} \in \mathfrak{M}_{\nu, t_2}$ by (i), (ii) M_1 , and (i) R_1 respectively. Using the abbreviated notation convention (§4) we have $\mu'_{E_2} \in \mathfrak{M}_{\nu, t_2}$ by the first two clauses of M_1 . We can then write $\mu'(t_2, E_2; t_3, \cdot)$ for $\mu'_{E_2} R_{t_2, t_3} = (\mu_{E_1} R_{t_1, t_2})_{E_2} R_{t_2, t_3} \in \mathfrak{M}_{\nu, t_3}$. The absolute radiative measure $(\mu_{E_1} R_{t_1, t_2})_{E_2}$ we may then designate, alternatively, by $\overline{\mu}_{E_2}$. Hence

$$\begin{aligned} \mu'(t_2, E_2; t_3, \cdot) &= \mu'_{E_2} R_{t_2, t_3} = \overline{\mu}_{E_2} R_{t_2, t_3} / \mu'(E_2) \\ &= \overline{\mu}(t_2, E_2; t_3, \cdot) / \mu(t_1, E_1; t_2, E_2). \end{aligned}$$

We observe that if $\mu \in \mathfrak{M}$ and $E \in \mathbf{S}$ and r is a positive real number, then $r\mu \in \mathfrak{M}$ and $(r\mu)_E = \mu_E$; for if E supports μ , we have for every $F \in \mathbf{S}$

$$(r\mu)_E(F) = (r\mu)(E \cap F) / (r\mu)(E) = \mu_E(F).$$

On the other hand if E does not support μ neither does E support $r\mu$, so then once again $(r\mu)_E = \mu_E (= 0)$. It follows, for example, that $(\mu_{E_1} R_{t_1, t_2})_{E_2} = (\mu_{E_1} R_{t_1, t_2})_{E_2}$, i.e., that μ'_{E_2} is independent of the choice of either the absolute or relative radiative measure associated with the pair (μ, E_1) where $\mu \in \mathfrak{M}_{\nu, t_1}$, $E_1 \in \mathbf{S}$, $\mu(E_1) > 0$.

Theorems A, B, and C of section 5 apply without qualification to $\mu' \in \mathfrak{M}_{\nu, t_2}$, so that we may speak of the G - and g -densities of μ' . Furthermore, by means of (A, §7) the existence of a γ -density connecting the G - and g -densities of μ' is asserted. Hence the results of the preceding sections on the three main types of densities may be carried over immediately to the transforms of μ' .

We now go on to consider certain properties which are peculiar to the specifically transformed radiative measures (i.e., of the μ' -type).

Theorem A. *For arbitrary but fixed $(t_1, t_2), (t_2, t_3) \in D, \mu \in \mathfrak{M}_{\nu, t_1}, E_1 \in S$, the corresponding set function $\overline{\mu'}(t_2, \cdot; t_3, E_3), E_3 \in S$, to $\overline{\mu'}(t_2, E_2; t_3, \cdot) \in \mathfrak{M}_{\nu, t_2}, E_2 \in S$, is a μ' -continuous measure on S . There exists a finite valued non-negative measurable function $g'(t_2, \cdot; t_3, E_3)$ on X unique $[\mu']$ such that for every $E_2 \in S$*

$$\overline{\mu'}(t_2, E_2; t_3, E_3) = \int_{E_2} g'(t_2, \cdot; t_3, E_3) d\mu'.$$

Proof: The proof of (B, §5) yields the fact that $\overline{\mu'}(t_2, \cdot; t_3, E_3)$ is a totally finite measure on S . The novel assertion of the theorem is that $\overline{\mu'}(t_2, \cdot; t_3, E_3)$ is μ' -continuous. If $F \in S$ and $\mu'(F) = 0$, then $\overline{\mu'_F} = 0$ (the zero measure) and so $\overline{\mu'_F} R_{t_2, t_3} = 0$, in particular

$$(\overline{\mu_{E_1} R_{t_1, t_2}})_F R_{t_2, t_3}(E_3) = \overline{\mu'}(t_2, F; t_3, E_3) = 0.$$

An R-N argument yields a finite-valued non-negative measurable function $g'(t_2, \cdot; t_3, E_3)$ unique $[\mu']$ which satisfies the remaining requirements of the theorem.

If we write $G(t_1, E_1; t_2, \cdot)$ for the R-N derivative $d(\mu(E_1)\mu')/d\nu$, then the real number $\overline{\mu'}(t_2, E_2; t_3, E_3)$ may always, by the preceding theorem and ([8], p. 134), be written as

$$\overline{\mu'}(t_2, E_2; t_3, E_3) = (1/\mu(E_1)) \int_{E_2} g'(t_2, \cdot; t_3, E_3) G(t_1, E_1; t_2, \cdot) d\nu.$$

If X supports μ' then by (A, §2) there is a unique $[\nu]$ maximal full support F for μ' . It follows that $G(t_1, E_1; t_2, x) > 0$ for every $x \in F$ except perhaps for a ν -null subset of F . In this case the restriction of the function $g'(t_2, \cdot; t_3, E_3)$ to F is unique $[\nu]$ since the only μ' -null subsets of F are the ν -null subsets of F . As before, we may write $G(t_1, E_1; t_2, x) = 0$ for every $x \in F'$. Since $g'(t_2, \cdot; t_3, E_3)$ is analytically linked to $G(t_1, E_1; t_2, \cdot)$ by means of the integral representation of $\overline{\mu'}(t_2, \cdot; t_3, E_3)$ defined above, and since $g'(t_2, \cdot; t_3, E_3)$ is in general unique $[\mu']$, it is permissible to adopt the following convention which will be used continually in the sequel: If X supports μ' and F is the unique $[\nu]$ maximal full support for μ' , then $g'(t_2, x; t_3, E_3) = 0$ for every $x \in F'$ and for every $E_3 \in S$. If X does not support μ' , then $g'(t_2, x; t_3, E_3) = 0$ for every $x \in X$ and for every $E_3 \in S$. We collect these observations and conventions in the

Corollary. *$g'(t_2, \cdot; t_3, E_3)$, the g -density with respect to μ' of the radiative measure $\overline{\mu'}(t_2, \cdot; t_3, E_3)$, is unique $[\nu]$ for every $E_3 \in S$.*

Each of the following two theorems summarizes a property of g -densities with respect to μ' , the latter being associated with the triple $((t_1, t_2) \in D, \mu \in \mathfrak{M}_{\nu, t_1}, E_1 \in \mathbf{S})$. These results properly belong to the present discussion, yet no immediate use will be made of them in the following section on the transition relations for the G -, g -, and γ -densities. Mention of this fact is made because the first of the following two theorems makes explicit use of clause (v) in R_1 , yet this property of the radiative process is not required for the transition relations for the various densities.

Lemma. For each $(t_1, t_2) \in D$, $E \in \mathbf{S}$, $\mu \in \mathfrak{M}_{\nu, t_1}$, and $F \in \mathbf{S}$,

$$\bar{\mu}_E R_{t_1, t_2}(F) \leq \mu(E).$$

Proof: By (v) R_1 , $\mu R_{t_1, t_2}(X) \leq \mu(X)$ for every $(t_1, t_2) \in D$. By (ii) M_1 , $\bar{\mu}_E \in \mathfrak{M}_{\nu, t_1}$, so that, for every $F \in \mathbf{S}$,

$$\bar{\mu}_E R_{t_1, t_2}(F) \leq \bar{\mu}_E R_{t_1, t_2}(X) \leq \bar{\mu}_E(X) = \mu(E).$$

Theorem B. If $g'(t_2, \cdot; t_3, E_3)$ is a g -density with respect to μ' of the radiative measure $\bar{\mu}(t_2, \cdot; t_3, E_3)$, then $0 \leq g'(t_2, x; t_3, E_3) \leq 1$ for almost every $[\nu] x \in X$.

Proof: If X does not support μ' , $g'(t_2, \cdot; t_3, E_3) = 0$ (the zero function) on X . If X supports μ' , then, if F is the unique $[\nu]$ maximal full support for μ' , $g'(t_2, \cdot; t_3, E_3) = 0$ on F' . If G is a measurable subset of F with positive ν -measure on which $g'(t_2, x; t_3, E_3) > 1$ for every $x \in G$, then, since G fully supports μ' ,

$$\bar{\mu}'(t_2, G; t_3, E_3) = \int_G g'(t_2, \cdot; t_3, E_3) d\mu' > \int_G d\mu' = \mu'(G),$$

contrary to the conclusion of the lemma.

For every subset $E \subset X$, let χ_E denote the characteristic function of E , defined, as usual, by $\chi_E(x) = 1$ or 0 according as $x \in E$ or $x \notin E$ respectively.

Theorem C. If $g'(t_2, \cdot; t_3, E_3)$ is a g -density with respect to μ' of the radiative measure $\bar{\mu}'(t_2, \cdot; t_3, E_3)$ and (i) $t_2 = t_3 = t$, (ii) F is the maximal full support for μ' , then $g'(t, \cdot; t, E_3) = \chi_{E_3 \cap F} [\nu]$.

Proof: By (A, §8) we may represent $\bar{\mu}'(t_2, E; t_3, E_3)$ by

$$(1/\mu(E_1)) \int_E g'(t_2, \cdot; t_3, E_3) G(t_1, E_1; t_2, \cdot) d\nu$$

for every $E \in \mathbf{S}$, where $G(t_1, E_1; t_2, \cdot)$ is the R-N derivative introduced in §5. If (i) above holds, then, by (iii) R_1 ,

$$\bar{\mu}'(t_2, E; t_3, E_3) = \bar{\mu}_E R_{t_2, t_3}(E_3) = \bar{\mu}_E R_{t, t}(E_3) = \mu'(E \cap E_3 \cap F),$$

and using (A, §5) this may be written as

$$(1/\mu(E_1)) \int_{E \cap E_3 \cap F} G(t_1, E_1; t, \cdot) d\nu = (1/\mu(E_1)) \int_E \chi_{E_3 \cap F} G(t_1, E_1; t, \cdot) d\nu$$

for every $E \in \mathbf{S}$. Hence for case (i) the two integrals are equal for every $E \in \mathbf{S}$, so that

$$g'(t, \cdot; t, E_3)G(t_1, E_1; t, \cdot) = \chi_{E_3 \cap F} G(t_1, E_1; t, \cdot)[\nu]$$

by ([8], p. 105). If (ii) above holds, $G(t_1, E_1; t, x) > 0$ for almost every $[\nu]$ x in the maximal full support F for μ' ; and $g'(t, x; t, E_3) = 0$ for every $x \in F'$. Hence if both (i) and (ii) hold, $g'(t, \cdot; t, E_3) = \chi_{E_3 \cap F} [\nu]$.

9. Transition Relations for Densities. Let $(t_1, t_2), (t_2, t_3) \in D$, $\mu \in \mathfrak{M}_{\nu, t_1}$, and $E_1 \in \mathbf{S}$ be arbitrary but fixed. Let $\{F_i\}$ be a partition of X ; then using the notations of section 4 and the appropriate support conventions the transition relation for radiative measures may be written

$$\mu(t_1, E_1; t_3, E_3) = \sum_{i=1}^n \mu'(F_i) \mu'(t_2, F_i; t_3, E_3), \quad E_3 \in \mathbf{S}.$$

By (A, §8) we may write

$$\begin{aligned} \mu(t_1, E_1; t_3, E_3) &= \sum_{i=1}^n \mu'(F_i) \left[(1/\mu'(F_i)) \int_{F_i} g'(t_2, \cdot; t_3, E_3) d\mu' \right] \\ &= \sum_{i=1}^n \int_{F_i} g'(t_2, \cdot; t_3, E_3) d\mu'. \end{aligned}$$

By (A, §5), ([8], p. 134), and the linearity of the radiative process,

$$(1/\mu(E_1)) \bar{\mu}(t_1, E_1; t_3, E_3) =$$

$$(1/\mu(E_1)) \cdot \sum_{i=1}^n \int_{F_i} g'(t_2, \cdot; t_3, E_3) G(t_1, E_1; t_2, \cdot) d\nu.$$

Hence

$$\bar{\mu}(t_1, E_1; t_3, E_3) = \int_X g'(t_2, \cdot; t_3, E_3) G(t_1, E_1; t_2, \cdot) d\nu$$

holds for arbitrary $(t_1, t_2), (t_2, t_3) \in D$, $E_1, E_3 \in \mathbf{S}$. By ((c)A, §7) and (B, §5),

$$\begin{aligned} \bar{\mu}(t_1, E_1; t_3, E_3) &= \int_{E_1} g(t_1, x_1; t_3, E_3) d\nu(x_1) \\ &= \int_X g'(t_2, x_2; t_3, E_3) \left[\int_{E_1} \gamma(t_1, x_1; t_2, x_2) d\nu(x_1) \right] d\nu(x_2) \\ &= \int_X \int_{E_1} g'(t_2, x_2; t_3, E_3) \gamma(t_1, x_1; t_2, x_2) d\nu(x_1) d\nu(x_2) < \infty. \end{aligned}$$

By a Fubini argument and the fact that $E_1 \in \mathbf{S}$ is initially arbitrary we have

$$g(t_1, x_1; t_3, E_3) = \int_X g'(t_2, x_2; t_3, E_3) \gamma(t_1, x_1; t_2, x_2) d\nu(x_2)$$

for almost every $[\nu] x_1 \in X$ and every $E_3 \in \mathbf{S}$. By ((b)A, §7) and ([8], p. 134)

$$g(t_1, x_1; t_3, E_3) = \int_X g'(t_2, \cdot; t_3, E_3) dg(t_1, x_1; t_2, \cdot),$$

for almost every $[\nu] x_1 \in X$ and every $E_3 \in \mathbf{S}$, or in alternate notation (intended to be equivalent) for the radiative measure $g(t_1, x_1; t_2, \cdot)$

$$g(t_1, x_1; t_3, E_3) = \int_X g(t_1, x_1; t_2, dx_2) g'(t_2, x_2; t_3, E_3).$$

This is the *transition relation for g -densities*.

By R_2 (in which m is now μ') and the corollary of (A, §8), the g -density with respect to μ' , $g'(t_2, \cdot; t_3, E_3)$, may for almost every $[\nu] x_2 \in X$ and every $E_3 \in \mathbf{S}$ be chosen as the corresponding function to $g'(t_2, x_2; t_3, \cdot) \in \mathfrak{M}_{\nu, t_3}$; so that, by ((b)A, §7) and the previously cited measure-theoretical arguments, we have for every $E_3 \in \mathbf{S}$

$$\begin{aligned} \int_{E_3} \gamma(t_1, x_1; t_3, x_3) d\nu(x_3) \\ &= \int_X g(t_1, x_1; t_2, dx_2) \left[\int_{E_3} \gamma'(t_2, x_2; t_3, x_3) d\nu(x_3) \right] \\ &= \int_{E_3} \int_X \gamma(t_1, x_1; t_2, x_2) \gamma'(t_2, x_2; t_3, x_3) d\nu(x_2) d\nu(x_3), \end{aligned}$$

where $\gamma'(t_2, x_2; t_3, \cdot)$ is the R-N derivative with respect to ν of $g'(t_2, x_2; t_3, \cdot)$. It follows that

$$\gamma(t_1, x_1; t_3, x_3) = \int_X \gamma(t_1, x_1; t_2, x_2) \gamma'(t_2, x_2; t_3, x_3) d\nu(x_2)$$

for almost every $[\nu] x_1 \in X$ and $x_3 \in X$. This is the *transition relation for γ -densities*.

Finally, for almost every $[\nu] x_3 \in X$ and every $E_1 \in \mathbf{S}$, by ((c)A, §7) and the usual integration arguments,

$$G(t_1, E_1; t_3, x_3) = \int_X G(t_1, E_1; t_2, x_2) G'(t_2, dx_2; t_3, x_3),$$

where $G'(t_2, \cdot; t_3, x_3)$ has $\gamma'(t_2, \cdot; t_3, x_3)$ as an R-N derivative with respect to ν . In an alternative notation (intended to be equivalent) for the radiative measure $G'(t_2, \cdot; t_3, x_3)$,

$$G(t_1, E_1; t_3, x_3) = \int_X G(t_1, E_1; t_2, \cdot) dG'(t_2, \cdot; t_3, x_3).$$

This is the *transition relation for G-densities*.

The transition relation for G -densities is an abstract counterpart of the classical equation of transfer for the radiative process $\mathcal{R}_\phi$ on a general carrier space $(X, \mathbf{S}, \nu)$. Within the present axiomatic framework, it yields a symbolic description of the evolution of the general radiant density function $G(t_1, E_1; t_3, \cdot)$ associated with $\mu(t_1, E_1; t_3, \cdot)$. That is, for fixed $t_1 \in T$ and $E_1 \in \mathbf{S}$ we have for each $\mu \in \mathfrak{M}_{r,t_1}$ the radiative measure $\mu_{E_1} \in \mathfrak{M}_{r,t_1}$, which serves to yield a given initial condition by means of its associated general radiant density function on X at t_1 . The transition relation for G -densities describes the values $G(t_1, E_1; t_3, x_3)$ of subsequent general radiant density functions in terms of the temporally intermediate densities of $\mu(t_1, E_1; t_2, \cdot)$, $t_1 \leq t_2 \leq t_3$.

We may say that the transition relations for radiative measures and their various densities yield a *static* description of the evolution of the abstract radiation field; the next step in the line of development would be to consider a *dynamic* description: one which involves time derivatives of the measures and densities. The step from the static to the dynamic forms of the transition relations is considered in the following sections. In these sections we will study one example of just how much additional structure must be given the radiative process and its carrier space in order to make possible a rigorous derivation of the dynamical forms of the transition relations. The question of minimal required structure is not germane to the present goals and will be reserved for subsequent study. It turns out that relatively small concessions must be made in order to obtain the dynamical descriptions of the radiative process on both the set and point levels in X .

10. t-differentiable Process Axiom. We begin by motivating the choice of the present objects of study: the transforms of μ' -type measures and their g -densities with respect to μ' . Let $\mu \in \mathfrak{M}_{r,t_1}$, $(t_1, t_2) \in D$, $E_1 \in \mathbf{S}$, and $\Delta \in T$, $\Delta > 0$. Let X support $\mu' = \mu_{E_1} R_{t_1,t_2} \in \mathfrak{M}_{r,t_2}$, and let F be the maximal full support for μ' . Then for every $E \in \mathbf{S}$, $E \subset F'$, $\mu(t_1, E_1; t_2, E) = 0$. If $\mu(t_1, \cdot; t_2, E)$ is the corresponding radiative measure, it follows that since $\mu(t_1, E_1; t_2, E) = 0$ its g -density $g(t_1, \cdot; t_2, E)$ vanishes a.e. $[\nu]$ on E_1 . Thus, by R_2 , the corresponding measure $g(t_1, x_1; t_2, \cdot)$ vanishes on E for almost every $[\nu]$ $x_1 \in E_1$. From this and ((b)A, §7),

$$\begin{aligned} g(t_1, x_1; t_2, E_3) &= \int_X \chi_{E_3}(x_2) g(t_1, x_1; t_2, dx_2) \\ &= \int_X \chi_{E_3}(x_2) \chi_F(x_2) g(t_1, x_1; t_2, dx_2) = \int_X \chi_{E_3 \cap F} dg(t_1, x_1; t_2, \cdot) \end{aligned}$$

for almost every $[\nu]$ $x_1 \in E_1$ and every $E_3 \in \mathbf{S}$.

The transition relation for g -densities may be written

$$g(t_1, x_1; t_2 + \Delta, E_3) = \int_X g'(t_2, x_2; t_2 + \Delta, E_3) g(t_1, x_1; t_2, dx_2),$$

so that a study of the quotient

$$[g(t_1, x_1; t_2 + \Delta, E_3) - g(t_1, x_1; t_2, E_3)]/\Delta$$

is equivalent to a study of

$$\int_X [(g'(t_2, x_2; t_2 + \Delta, E_3) - \chi_{E_3 \cap F}(x_2))/\Delta] g(t_1, x_1; t_2, dx_2)$$

in which we naturally choose to consider the quotients

$$[g'(t_2, x_2; t_2 + \Delta, E_3) - \chi_{E_3 \cap F}(x_2)]/\Delta,$$

and their μ' -integrals

$$\left(\int_{E_2} g'(t_2, \cdot; t_2 + \Delta, E_3) d\mu' - \int_{E_2} \chi_{E_3 \cap F} d\mu' \right) / \Delta, \quad E_2 \in \mathbf{S},$$

which, by (A, §8) and the fact that F is the maximal full support for μ' , may be written

$$\begin{aligned} (\overline{\mu'_F}(t_2, E_2; t_2 + \Delta, E_3) - \overline{\mu'_F}(E_2 \cap E_3))/\Delta \\ = (\overline{\mu'}(t_2, E_2; t_2 + \Delta, E_3) - \mu'(E_2 \cap E_3))/\Delta. \end{aligned}$$

Let $\{\Delta_n\}$ be a decreasing null sequence of positive elements in T . If (E_2, E_3) is an arbitrary pair of measurable sets it is then clear from the preceding sections that (C, §8) in general permits, but does not insure, the possibility that

$$\lim_n (g'(t_2, x_2; t_2 + \Delta_n, E_3) - \chi_{E_3 \cap F}(x_2))/\Delta_n$$

exists and is finite for almost every $[\nu] x_2 \in X$. Finally, if X does not support μ' , it is also clear that the above limit is zero for almost every $[\nu] x_2 \in X$. This discussion indicates that, in the case of X supporting a radiative measure, the preceding set of axioms neither insures—but yet does not preclude—the possibility of the existence of finite limits of sequences of the kind exhibited above. Hence the preceding set of axioms will in general be independent of, but yet consistent with, the following regularity axiom.

R_3 : Let $(t_1, t_2), (t_2, t_3) \in D$, $\mu \in \mathfrak{M}_{r, t_1}$, and $E_1, E_3 \in \mathbf{S}$. If X supports $\mu' = \mu_{E_1} R_{t_1, t_2}$, let F denote the maximal full support for μ' . If $g'(t_2, \cdot; t_3, E_3)$ is the g -density with respect to μ' of $\mu'(t_2, \cdot; t_3, E_3)$ and $\{\Delta_n\}$ is any decreasing null sequence of positive elements in T , then:

(i) The sequence

$$\{h'_{\star n}(t_2, \cdot, E_3)\} = \{(\chi_{E_3 \cap F} - g'(t_2, \cdot; t_2 + \Delta_n, E_3))/\Delta_n\}$$

converges uniformly on X to a unique $[\nu]$ essentially bounded function $h'_\star(t_2, \cdot, E_3)$.

(ii) The corresponding function $h'_\star(t_2, x_2, \cdot)$ is countably additive on $\mathbf{S}$ for almost $[\nu] x_2 \in X$.

(iii)

$$\{\gamma_n\} = \{(\gamma(t_1, x_1; t_2 + \Delta_n, \cdot) - \gamma(t_1, x_1; t_2, \cdot))/\Delta_n\}$$

is fundamental in measure for almost every $[\nu] x_1 \in X$.

(iv)

$$\{G_n\} = \{(G(t_1, E_1; t_2 + \Delta_n, \cdot) - G(t_1, E_1; t_2, \cdot))/\Delta_n\}$$

is fundamental in measure.

The references in R_3 to the concepts of essential boundedness and fundamentality in measure are, of course, with respect to the carrier measure. Clauses (iii) and (iv) will not be needed until section 15 at which time their motivation (which was not included in the preceding discussion) will be supplied. We observe that all three types of densities (G -, g -, and γ -densities) are represented in R_3 . It turns out that in order to allow the present development of the foundations of radiative transfer theory to proceed rigorously, we require of the g -densities (in the form of g') the strongest, and of the G - and γ -densities the weakest, properties of convergence in general measure theory ([15], p. 237). However, it is worthwhile to observe that the attainment of the final goal of an abstract equation of transfer is possible on a *formal* level using only the three structure axioms C_1 , M_1 , and R_1 , along with the final structure axiom T_1 introduced in section 18. Henceforth the radiative process $\mathcal{R}_\phi$ will be considered to obey R_3 .

11. Measure Attenuating Functions.

Theorem A. *Let the hypothesis be as in R_3 . If we write*

$$\int_{E_2} h'_{*n}(t_2, \cdot, E_3) d\mu' = \mathfrak{A}_{*n}(t_2, E_2, E_3), \quad n = 1, 2, 3, \dots,$$

and similarly

$$\int_{E_2} h_{*}(t_2, \cdot, E_3) d\mu' = \mathfrak{A}_{*}(t_2, E_2, E_3), \quad \text{for every } E_2 \in \mathbf{S},$$

then:

(a)

$$\lim_n \mathfrak{A}_{*n}(t_2, E_2, E_3) = \mathfrak{A}_{*}(t_2, E_2, E_3) < \infty.$$

(b) *The set function $\mathfrak{A}_{*}(t_2, \cdot, E_3)$ is a totally finite μ' -continuous signed measure on $\mathbf{S}$.*

(c) *The set function $\mathfrak{A}_{*}(t_2, E_2, \cdot)$ is a totally finite ν -continuous signed measure on $\mathbf{S}$.*

Proof: (a) By (i) R_3 the sequence $\{h'_{*n}(t_2, \cdot, E_3)\}$ converges uniformly on X to the function $h'_{*}(t_2, \cdot, E_3)$. For every $E_2 \in \mathbf{S}$, $\mu'(E_2) < \infty$ so that the sequence

$\{h'_{*n}(t_2, \cdot, E_3)\}$ converges in the mean ([8], p. 111) to the μ' -integrable function $h'_*(t_2, \cdot, E_3)$ on E_2 , i.e.,

$$\int_{E_2} |h'_{*n}(t_2, \cdot, E_3) - h'_*(t_2, \cdot, E_3)| d\mu' \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies

$$\mathfrak{A}_{*n}(t_2, E_2, E_3) \rightarrow \mathfrak{A}_*(t_2, E_2, E_3) < \infty \quad \text{as } n \rightarrow \infty.$$

(b) By definition $\mathfrak{A}_*(t_2, E_2, E_3) = \int_{E_2} h'_*(t_2, \cdot, E_3) d\mu'$, $t_2 \in T$ and $E_2, E_3 \in \mathbf{S}$. By (a), $h'_*(t_2, \cdot, E_3)$ is a μ' -integrable function so that in particular $\int_X h'_*(t_2, \cdot, E_3) d\mu' = \mathfrak{A}_*(t_2, X, E_3) < \infty$. In general ([8], p. 98) the indefinite integral of an integrable function is countably additive. Further, if $\mu'(E) = 0$, then $\int_E h'_*(t_2, \cdot, E_3) d\mu' = 0$ ([8], p. 104), i.e., $\mathfrak{A}_*(t_2, E, E_3) = 0$.

(c) The finiteness of $\mathfrak{A}_*(t_2, E_2, \cdot)$ on $\mathbf{S}$ is implied by the μ' -integrability of $h'_*(t_2, \cdot, E_3)$ for every $E_3 \in \mathbf{S}$. If $\nu(E) = 0$, then, since $\mu'(t_2, E_2; t_2 + \Delta_n, \cdot) \in \mathfrak{M}_{\nu, t_2 + \Delta_n}$, $n = 1, 2, 3, \dots$, $g'(t_2, \cdot; t_2 + \Delta_n, E)$ vanishes almost everywhere $[\nu]$ on $E_2 \in \mathbf{S}$, so that $h'_{*n}(t_2, \cdot, E)$, $n = 1, 2, 3, \dots$, vanishes almost everywhere $[\nu]$ on $E_2 \in \mathbf{S}$, which implies $\mathfrak{A}_{*n}(t_2, E_2, E) = 0$. Let $\{D_n\}$ be a decreasing sequence of sets in $\mathbf{S}$ with $\lim_n D_n = 0 \in \mathbf{S}$. Now $h'_*(t_2, x, \cdot)$ vanishes at $0 \in \mathbf{S}$, is finite-valued and countably additive on $\mathbf{S}$ for almost every $[\nu]$ $x \in X$. $h'_*(t_2, x_2, \cdot)$ is therefore a signed measure on $\mathbf{S}$ and is continuous from above at $0 \in \mathbf{S}$ for almost every $x_2 \in X$. It follows that $\lim_n h'_{*n}(t_2, x_2, D_n) = 0$ for almost every $[\nu]$ $x_2 \in X$ and that the sequence $\{\int_{E_2} h'_{*n}(t_2, \cdot, D_n) d\mu'\}$ has the limit 0 for every $E_2 \in \mathbf{S}$. Therefore $\mathfrak{A}_*(t_2, E_2, \cdot)$, $E_2 \in \mathbf{S}$, is continuous from above at $0 \in \mathbf{S}$. Let $\{F_i\}$ be a disjoint sequence of measurable sets with union F'' . Let $H_n = \bigcup_{i=1}^n F_i$ and $D_n = F'' - H_n$. By the finite additivity of $\mathfrak{A}_*(t_2, E_2, \cdot)$ we have

$$\begin{aligned} \mathfrak{A}_*(t_2, E_2, F'') &= \sum_{i=1}^n \mathfrak{A}_*(t_2, E_2, F_i) + \mathfrak{A}_*(t_2, E_2, D_n) \\ &= \lim_n \sum_{i=1}^n \mathfrak{A}_*(t_2, E_2, F_i) + \lim_n \mathfrak{A}_*(t_2, E_2, D_n) = \sum_{i=1}^{\infty} \mathfrak{A}_*(t_2, E_2, F_i), \end{aligned}$$

which completes the proof of the theorem.

Letting $t_1 \in T$, $\mu \in \mathfrak{M}_{\nu, t_1}$, and $E_1 \in \mathbf{S}$ be arbitrary but fixed, the t -differentiable Process Axiom permits the definition of a real finite-valued function $\mathfrak{A}_*$ on the Cartesian product $T \times \mathbf{S} \times \mathbf{S}$. Some of its properties are summarized in Theorem A. In order to continue with the study of the dynamic aspects of the radiative process on X , we consider four restrictions of $\mathfrak{A}_*$ to certain subsets of $T \times \mathbf{S} \times \mathbf{S}$. These restrictions are as follows: Let $\mathbf{U} = \{(t_2, E_2, E_3) \in T \times \mathbf{S} \times \mathbf{S} : E_3 = X\}$. We will denote by A_* the restriction $\mathfrak{A}_*|_{\mathbf{U}}$ of $\mathfrak{A}_*$ to $\mathbf{U}$ and write $(\mathfrak{A}_*|_{\mathbf{U}})(t_2, E_2, X) = A_*(t_2, E_2)$ for every $(t_2, E_2, X) \in \mathbf{U}$. Let $T \times \Delta = \{(t_2, E_2, E_3) \in T \times \mathbf{S} \times \mathbf{S} : E_2 = E_3\}$. We will denote by $\mathfrak{Q}_*$ the restriction $\mathfrak{A}_*|_{T \times \Delta}$ of $\mathfrak{A}_*$ to $T \times \Delta$ and write $(\mathfrak{A}_*|_{T \times \Delta})(t_2, E_2, E_3) = \mathfrak{Q}_*(t_2, E_2)$ for every $(t_2, E_2, E_3) \in T \times \Delta$. Let $\mathbf{V} = \{(t_2, E_2, E_3) \in T \times \mathbf{S} \times \mathbf{S} : E_3 \subset E'_2\}$. We will denote by Σ_* the

restriction $\mathfrak{A}_*|V$ of $\mathfrak{A}_*$ to V and write $(\mathfrak{A}_*|V)(t_2, E_2, E_3) = -\Sigma_*(t_2, E_2, E_3)$ for every $(t_2, E_2, E_3) \in V$. Finally, it will be necessary specifically to consider the restriction of Σ_* to the subset W of V , where $W = \{(t_2, E_2, E_3) \in V : E_2 = E_3\}$. We will denote by S_* the restriction $\Sigma_*|W$ of Σ_* to W and write $(\Sigma_*|W)(t_2, E_2, E_3) = S_*(t_2, E_2)$ for every $(t_2, E_2, E_3) \in W$.

The functions A_* , $\mathfrak{A}_*$, Σ_* , and S_* will be referred to collectively as the *measure attenuating functions* for $\mathfrak{R}_\phi$, where $\mathfrak{R}_\phi$ is the radiative process on the carrier space X ; they will be referred to individually as the *measure-absorption*, *-attenuation*, *-scattering*, and *total measure-scattering function* respectively. Further restrictions but of an obvious kind will be made of some of these functions. It will be clear from the context what the domains of the restrictions are in each case.

Theorem B. *Let the hypothesis be as in R_3 , then:*

- (a) $A_*(t_2, \cdot)$ is a totally finite μ' -continuous measure on S .
- (b) $\mathfrak{A}_*(t_2, \cdot)$ is a finite-valued non-negative μ' -continuous function on S .
- (c) $\Sigma_*(t_2, \cdot, \cdot)$ is a finite-valued non-negative function on $S \times S$. In particular, for fixed $E_2 \in S$, $\Sigma_*(t_2, E_2, \cdot)$ is ν -continuous on S , and if $\{(t_2, E_2, F_i)\}$ is a disjoint sequence in V ,

$$\Sigma_*\left(t_2, E_2, \bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} (\Sigma_*(t_2, E_2, F_i)).$$

On the other hand, for fixed $E_3 \in S$, $\Sigma_*(t_2, \cdot, E_3)$ is μ' -continuous, and if $\{(t_2, F_i, E_3)\}$ is a disjoint sequence in V ,

$$\Sigma_*\left(t_2, \bigcup_{i=1}^{\infty} F_i, E_3\right) = \sum_{i=1}^{\infty} (\Sigma_*(t_2, F_i, E_3)).$$

- (d) $S_*(t_2, \cdot)$ is a finite-valued non-negative μ' -continuous function on S .

(e)
$$\mathfrak{A}_*(t_2, \cdot) = A_*(t_2, \cdot) + S_*(t_2, \cdot).$$

Proof: (a) From (i) R_3 , the definition of A_* , and (a) of Theorem A we may write

$$A_*(t_2, E_2) = \lim_n (\mu'(E_2) - \overline{\mu'}(t_2, E_2; t_2 + \Delta_n, X))/\Delta_n$$

for every $E_2 \in S$. With $E_3 = X$ in (b) of Theorem A, $A_*(t_2, \cdot)$ is seen to be totally finite, μ' -continuous, and countably additive on S . From the lemma of §8 and the limit form above, $A_*(t_2, \cdot)$ is non-negative on S .

(b) From (i) R_3 , the definition of $\mathfrak{A}_*$, and (a) of Theorem A we may write

$$\mathfrak{A}_*(t_2, E_2) = \lim_n (\mu'(E_2) - \overline{\mu'}(t_2, E_2; t_2 + \Delta_n, E_2))/\Delta_n$$

for every $E_2 \in S$. Hence $\mathfrak{A}_*(t_2, E_2) < \infty$ for every $E_2 \in S$. Non-negativity follows from the limit form above and the lemma of §8. μ' -continuity follows from the limit form above.

(c) From (i)R₃, the definition of Σ_* , and (a) of Theorem A we may write

$$\Sigma_*(t_2, E_2, E_3) = \lim_n \overline{\mu'}(t_2, E_2; t_2 + \Delta_n, E_3)/\Delta_n$$

for every $(t_2, E_2, E_3) \in V$. Hence $0 \leq \Sigma_*(t_2, E_2, E_3) < \infty$ for every $(t_2, E_2, E_3) \in V$. For fixed $E_2 \in S$, the ν -continuity of $\Sigma_*(t_2, E_2, \cdot)$ is directly implied by the limit form above. If $\{(t_2, E_2, F_i)\}$ is a disjoint sequence in V , $\{F_i\}$ is a disjoint sequence in S ([8], p. 137). Hence for $x_2 \in X \cap E_2 = E_2$, by (i)R₃,

$$-h'_{*n}(t_2, x_2, F_i) = g'(t_2, x_2; t_2 + \Delta_n, F_i)/\Delta_n \geq 0, \quad n = 1, 2, 3, \dots$$

From (ii)R₃ it follows that on E_2 , except perhaps for a ν -null subset, $\{\sum_{i=1}^n -h'_*(t_2, \cdot, F_i)\}$ is an increasing sequence of non-negative μ' -integrable functions with the limit $-h'_*(t_2, \cdot, \bigcup_{i=1}^\infty F_i)$. Hence ([8], p. 112)

$$\begin{aligned} \lim_n \int_{E_2} \left(\sum_{i=1}^n -h'_*(t_2, \cdot, F_i) \right) d\mu' &= \lim_n \sum_{i=1}^n (\Sigma_*(t_2, E_2, F_i)) \\ &= \int_{E_2} -h'_*\left(t_2, \cdot, \bigcup_{i=1}^\infty F_i\right) d\mu' = \Sigma_*\left(t_2, E_2, \bigcup_{i=1}^\infty F_i\right). \end{aligned}$$

Finally, for fixed E_3 in S , μ' -continuity and countable additivity for $\Sigma_*(t_2, \cdot, E_3)$ (in the indicated sense) follow from (b) of Theorem A.

(d) From (i)R₃, the definition of S_* , and ((a) of Theorem A) we may write

$$S_*(t_2, E_2) = \lim_n \overline{\mu'}(t_2, E_2; t_2 + \Delta_n, E'_2)/\Delta_n.$$

μ' -continuity follows from this form; finite-valuedness and non-negativity follow immediately from the fact that S_* is a restriction of Σ_* .

(e) From (c) of Theorem A,

$$\mathfrak{A}_*(t_2, E_2, X) = \mathfrak{A}_*(t_2, E_2, E_2) + \mathfrak{A}_*(t_2, E_2, E'_2)$$

which with the definitions of A^* , $\mathfrak{A}_*$, and S_* yields the desired conclusion. This ends the proof.

12. Dynamic Transition Relation for Radiative Measures. The following theorem asserts the existence of the dynamic version of the transition relation for radiative measures and gives, on a set level, the first recognizable counterpart to the classical equation of transfer.

Theorem A. *Let $(t_1, t_2) \in D$, $\mu \in \mathfrak{M}_{\nu, t_1}$, $E_1 \in S$ be fixed. If $E_2 \in S$ and if $\{\Delta_n\}$ is any decreasing null sequence of positive elements in T , then:*

$$\begin{aligned} \text{(a)} \quad \lim_n (\mu(t_1, E_1; t_2 + n, E_2) - \mu(t_1, E_1; t_2, E_2))/\Delta_n \\ \equiv \partial^+ \mu(t_1, E_1; t_2, E_2)/\partial t_2 \end{aligned}$$

exists and is finite.

$$(b) \quad \partial^+ \mu(t_1, E_1; t_2, E_2) / \partial t_2 = -\mathfrak{A}_*(t_2, X, E_2) = -\mathfrak{Q}_*(t_2, E_2) + S_*(t_2, E'_2).$$

Proof: Let $\{F_i\}$ be a partition of X . The appropriate support conventions being in force, we write the transition relation for radiative measures as follows:

$$\mu(t_1, E_1; t_2 + \Delta_n, E_2) = \sum_{i=1}^m \mu(t_1, E_1; t_2, F_i) \mu'(t_2, F_i; t_2 + \Delta_n, E_2),$$

for $n = 1, 2, 3, \dots$. Using the μ' -notation, we have

$$\begin{aligned} \mu(t_1, E_1; t_2 + \Delta_n, E_2) &= \sum_{i=1}^m \mu'(F_i) \mu'(t_2, F_i; t_2 + \Delta_n, E_2) \\ &= \sum_{i=1}^m \overline{\mu'}(t_2, F_i; t_2 + \Delta_n, E). \end{aligned}$$

Further, $\mu(t_1, E_1; t_2, E_2) = \sum_{i=1}^m \mu'(F_i \cap E_2)$, so that

$$\begin{aligned} \lim_n (\mu(t_1, E_1; t_2 + \Delta_n, E_2) - \mu(t_1, E_1; t_2, E_2)) / \Delta_n \\ = \sum_{i=1}^m [\lim_n (\overline{\mu'}(t_1, F_i; t_2 + \Delta_n, E_2) - \mu'(F_i \cap E_2)) / \Delta_n] \\ = \sum_{i=1}^m -\mathfrak{A}_*(t_2, F_i, E_2) = -\mathfrak{A}_*(t_2, X, E_2) < \infty, \end{aligned}$$

by ((b)A, §11). Hence $\partial^+ \mu(t_1, E_1; t_2, E_2) / \partial t_2$ exists and is finite. Again by ((b)A, §11) and the definitions of the measure attenuating functions,

$$\begin{aligned} \partial^+ \mu(t_1, E_1; t_2, E_2) / \partial t_2 &= -\mathfrak{A}_*(t_2, E_2, E_2) - \mathfrak{A}_*(t_2, E'_2, E_2) \\ &= -\mathfrak{Q}_*(t_2, E_2) + S_*(t_2, E'_2), \end{aligned}$$

which proves (b).

We remark that the symbol $\partial^+ \mu(t_1, E_1; t_2, E_2) / \partial t_2$ is to denote a right hand derivative with respect to t evaluated at t_2 . Inasmuch as $\mu \in \mathfrak{M}_{\nu, t_1}$, $E_1 \in \mathbf{S}$, and $E_2 \in \mathbf{S}$ are arbitrary but fixed throughout the discussion, the right hand derivative is essentially a total derivative, t_1 , E_1 , and E_2 playing the roles of parameters. We observe that the form of the t -differentiability Axiom allows only derivatives of this kind. It should be evident after some reflection that t -differentiability of $\mathfrak{Q}_\nu$ in the other direction is, in general, too much to hope for. While such differentiability can actually be demanded in R_3 and resulting two-sided derivatives be obtained, this additional regularity condition adds nothing radically different to the present store of available techniques and will not materially affect the general symbolic structure of the subsequent theorems. For the present study we will continue with the adopted form of the radiative process.

For arbitrary $\mu \in \mathfrak{M}_{\nu, t}$ and $E \in \mathbf{S}$, the value $\mu(E)$ was defined earlier as the radiant energy in E (at $t \in T$ and $\lambda \in \Lambda$), μ by fiat being a ν -continuous totally finite measure on $\mathbf{S}$. The set function $-\mathfrak{A}_*(t_2, X, \cdot)$ by ((c)A, §11) is a ν -continuous totally finite signed measure on $\mathbf{S}$. Thus the properties of $-\mathfrak{A}_*(t_2, X, \cdot)$

parallel those of $\mu' = \overline{\mu'_X} \in \mathfrak{M}_{\nu, t_2}$, and the value $\mu'(E_2)$ being the radiant energy in E_2 at t_2 (relative, in this case, to $\mu(E_1)$) has a counterpart in the value $-\mathfrak{A}_*(t_2, X, E_2)$ which we shall define as the *general radiant (energy) flux* into $E_2 \in \mathbf{S}$ at $t_2 \in T$ (and at $\lambda \in \Lambda$) for the radiative process $\mathfrak{R}_\phi$. Thus while the radiant energy in a measurable set has been fixed as non-negative, its associated radiant flux (change in time) may well be negative. This fact is reflected formally in the *signed* character of the set function $-\mathfrak{A}_*(t_2, X, \cdot)$. We observe that the preceding theorem could have been proved using a much weaker form of R_3 (e.g., using measures instead of functions on (points of) X and less stringent convergence requirements). However, since one of the central aims of this work is to develop a radiative transfer theory from a unified set of axioms, we retain the present form of R_3 . Axiom R_3 will be used to its fullest extent somewhat later in the study.

Corollary. *If $A_*(t_2, E_2) = 0$, then*

$$\partial^+ \mu(t_1, E_1; t_2, E_2) / \partial t_2 = -S_*(t_2, E_2) + S_*(t_2, E'_2).$$

This corollary indicates that if the measure absorption function vanishes for $E_2 \in \mathbf{S}$ at $t_2 \in T$, the phraseology of the definition of general radiant flux receives special justification. If, in addition, X has a topology commensurate with $\mathbf{S}$ (as is the case for the classical carrier space) it would be meaningful, as in the classical case, to say that the net rate of energy flow into E_2 takes place across the boundary of E_2 . If $A_*(t_2, E_2) > 0$, then, in addition to the net flow of energy into E_2 across its boundary, there is a rate of loss of radiant energy in E_2 induced by a mechanism of $\mathfrak{R}_\phi$ which we have summarized analytically by the function $A_*(t_2, \cdot)$ on $\mathbf{S}$. This points up the nature of the *general* radiant flux into E_2 and attempts to draw an abstract parallel to the classical concept of absorption of radiant energy. It is clear from the definition of the measure absorption function that a sufficient condition that it vanish on $\mathbf{S}$ is that the contraction property of the radiative process exhibits an equality (cf. (v) R_1).

13. μ' -differentiable Process Axiom. In order to derive the dynamic transition relations for the various densities of radiative measures, we require further regularity properties of the radiative measures, the radiative process, and the carrier space. We now set about to motivate the μ' -differentiability Axiom, the last of the regularity axioms required for the development of the theory of radiative processes. We begin by proving a simple but important lemma.

Lemma. *Let $(t_1, t_2) \in D$, $\mu \in \mathfrak{M}_{\nu, t_1}$, and $E_1 \in \mathbf{S}$ be arbitrary but fixed. If $E_2, E_3 \in \mathbf{S}$, then*

$$\begin{aligned} \mathfrak{A}_*(t_2, E_2, E_3) &= \mathfrak{A}_*(t_2, E_2 \cap E_3) - \Sigma_*(t_2, E_2 \cap E_3, E_3 - E_2) \\ &\quad - \Sigma_*(t_2, E_2 - E_3, E_3 - E_2) - \Sigma_*(t_2, E_2 - E_3, E_2 \cap E_3). \end{aligned}$$

Proof: By ((c)A, §11),

$$\mathfrak{A}_*(t_2, E_2, E_3) = \mathfrak{A}_*(t_2, E_2, E_3 - E_2) + \mathfrak{A}_*(t_2, E_2, E_2 \cap E_3).$$

By ((b)A, §11),

$$\mathfrak{V}_*(t_2, E_2, E_3 - E_2) = \mathfrak{V}_*(t_2, E_2 \cap E_3, E_3 - E_2) \\ + \mathfrak{V}_*(t_2, E_2 - E_3, E_3 - E_2).$$

Similarly,

$$\mathfrak{V}_*(t_2, E_2, E_2 \cap E_3) = \mathfrak{V}_*(t_2, E_2 \cap E_3, E_2 \cap E_3) \\ + \mathfrak{V}_*(t_2, E_2 - E_3, E_2 \cap E_3).$$

Since $(E_2 \cap E_3, E_3 - E_2)$, $(E_2 - E_3, E_3 - E_2)$, and $(E_2 - E_3, E_2 \cap E_3)$ are elements in the T -section of $\mathbf{V}$ determined by $t_2 \in T$ and $(t_2, E_2 \cap E_3, E_2 \cap E_3)$ is an element of $T \times \Delta$, the lemma now follows from the definitions of $\mathfrak{Q}_*$ and Σ_* .

From (i)R₃ with given fixed E_3 and F ,

$$\chi_{E_3 \cap F} - g'(t_2, \cdot; t_2 + \Delta_n, E_3) = h'_*(t_2, \cdot, E_3)\Delta_n + o(\Delta_n, \cdot) [\nu]$$

for every positive integer n . $o(\Delta_n, \cdot)$ is a μ' -integrable function on X for every positive integer n . Further, $o(\Delta_n, \cdot)/\Delta_n \rightarrow 0$ (the zero function) uniformly as $n \rightarrow \infty$. Hence

$$\mu'(E_2 \cap E_3) - \overline{\mu'}(t_2, E_2; t_2 + \Delta_n, E_3) = \mathfrak{V}_*(t_2, E_2, E_3)\Delta_n + \int_{E_3} o(\Delta_n, \cdot) d\mu',$$

$E_2 \in \mathbf{S}$, for $n = 1, 2, 3, \dots$. By the preceding lemma this may be written as

$$\mu'(E_2 \cap E_3) - \overline{\mu'}(t_2, E_2; t_2 + \Delta_n, E_3) = [\mathfrak{Q}_*(t_2, E_2 \cap E_3) \\ - \Sigma_*(t_2, E_2 \cap E_3, E_3 - E_2) - \Sigma_*(t_2, E_2 - E_3, E_3 - E_2) \\ - \Sigma_*(t_2, E_2 - E_3, E_2 \cap E_3)]\Delta_n + \int_{E_3} o(\Delta_n, \cdot) d\mu',$$

$n = 1, 2, 3, \dots$. In order to obtain the dynamic transition relations for the various densities it is clear from this equation that it must be possible to perform set differentiations on the indicated set functions. The ground work for these operations is prepared by the following discussion.

A *net* in X ([15], p. 296) is a countable class of disjoint measurable sets of finite ν -measure whose union is X . Since $(X, \mathbf{S}, \nu)$ is a totally σ -finite measure space, the collection of all nets in X is not empty. This is true also for the measure spaces $(X, \mathbf{S}, \mu)$, $\mu \in \mathfrak{M}_{\nu, t}$, $t \in T$. A sequence $\{\mathfrak{N}_i\}$ of nets in X is *monotone* ([15], p. 296) if, for each positive integer n , every set of $\mathfrak{N}_{n+1}$ is a subset of some set of $\mathfrak{N}_n$. Let $\mu \in \mathfrak{M}_{\nu, t}$ and let m be a μ -continuous set function on $\mathbf{S}$. We define for each positive integer n the point function d_n on X by choosing for each $x \in X$ the (unique) set $F_n \in \mathfrak{N}_n$ such that $x \in F_n$ and setting

$$d_n(x) = \begin{cases} \frac{m(F_n)}{\mu(F_n)} & \text{for } \mu(F_n) > 0 \\ 0 & \text{for } \mu(F_n) = 0. \end{cases}$$

The *upper and lower derivates of m with respect to $\mathfrak{N} = \{\mathfrak{N}_i\}$ and μ* will be denoted by

$$\overline{D}[\mathfrak{N}, \mu]m(x) = \limsup_n d_n(x),$$

$$\underline{D}[\mathfrak{N}, \mu]m(x) = \liminf_n d_n(x),$$

respectively. m is said to be *differentiable at x with respect to $\mathfrak{N}$ and μ* if

$$-\infty < \underline{D}[\mathfrak{N}, \mu]m(x) = \overline{D}[\mathfrak{N}, \mu]m(x) < \infty,$$

and the common value will be the *derivative of m at x with respect to $\mathfrak{N}$ and μ* and will be designated by $D[\mathfrak{N}, \mu]m(x)$. The function d_n is clearly measurable, and so are $\overline{D}[\mathfrak{N}, \mu]m$, $\underline{D}[\mathfrak{N}, \mu]m$, and $D[\mathfrak{N}, \mu]m$ ([8], p. 84). If $\mathfrak{N}$ is a sequence of nets, let $\mathfrak{N}_0 = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$. $\mathfrak{N}$ is said to be *regular on $E \in \mathfrak{S}$ with respect to μ* ([15], p. 297) if, given $\epsilon > 0$, there is a sequence K of sets from $\mathfrak{N}_0$ such that $\mu(E - \bigcup_{i=1}^{\infty} K_i) = 0$, $\mu(\bigcup_{i=1}^{\infty} K_i) \leq \mu(E) + \epsilon$.

With these preliminaries in mind we return to consider the family of measure attenuating functions. By definition,

$$A_*(t_2, E_2) = \mathfrak{A}_*(t_2, E_2, X) = \int_{E_*} h'_*(t_2, \cdot, X) d\mu',$$

where μ' is defined by the fixed elements of the lemma. The set function $A_*(t_2, \cdot)$ is μ' -continuous, and if $\mathfrak{N}$ is a monotone sequence of nets regular with respect to μ' on each measurable set then ([15], p. 299) $D[\mathfrak{N}, \mu']A_*(t_2, \cdot) = h'_*(t_2, \cdot, X)$ [ν]. (Clearly, by the source cited, there is equality [μ']; however, on the maximal full support of μ' there is equality [ν] (§8), and, by earlier conventions, there is also equality [ν] on F' .) We write $a_*(t_2, \cdot) \equiv h'_*(t_2, \cdot, X)$. Again, by definition,

$$\begin{aligned} A_*(t_2, E_2 \cap E_3) &= \mathfrak{A}_*(t_2, E_2 \cap E_3, X) \\ &= \int_{E_2 \cap E_*} h'_*(t_2, \cdot, X) d\mu' = \int_{E_*} \chi_{E_2} h'_*(t_2, \cdot, X) d\mu', \end{aligned}$$

so that $A_*(t_2, E_3 \cap \cdot)$, the restriction of $A_*(t_2, \cdot)$ to $E_3 \in \mathfrak{S}$, has the μ' -derivative

$$D[\mathfrak{N}, \mu']A_*(t_2, E_3 \cap \cdot) = \chi_{E_3} a_*(t_2, \cdot) [\nu].$$

An examination of the set functions $\alpha_*(t_2, E_3 \cap \cdot)$, $\Sigma_*(t_2, E_3 \cap \cdot, E_3 \cap (\cdot)')$, $\Sigma_*(t_2, (E_3)' \cap \cdot, E_3 \cap (\cdot)')$, and $\Sigma_*(t_2, (E_3)' \cap \cdot, E_3 \cap \cdot)$ appearing in the decomposition of $\mathfrak{A}_*(t_2, \cdot, E_3)$ yields the realization that the use of the theorem (implicitly cited) above for obtaining the derivative of $A_*(t_2, E_3 \cap \cdot)$ with respect to $\mathfrak{N}$ and μ' cannot be repeated. For example,

$$\begin{aligned} \alpha_*(t_2, E_2 \cap E_3) &= \mathfrak{A}_*(t_2, E_2 \cap E_3, E_2 \cap E_3) \\ &= \int_{E_2 \cap E_*} h'_*(t_2, \cdot, E_2 \cap E_3) d\mu', \end{aligned}$$

so that if $\mathfrak{N}$ is a monotone sequence of nets regular with respect to μ' on each measurable set,

$$\lim_n \mathfrak{Q}_*(t_2, E_3 \cap F_n) / \mu'(F_n) = \lim_n \int_{F_n} \chi_{E_3} h'_*(t_2, \cdot, F_n \cap E_3) d\mu' / \mu'(F_n)$$

involves the elements of the members of the monotone sequence in the integrand. The μ' -continuity of the above set functions allows but does not insure the existence of derivatives with respect to $\mathfrak{N}$ and μ' . We also observe that

$$\begin{aligned} \Sigma_*(t_2, E_3 \cap \cdot, E_3 \cap (\cdot)') + \Sigma_*(t_2, (E_3)' \cap \cdot, E_3 \cap (\cdot)') \\ = \Sigma_*(t_2, \cdot, E_3 \cap (\cdot)'), \end{aligned}$$

hence its corresponding function $\Sigma_*(t_2, E_2, (E_2)' \cap \cdot)$ by ((c)B, §11) is countably additive. It is desirable that the derivative of the latter set function retain this property. An examination of this situation would be analogous to that given in detail in (C, §5) for g -densities, and the conclusions would be similar. This hair's breadth gap between countable additivity almost everywhere $[\nu]$ and "almost countable additivity" is harmlessly bridged as before by axiomatic dictum. Finally, the property exhibited by the derivative of $A_*(t_2, E_3 \cap \cdot)$ is needed for the derivative of $S_*(t_2, E_3 \cap \cdot)$. Similar properties are also required for the Σ_* functions appearing in the decomposition of $\mathfrak{A}_*(t_2, \cdot, E_3)$. Such properties are generalizations of the properties possessed by metric densities in the (Euclidean) carrier spaces of the form E_n . With these preliminaries we state the following axiom.

R_4 : Let $(t_1, t_2) \in D$, $\mu \in \mathfrak{M}_{r, t_1}$, and $E_1 \in \mathbf{S}$ be arbitrary but fixed.

- (i) There exists a monotone sequence $\mathfrak{N}$ of nets in X regular with respect to μ' on each measurable set.
- (ii) Each of the following derivatives with respect to $\mathfrak{N}$ and μ' exists and has an essential bound $[\nu]$ independent of $E_3 \in \mathbf{S}$.

$$(a) \quad D[\mathfrak{N}, \mu'] S_*(t_2, \cdot) \equiv s_*(t_2, \cdot);$$

$$D[\mathfrak{N}, \mu'] S_*(t_2, E_3 \cap \cdot) = \chi_{E_3} s_*(t_2, \cdot) [\nu].$$

$$(b) \quad D[\mathfrak{N}, \mu'] \Sigma_*(t_2, E_3 \cap \cdot, E_3 \cap (\cdot)') \equiv {}^1 \bar{\sigma}_*(t_2, \cdot, E_3)$$

and vanishes $[\nu]$ on E'_3 .

$$(c) \quad D[\mathfrak{N}, \mu'] \Sigma_*(t_2, (E_3)' \cap \cdot, E_3 \cap (\cdot)') \equiv {}^2 \bar{\sigma}_*(t_2, \cdot, E_3)$$

and vanishes $[\nu]$ on E_3 .

$$(d) \quad D[\mathfrak{N}, \mu'] \Sigma_*(t_2, (E_3)' \cap \cdot, E_3 \cap \cdot) \equiv {}^3 \bar{\sigma}_*(t_2, \cdot, E_3)$$

and vanishes $[\nu]$ on X .

- (iii) The corresponding set function ${}^1 \bar{\sigma}_*(t_2, x, \cdot) + {}^2 \bar{\sigma}_*(t_2, x, \cdot)$ is countably additive for almost every $[\nu] x \in X$.

Henceforth the process $\mathcal{R}_\phi$ will be considered to obey R_4 . The derivatives in R_4 are unique $[\nu]$ ([15], p. 302). Since ((e)B, §11)

$$\mathcal{R}_*(t_2, E_3 \cap \cdot) = A_*(t_2, E_3 \cap \cdot) + S_*(t_2, E_3 \cap \cdot),$$

we have

$$D[\mathcal{R}, \mu']\mathcal{R}_*(t_2, E_3 \cap \cdot) = \chi_{E_3}a_*(t_2, \cdot) + \chi_{E_3}s_*(t_2, \cdot) [\nu].$$

Let $D[\mathcal{R}, \mu']\mathcal{R}_*(t_2, \cdot) \equiv \alpha_*(t_2, \cdot)$. Hence

$$\alpha_*(t_2, \cdot) = a_*(t_2, \cdot) + s_*(t_2, \cdot) [\nu].$$

The combination occurring in (iii) R_4 is to be used repeatedly. For brevity, we define

$$\bar{\sigma}_*(t_2, \cdot, E_3) \equiv {}^1\bar{\sigma}_*(t_2, \cdot, E_3) + {}^2\bar{\sigma}_*(t_2, \cdot, E_3).$$

The correspondent to this is the combination occurring in (iii) R_4 . As an immediate consequence of this definition, we have

$$\bar{\sigma}_*(t_2, \cdot, X) = {}^1\bar{\sigma}_*(t_2, \cdot, X) = s_*(t_2, \cdot) [\nu].$$

14. Density Attenuating Functions. The functions $\bar{\sigma}_*(t, \cdot, E)$, $s_*(t, \cdot)$, $a_*(t, \cdot)$, and $\alpha_*(t, \cdot)$ will be referred to collectively as the *density attenuating functions* for $\mathcal{R}_\phi$, where $\mathcal{R}_\phi$ is the radiative process. $\bar{\sigma}_*(t, \cdot, E)$ is the *density scattering function into $E \in \mathbf{S}$* ; $s_*(t, \cdot)$ is the *total density scattering function*; $a_*(t, \cdot)$ is the *density absorption function*; and $\alpha_*(t, \cdot)$ is the *density attenuation function*.

The non-negative, ν -continuous ((c)B, §11), countably additive ((iii) R_4), a.e. $[\nu]$ totally finite ((iii) R_4) set function $\bar{\sigma}_*(t, x, \cdot)$ has an R-N derivative $\sigma_*(t, x, \cdot)$ with respect to ν for almost every $x \in X$. We will call $\sigma_*(t, x, \cdot)$ the *density scattering function*. This function may be viewed as the X -section determined by $x \in X$ of the function $\sigma_*(t, \cdot, \cdot)$ defined on the measurable space $(X \times X, \mathbf{S} \times \mathbf{S})$. Further, by introducing the product measure $\nu \times \nu$, we observe that $\sigma_*(t, \cdot, \cdot)$ is determined uniquely $[\nu \times \nu]$; there is no general guarantee ([8], p. 142), however, that $\sigma_*(t, \cdot, \cdot)$ has the respectability of measurability on $(X \times X, \mathbf{S} \times \mathbf{S})$, and this indispensable property will be endowed by fiat, and considered in force in the sequel (cf. (i) R_2). Its formal announcement rightly belongs in (iii) R_4 .

Since $\bar{\sigma}_*(t, x, E) = \int_E \sigma_*(t, x, \cdot) d\nu$, we see from the definitions above that the family of density attenuating functions is thus describable in terms of two functions: $\sigma_*(t, \cdot, \cdot)$ and $a_*(t, \cdot)$; one characterizes the scattering, the other the absorption phenomena of $\mathcal{R}_\phi$, respectively. We will consider $\sigma_*(t, \cdot, \cdot)$ a member of the family of density attenuating functions.

15. Dynamic Transition Relations for Densities.

Theorem A. Let $(t_1, t_2) \in D$, $\mu \in \mathcal{M}_{\nu, t_1}$, and $E_1 \in \mathbf{S}$ be arbitrary but fixed. Let $\mu' = \mu_{E_1}R_{t_1, t_2}$, and, for $(t_2, t_3) \in D$, let $g(t_1, \cdot; t_3, E_2)$ be the g -density associated

with $\overline{\mu'}(t_1, \cdot; t_2, E_2)$, $E_2 \in \mathbf{S}$. If $\{\Delta_n\}$ is any decreasing null sequence of positive elements in T , then:

$$(a) \quad \lim_n (g(t_1, x_1; t_2 + \Delta_n, E_2) - g(t_1, x_1; t_2, E_2))/\Delta_n \\ \equiv \partial^+ g(t_1, x_1; t_2, E_2)/\partial t_2$$

exists and is finite for every $E_2 \in \mathbf{S}$ and almost every $[\nu]$ $x_1 \in X$.

$$(b) \quad \partial^+ g(t_1, x_1; t_2, E_2)/\partial t_2 = - \int_{E_2} \alpha_*(t_2, x_2) g(t_1, x_1; t_2, dx_2) \\ + \int_X \bar{\sigma}_*(t_2, x_2, E_2) g(t_1, x_1; t_2, dx_2).$$

Proof: By (i)R₃,

$$\chi_{E_2 \cap F} - g'(t_2, \cdot; t_2 + \Delta_n, E_2) = h'_*(t_2, \cdot, E_2)\Delta_n + o(\Delta_n, \cdot) [\nu]$$

for each positive integer n . For each n , $o(\Delta_n, \cdot)$ is a μ' -integrable function on X , and $o(\Delta_n, \cdot)/\Delta_n \rightarrow 0$ uniformly on X as $n \rightarrow \infty$. For each $E \in \mathbf{S}$ and positive integer n ,

$$\mu'(E \cap E_2) - \overline{\mu'}(t_2, E; t_2 + \Delta_n, E_2) = \mathfrak{X}_*(t_2, E, E_2)\Delta_n + \int_E o(\Delta_n, \cdot) d\mu'$$

which, by lemma §13, may be written

$$\mu'(E \cap E_2) - \overline{\mu'}(t_2, E; t_2 + \Delta_n, E_2) = [\mathfrak{Q}_*(t_2, E, E_2) \\ - \Sigma_*(t_2, E \cap E_2, E_2 - E) - \Sigma_*(t_2, E - E_2, E_2 - E) \\ - \Sigma_*(t_2, E - E_2, E \cap E_2)]\Delta_n + \int_E o(\Delta_n, \cdot) d\mu'.$$

Using R₄, ([15], p. 299), and the definitions of the density attenuating functions,

$$\chi_{E_2 \cap F} - g'(t_2, \cdot; t_2 + \Delta_n, E_2) = \\ [\chi_{E_2}\alpha_*(t_2, \cdot) - \bar{\sigma}_*(t_2, \cdot, E_2)]\Delta_n + o(\Delta_n, \cdot) [\nu].$$

Hence

$$(\chi_{E_2 \cap F} - g'(t_2, \cdot; t_2 + \Delta_n, E_2))/\Delta_n \rightarrow \chi_{E_2}\alpha_*(t_2, \cdot) - \bar{\sigma}_*(t_2, \cdot, E_2)$$

uniformly a.e. $[\nu]$ on X . Using the transition relations for g -densities and observing that $g(t_1, x_1; t_2, X) < \infty$ for almost every $[\nu]$ $x_1 \in X$, we have

$$\lim_n (g(t_1, x_1; t_2 + \Delta_n, E_2) - g(t_1, x_1; t_2, E_2))/\Delta_n \\ = \lim_n \int_X [(g'(t_2, x_2; t_2 + \Delta_n, E_2) - \chi_{E_2 \cap F}(x_2))/\Delta_n] g(t_1, x_1; t_2, dx_2) \\ = \int_X [-\chi_{E_2}\alpha_*(t_2, x_2) + \bar{\sigma}_*(t_2, x_2, E_2)] g(t_1, x_1; t_2, dx_2) < \infty$$

for almost every $[\nu]$ $x_1 \in X$. This ends the proof.

Theorem B. *Retaining the hypothesis of Theorem A, then:*

$$(a) \quad \lim_n (\gamma(t_1, x_1; t_2 + \Delta_n, \cdot) - \gamma(t_1, x_1; t_2, \cdot))/\Delta_n \equiv \partial^+ \gamma(t_1, x_1; t_2, \cdot)/\partial t_2$$

exists and is a.e. $[\nu]$ finite for almost every $[\nu]$ $x_1 \in X$.

$$(b) \quad \partial^+ \gamma(t_1, x_1; t_2, x_2)/\partial t_2 = -\alpha_*(t_2, x_2)\gamma(t_1, x_1; t_2, x_2) \\ + \int_X \gamma(t_1, x_1; t_2, x')\sigma_*(t_2, x', x_2) d\nu(x') [\nu].$$

Proof: Let

$$\gamma_n = (\gamma(t_1, x_1; t_2 + \Delta_n, \cdot) - \gamma(t_1, x_1; t_2, \cdot))/\Delta_n.$$

By ((b)A, §7) and Theorem A, for almost every $[\nu]$ $x_1 \in X$

$$\lim_n \int_{E_2} \gamma_n d\nu = \int_{E_2} \alpha_*(t_2, x_2)g(t_1, x_1; t_2, dx_2) \\ + \int_X \bar{\sigma}_*(t_2, x_2, E_2)g(t_1, x_1; t_2, dx_2) < \infty.$$

Thus the sequence $\{\int_{E_2} \gamma_n d\nu\}$ of real numbers is fundamental for almost every $[\nu]$ $x_1 \in X$ and every $E_2 \in \mathbf{S}$. By (iii)R₃, $\{\gamma_n\}$ is fundamental in measure for almost every $[\nu]$ $x_1 \in X$, so that ([15], p. 236) $\{\gamma_n\}$ is mean fundamental and there exists ([8], p. 107) a ν -integrable function $\gamma^+ \equiv \partial^+ \gamma(t_1, x_1; t_2, \cdot)/\partial t_2$ to which $\{\gamma_n\}$ converges in the mean for almost every $[\nu]$ $x_1 \in X$. Hence, for every $E_2 \in \mathbf{S}$, by a Fubini argument,

$$\int_{E_2} \gamma^+(x_2) d\nu(x_2) = \int_{E_2} \left[-\alpha_*(t_2, x_2)\gamma(t_1, x_1; t_2, x_2) \right. \\ \left. + \int_X \sigma_*(t_2, x', x_2)\gamma(t_1, x_1; t_2, x') d\nu(x') \right] d\nu(x_2).$$

An application of ([8], p. 105) yields (b) and completes the proof.

The result in part (b) of the preceding theorem is the *dynamic transition relation for γ -densities*. (b) of Theorem A gives the *dynamic transition relation for g -densities*. Part (b) of the following theorem gives the *dynamic transition relation for G -densities*.

Theorem C. *Retaining the hypothesis of Theorem A, then:*

(a)

$$\lim_n (G(t_1, E_1; t_2 + \Delta_n, \cdot) - G(t_1, E_1; t_2, \cdot))/\Delta_n \equiv \partial^+ G(t_1, E_1; t_2, \cdot)/\partial t_2$$

exists, is a.e. $[\nu]$ finite, and

(b)

$$\begin{aligned} \partial^+ G(t_1, E_1; t_2, x_2) / \partial t_2 &= -\alpha_*(t_2, x_2) G(t_1, E_1; t_2, x_2) \\ &+ \int_X G(t_1, E_1; t_2, x') \sigma_*(t_2, x', x_2) d\nu(x') [\nu]. \end{aligned}$$

Proof: Let

$$g_n = (g(t_1, \cdot; t_2 + \Delta_n, E_2) - g(t_1, \cdot; t_2, E_2)) / \Delta_n.$$

From the proof of Theorem A,

$$\begin{aligned} g_n(x_1) &= \int_X [-\chi_{E_2} \alpha_*(t_2, x_2) + \bar{\sigma}_*(t_2, x_2, E_2)] g(t_1, x_1; t_2, dx_2) \\ &+ \int_X (o(\Delta_n, x_2) / \Delta_n) g(t_1, x_1; t_2, dx_2) \end{aligned}$$

for every positive integer n . For any given $\epsilon > 0$, there exists a positive integer n_0 such that for $n \geq n_0$

$$|g_n(x_1) - \int_X [] g(t_1, x_1; t_2, dx_2)| < \epsilon g(t_1, x_1; t_2, X) < \infty$$

for almost every $[\nu] x_1 \in X$. Let

$$H_n(x_1) = g_n(x_1) - \int_X [] g(t_1, x_1; t_2, dx_2), \quad n = 1, 2, 3, \dots$$

Since each H_n and $g(t_1, \cdot; t_2, X)$ are ν -integrable, and since $H_n \rightarrow 0$ a.e. $[\nu]$, we conclude that $H_n \rightarrow 0$ in the mean by the Lebesgue bounded convergence theorem. That is, for $E_1 \in \mathbf{S}$,

$$\int_{E_1} g_n(x_1) d\nu(x_1) \rightarrow \int_{E_1} \left(\int_X [] g(t_1, x_1; t_2, dx_2) \right) d\nu(x_1),$$

as $n \rightarrow \infty$. This limit is finite since $G(t_1, E_1; t_2, \cdot)$ is ν -integrable and the density attenuating functions are essentially bounded $[\nu]$. By (A, §7) we may write for arbitrary $E_2 \in \mathbf{S}$

$$\int_{E_1} g_n(x_1) d\nu(x_1) = \int_{E_2} G_n(x_2) d\nu(x_2),$$

where G_n is defined in (iv)R₃ for every positive integer n . By integrating the limit in (b) of Theorem A over $E_1 \in \mathbf{S}$ and applying a Fubini argument, the result may be written as

$$\begin{aligned} - \int_{E_2} \alpha_*(t_2, x_2) G(t_1, E_1; t_2, x_2) d\nu(x_2) \\ + \int_{E_2} \left(\int_X G(t_1, E_1; t_2, x') \sigma_*(t_2, x', x_2) d\nu(x') \right) d\nu(x_2) < \infty. \end{aligned}$$

Since $\{G_n\}$ is fundamental in measure ((iv) R_3) and since, for every $E_2 \in S$, $\{\int_{E_2} G_n d\nu\}$ is a fundamental sequence of real numbers, it follows that $\{G_n\}$ is mean fundamental and therefore converges in the mean to a ν -integrable function $G^+ \equiv \partial^+ G(t_1, E_1; t_2, \cdot)/\partial t_2$. (b) now follows by the same arguments as those used to conclude the proof of Theorem B.

16. Restricted Processes on General Carrier Spaces. Let (X, S, ν) be a carrier space. Let $S_x = \{E : x \in E \in S\}$ for every $x \in X$. The radiative process $\mathcal{R}_\phi$ is said to be *restricted at* $(t, x) \in T \times X$ iff $\bar{\sigma}_*(t, x, \cdot)$ is countably additive and $\bar{\sigma}_*(t, x, E') = 0$ for every $E \in S_x$, otherwise it is *unrestricted at* (t, x) . Since $\bar{\sigma}_*(t, x, E) + \bar{\sigma}_*(t, x, E') = s_*(t, x)$, the above condition may be replaced by $\bar{\sigma}_*(t, x, E) = s_*(t, x)$. $\mathcal{R}_\phi$ is said to be *restricted on* $(t, R) \in T \times S$ iff it is restricted at (t, x) for almost every $x \in R$, otherwise it is *unrestricted on* (t, R) .

Theorem A. *Retaining the hypothesis of (A, §15), let $\mathcal{R}_\phi$ be restricted on $(t_2, R) \in T \times S$; then*

$$\begin{aligned} \partial^+ G(t_1, E_1; t_2, x_2)/\partial t_2 = & -(a_*(t_2, x_2) + \chi_{R'}(x_2)s_*(t_2, x_2))G(t_1, E_1; t_2, x_2) \\ & + \int_{R'} G(t_1, E_1; t_2, x')\sigma_*(t_2, x', x_2) d\nu(x') [\nu]. \end{aligned}$$

Proof: For every $E_2 \in S$, we may write (proof, (C, §15))

$$\begin{aligned} \int_{E_2} G^+ d\nu = & - \int_{E_2} \alpha_*(t_2, \cdot)G(t_1, E_1; t_2, \cdot) d\nu \\ & + \int_X G(t_1, E_1; t_2, \cdot)\bar{\sigma}_*(t_2, \cdot, E_2) d\nu < \infty. \end{aligned}$$

Partitioning X so that

$$\int_X = \int_{R \cap E_2} + \int_{R \cap E_2'} + \int_{R'}$$

we observe that

$$\begin{aligned} (i) \quad & \int_{E_2 \cap R} G\bar{\sigma}_* d\nu = \int_{E_2 \cap R} Gs_* d\nu; \\ (ii) \quad & \int_{R \cap E_2'} G\bar{\sigma}_* d\nu = 0; \\ (iii) \quad & \int_{R'} G\bar{\sigma}_* d\nu = \int_{E_2} \left(\int_{R'} G\sigma_* d\nu \right) d\nu. \end{aligned}$$

For (i), by the definition of restrictedness, $\bar{\sigma}_*(t_2, x, E_2) = s_*(t_2, x)$ for almost every $x \in E_2 \cap R$. For (ii), $\bar{\sigma}_*(t_2, x, E_2) = \bar{\sigma}_*(t_2, x, (E_2')') = 0$ for almost every $x \in E_2' \cap R$. (iii) follows by a Fubini argument using $\sigma_*(t, \cdot, \cdot)$ and the finiteness of $\int_{R'} G\bar{\sigma}_* d\nu$. Hence, for every $E_2 \in S$, the integral over X may be written

$$\int_{E_2} \chi_R s_*(t_2, \cdot) G(t_1, E_1; t_2, \cdot) d\nu + \int_{E_2} \left(\int_{R'} G(t_1, E_1; t_2, x') \sigma_*(t_2, x', \cdot) d\nu(x') \right) d\nu.$$

By ([8], p. 105), the identity $1 = \chi_E + \chi_{E'}$, $E \subset X$, and the decomposition of $\alpha_*(t_2, \cdot)$ given after R_4 , the assertion of the theorem follows.

Analogous dynamic transition relations for γ - and g -densities may be derived, but we shall not carry out the details in this study. The G -density has been singled out for specific study mainly for the reasons cited in §9, and results obtained for it will be typical of the results obtainable for the other densities.

Corollary. (a) If $\mathcal{R}_\phi$ is restricted on (t_2, R) and $\nu(R) = 0$, then ((b)C, §15) holds.

(b) If $\mathcal{R}_\phi$ is restricted on (t_2, X) , then

$$\partial^+ G(t_1, E_1; t_2, x_2) / \partial t_2 = -a_*(t_2, x_2) G(t_1, E_1; t_2, x_2) [\nu].$$

17. Restricted Processes on Product Carrier Spaces. Let $(X, \mathbf{S}, \nu)$ be the Cartesian product of two carrier spaces $(X_i, \mathbf{S}_i, \nu_i)$, $i = 1, 2$, with M_1 valid on each. $\mathbf{S}_1 \times \mathbf{S}_2$ is the σ -algebra of subsets of $X = X_1 \times X_2$ generated by the class of all sets of the form $A_1 \times A_2$, $A_i \in \mathbf{S}_i$, $i = 1, 2$, which are the *measurable rectangles* in X . In addition, the product measure $\nu = \nu_1 \times \nu_2$ is totally σ -finite ([8], p. 144) with $\nu(X_1 \times X_2) = \nu_1(X_1)\nu_2(X_2) > 0$, so that $(X, \mathbf{S}, \nu)$ is a carrier space. The collections of ν_i -continuous measures on $\mathbf{S}_i$, $i = 1, 2$, give rise to ν -continuous product measures on $\mathbf{S}$ ([8], p. 144) and the latter can be defined in a natural way so as to satisfy the conditions of M_1 . A carrier space $(X, \mathbf{S}, \nu)$ constructed in this way will be called a *product carrier space* and each coordinate space of $(X, \mathbf{S}, \nu)$ a *coordinate carrier space*. It is clear that product carrier spaces can be compounded of an arbitrary finite number of coordinate carrier spaces to form finite dimensional carrier spaces of dimension greater than 2 with M_1 valid on the product.

Let $\mathcal{R}_\phi$ be a radiative process on the product carrier space $(X, \mathbf{S}, \nu) = (X_1 \times X_2, \mathbf{S}_1 \times \mathbf{S}_2, \nu_1 \times \nu_2)$. The process $\mathcal{R}_\phi$ is said to be *X_1 -restricted at $(t, (x_1, x_2)) \in T \times X$* iff $\bar{\sigma}_*(t, x, \cdot)$ is countably additive and $\bar{\sigma}_*(t, x, E'_1 \times E_2) = 0$ for every measurable rectangle $E_1 \times E_2 \in \mathbf{S}_x$, where $x = (x_1, x_2) \in X_1 \times X_2$. The process $\mathcal{R}_\phi$ is said to be *X_2 -restricted at $(t, (x_1, x_2)) \in T \times X$* iff $\bar{\sigma}_*(t, x, \cdot)$ is countably additive and $\bar{\sigma}_*(t, x, E_1 \times E'_2) = 0$ for every measurable rectangle $E_1 \times E_2 \in \mathbf{S}_x$. $\mathcal{R}_\phi$ on X is *X_i -restricted on $(t, R) \in T \times \mathbf{S}$* iff it is X_i -restricted at (t, x) for almost every $[\nu] x \in R$, $i = 1, 2$. The concepts on X_i -restrictedness of $\mathcal{R}_\phi$ on the coordinate carrier spaces are sufficient—when taken together—to characterize the concept of restrictedness of $\mathcal{R}_\phi$ on the product carrier space, as is proved in the following theorem.

Theorem A. Let $(X, \mathbf{S}, \nu) = (X_1 \times X_2, \mathbf{S}_1 \times \mathbf{S}_2, \nu_1 \times \nu_2)$ be a product carrier space with radiative process $\mathcal{R}_\phi$. Then the following two conditions are equivalent:

- (a) $\mathcal{R}_\phi$ is restricted at $(t, x) \in T \times X$.
 (b) $\mathcal{R}_\phi$ is both X_1 -restricted and X_2 -restricted at $(t, x) \in T \times X$.

Proof: (a) $\Rightarrow$ (b). Only X_1 -restrictedness will be demonstrated, the proof of X_2 -restrictedness is similar. If $A \in \mathbf{S}_{x_1}$, $B \in \mathbf{S}_{x_2}$, $x = (x_1, x_2) \in X_1 \times X_2$, then $A \times B \in \mathbf{S}_x$, and the measurable set $(A \times B)' \supset A' \times B \in \mathbf{S}$. Hence

$$0 = \bar{\sigma}_*(t, x, (A \times B)') \geq \bar{\sigma}_*(t, x, A' \times B) \geq 0.$$

(b) $\Rightarrow$ (a). For any $x \in X$, $\mathbf{S}$ may be partitioned into $\mathbf{S}_x$ and its complement $\mathbf{S}'_x$, where $\mathbf{S}_x = \{E : x \in E \in \mathbf{S}\}$. Define

$$\mathbf{R}_x = \{E : E \in \mathbf{S}_x, \bar{\sigma}_*(t, x, E') = 0\}$$

and

$$\mathbf{T}_x = \{E : E \in \mathbf{S}'_x, \bar{\sigma}_*(t, x, E) = 0\}.$$

Clearly $\mathbf{R}_x \subset \mathbf{S}_x$, $\mathbf{T}_x \subset \mathbf{S}'_x$, and $\mathbf{R}_x \cap \mathbf{T}_x = \emptyset \in \mathbf{S}$. The implication (b) $\Rightarrow$ (a) will be proved if the following three statements are shown to be true:

- (i) $\mathbf{R}_x \cup \mathbf{T}_x$ contains all measurable rectangles in $\mathbf{S}$.
 (ii) $\mathbf{R}_x \cup \mathbf{T}_x$ is a σ -algebra.
 (iii) $\mathbf{R}_x \cup \mathbf{T}_x = \mathbf{S}$ iff $\mathcal{R}_\phi$ is restricted at (t, x) .

By (i) and (ii), since $\mathbf{S}$ is by definition the smallest σ -algebra containing all measurable rectangles, the union of classes in (i) must contain $\mathbf{S}$, so that $\mathbf{R}_x \cup \mathbf{T}_x = \mathbf{S}$. (iii) would then complete the proof.

Proof of (i). First, if $A \times B \in \mathbf{S}_x$, $A \in \mathbf{S}_{x_1}$, $B \in \mathbf{S}_{x_2}$, then, since $(A \times B)' \subset (A' \times X_2) \cup (X_1 \times B')$,

$$0 \leq \bar{\sigma}_*(t, x, (A \times B)') \leq \bar{\sigma}_*(t, x, A' \times X_2) + \bar{\sigma}_*(t, x, X_1 \times B').$$

Each summand, by hypothesis (b), is zero, so that $A \times B \in \mathbf{R}_x$. Second, if $A \times B \in \mathbf{S}'_x$, then, $\{(A' \times B), (A \times B'), (A' \times B')'\}$ being a partition of $(A \times B)' \in \mathbf{S}_x$, precisely one of the following three statements must hold: $A' \times B \in \mathbf{S}_x$, $A \times B' \in \mathbf{S}_x$, $A' \times B' \in \mathbf{S}_x$. If $A' \times B \in \mathbf{S}_x$, then, since $A \times B = (A')' \times B$, by the X_1 -restrictedness of $\mathcal{R}_\phi$ at (t, x) , $\bar{\sigma}_*(t, x, A \times B) = \bar{\sigma}_*(t, x, (A')' \times B) = 0$. If, on the other hand, $A \times B' \in \mathbf{S}_x$, $\bar{\sigma}_*(t, x, A \times B) = \bar{\sigma}_*(t, x, A \times (B')') = 0$ by the X_2 -restrictedness of $\mathcal{R}_\phi$ at (t, x) . Finally, if $A' \times B' \in \mathbf{S}_x$, then $(A' \times B')' \supset A \times B$, but by the first step above we conclude that $A' \times B' \in \mathbf{R}_x$. The monotonicity of $\bar{\sigma}_*(t, x, \cdot)$ then implies $\bar{\sigma}_*(t, x, A \times B) = 0$. In each case it follows that $A \times B \in \mathbf{T}_x$.

Proof of (ii). Clearly $X \in \mathbf{R}_x$, so that $\mathbf{R}_x \cup \mathbf{T}_x$ is not empty. If $E \in \mathbf{R}_x \cup \mathbf{T}_x$, and, in particular, $E \in \mathbf{R}_x$, then $E' \in \mathbf{S}'_x$ and $\bar{\sigma}_*(t, x, E') = 0$; hence $E' \in \mathbf{T}_x$. On the other hand, if $E \in \mathbf{T}_x$, then $E' \in \mathbf{S}_x$ and $\bar{\sigma}_*(t, x, (E')') = \bar{\sigma}_*(t, x, E) = 0$, so that $E' \in \mathbf{R}_x$. Thus we have shown that $\mathbf{R}_x \cup \mathbf{T}_x$ is closed under the operation of complementation. It remains to show that $\mathbf{R}_x \cup \mathbf{T}_x$ is closed under the operation of countable union. Since $\emptyset \in \mathbf{T}_x$, we consider a non-empty countable subset

$\mathbf{Q}$ of $\mathbf{R}_x \cup \mathbf{T}_x$. $\{\mathbf{Q} \cap \mathbf{R}_x, \mathbf{Q} \cap \mathbf{T}_x\}$ is (in the extended sense) a partition of $\mathbf{Q}$. If $\mathbf{Q} \cap \mathbf{R}_x$ is not empty then $\bigcup (\mathbf{Q} \cap \mathbf{R}_x) \in \mathbf{S}_x$ and, since $(\bigcup (\mathbf{Q} \cap \mathbf{R}_x))' < \mathbf{Q}'_n$ for some $\mathbf{Q}_n \in \mathbf{Q} \cup \mathbf{R}_x$, we have

$$0 \leq \bar{\sigma}_*(t, x, (\bigcup (\mathbf{Q} \cap \mathbf{R}_x))') \leq \bar{\sigma}_*(t, x, \mathbf{Q}'_n) = 0,$$

so that $\bigcup (\mathbf{Q} \cap \mathbf{R}_x) \in \mathbf{R}_x$. If $\mathbf{Q} \cap \mathbf{R}_x$ is empty, $\mathbf{Q} \cap \mathbf{T}_x$ is not empty, and $\bigcup (\mathbf{Q} \cap \mathbf{T}_x)$ is an element of $\mathbf{S}'_x$. Further,

$$0 \leq \bar{\sigma}_*(t, x, \bigcup (\mathbf{Q} \cap \mathbf{T}_x)) \leq \sum_n \bar{\sigma}_*(t, x, \mathbf{Q}_n) = 0,$$

where the summation is over all $\mathbf{Q}_n \in \mathbf{Q} \cap \mathbf{T}_x$. Hence $\bigcup (\mathbf{Q} \cap \mathbf{T}_x) \in \mathbf{T}_x$. Thus, in any case,

$$\bigcup \mathbf{Q} = \bigcup (\mathbf{Q} \cap \mathbf{R}_x) \cup \bigcup (\mathbf{Q} \cap \mathbf{T}_x) \in \mathbf{R}_x \cup \mathbf{T}_x.$$

Proof of (iii). If $\mathcal{R}_\phi$ is restricted at (t, x) , then, for every $E \in \mathbf{S}_x$, $\bar{\sigma}_*(t, x, E') = 0$, so that $\mathbf{S}_x \subset \mathbf{R}_x$. Hence $\mathbf{S}_x = \mathbf{R}_x$. Further, if $E \in \mathbf{S}'_x$, $E' \in \mathbf{S}_x$, so that $\bar{\sigma}_*(t, x, (E')') = 0$; thus $\mathbf{S}'_x \subset \mathbf{T}_x$. Hence $\mathbf{S}'_x = \mathbf{T}_x$. It follows that $\mathbf{R}_x \cup \mathbf{T}_x = \mathbf{S}$. Conversely, if $\mathbf{R}_x \cup \mathbf{T}_x = \mathbf{S} = \mathbf{S}_x \cup \mathbf{S}'_x$, then, in particular, $\mathbf{R}_x = \mathbf{S}_x$ which, by the definition, implies $\mathcal{R}_\phi$ is restricted at (t, x) . This completes the proof of the theorem.

Let us say that a measurable space $(X, \mathbf{S})$ is *discrete on* $Y \subset X$ iff $\{y\} \in \mathbf{S}$ for every $y \in Y$; and that $(X, \mathbf{S})$ is *discrete* iff $(X, \mathbf{S})$ is discrete on X . (b) of Theorem A cannot in general be weakened to have, say, X_1 -restrictedness at (t, x) imply $\mathcal{R}_\phi$ restricted at (t, x) . Since the definition of the three types of restrictedness are purely measure-theoretical and independent of the origin of the measure $\bar{\sigma}_*(t, x, \cdot)$, the following example will illustrate this fact.

Let $(X, \mathbf{S}) = (X_1 \times X_2, \mathbf{S}_1 \times \mathbf{S}_2)$ be a measurable space formed from the discrete measurable spaces $(X_1, \mathbf{S}_1)$ and $(X_2, \mathbf{S}_2)$ such that $X_1 = \{x_{11}, x_{12}, x_{13}\}$, $X_2 = \{x_{21}, x_{22}, x_{23}\}$. For the point (x_{12}, x_{21}) , let $\bar{\sigma}_*((x_{12}, x_{21}), \cdot)$ be an additive set function on $\mathbf{S}_1 \times \mathbf{S}_2$ such that

$$\bar{\sigma}_*((x_{12}, x_{21}), \{x_{1i}\} \times \{x_{2j}\}) = \begin{cases} 0 & \begin{cases} i = 1, 3 \\ j = 1, 2, 3 \end{cases} \\ 1 & \begin{cases} i = 2 \\ j = 1, 2, 3 \end{cases} \end{cases}$$

$\bar{\sigma}_*((x_{12}, x_{21}), \cdot)$ defined in this way is additive on $\mathbf{S}$ and satisfies the measure-theoretic requirements of the definition of X_1 -restrictedness at (x_{12}, x_{21}) but not those of X_2 -restrictedness or restrictedness at the given point.

An important two-way tie exists between the concept of restrictedness of $\mathcal{R}_\phi$ at (t, x) and the vanishing of $\sigma_*(t, x, \cdot)$ for carrier spaces which are discrete on $\{x\} \subset X$. This is given in the following theorem.

Theorem B. *If the carrier space $(X, \mathbf{S}, \nu)$ is discrete on $\{x\} \subset X$ and $\nu(\{x\}) = 0$, then $\mathcal{R}_\phi$ is restricted at $(t, x) \in T \times X$ if and only if $\sigma_*(t, x, \cdot)$ vanishes a.e. $[\nu]$.*

Proof: If $\mathcal{R}_\phi$ is restricted at (t, x) , then

$$\bar{\sigma}_*(t, x, X) = \bar{\sigma}_*(t, x, X - \{x\}) + \bar{\sigma}_*(t, x, X \cap \{x\})$$

so that, by hypothesis, the first and second summands vanish. Thus the R-N derivative $\sigma_*(t, x, \cdot)$ of $\bar{\sigma}_*(t, x, \cdot)$ must vanish a.e. $[\nu]$ ([8], p. 104). Conversely, if $E \in \mathcal{S}_x$, then in general $\bar{\sigma}_*(t, x, \cdot) = \int \sigma_*(t, x, \cdot) d\nu$ is countably additive, and, in particular, $\bar{\sigma}_*(t, x, E') = \int_{E'} \sigma_*(t, x, \cdot) d\nu = 0$. This completes the proof.

Observe that the "if" statement of the preceding theorem is valid without the two special provisos on the structure of the carrier space.

Product carrier spaces with variously restricted radiative processes, while still rather abstract, already possess features which allow the resulting dynamic relations for G -densities to be nearly indistinguishable (in form) from the classical equations of transfer. The groundwork will now be prepared for the theorem which is a precise statement of this assertion. This theorem will represent the culmination of the present study of radiative processes.

Lemma. Let $(X, \mathcal{S}, \nu) = (X_1 \times X_2, \mathcal{S}_1 \times \mathcal{S}_2, \nu_1 \times \nu_2)$ be a product carrier space.

(a) If $\mathcal{R}_\phi$ is X_1 -restricted on $(t, R_1) \in T \times \mathcal{S}$, then

$$\bar{\sigma}_*(t, \cdot, A \times B) = \chi_{A \times X_2} \bar{\sigma}_*(t, \cdot, X_1 \times B) [\nu]$$

on R_1 for every measurable rectangle $A \times B \in \mathcal{S}$.

(b) If $\mathcal{R}_\phi$ is X_2 -restricted on $(t, R_2) \in T \times \mathcal{S}$, then

$$\bar{\sigma}_*(t, \cdot, A \times B) = \chi_{X_1 \times B} \bar{\sigma}_*(t, \cdot, A \times X_2) [\nu]$$

on R_2 for every measurable rectangle $A \times B \in \mathcal{S}$.

Proof: Only (a) will be proved; the proof of (b) is similar. For every $A \times B \in \mathcal{S}$,

$$\bar{\sigma}_*(t, \cdot, A \times B) + \bar{\sigma}_*(t, \cdot, A' \times B) = \bar{\sigma}_*(t, \cdot, X_1 \times B) [\nu].$$

If $x \in (A \times X_2) \cap R_1$, then $\bar{\sigma}_*(t, x, A' \times B) = 0$. If $x \in (A \times X_2)' \cap R_1 = (A' \times X_2) \cap R_1$, then

$$\bar{\sigma}_*(t, x, (A')' \times B) = 0 = \chi_{A \times X_2}(x) \bar{\sigma}_*(t, x, X_1 \times B).$$

This completes the proof of (a).

The density scattering function was defined earlier (§14) by the relation $\int_{\mathcal{S}} \sigma_*(t, x, \cdot) d\nu = \bar{\sigma}_*(t, x, E) < \infty$, $E \in \mathcal{S}$. If $X = X_1 \times X_2$ is a product carrier space, then, for each $t \in T$, $\sigma_*(t, \cdot, \cdot)$ has as its domain $X \times X = (X_1 \times X_2) \times (X_1 \times X_2)$. We will have occasion to work with certain iterated integrals of the multiple integral $\int_{\mathcal{S}} \sigma_*(t, x, \cdot) d\nu$. If $E = A_1 \times X_2$,

$$\begin{aligned} \int_{A_1 \times X_2} \sigma_*(t, (x_1, x_2), (x'_1, x'_2)) d\nu((x'_1, x'_2)) \\ = \int_{A_1} d\nu_1(x'_1) \int_{X_2} \sigma_*(t, (x_1, x_2), (x'_1, x'_2)) d\nu_2(x'_2), \end{aligned}$$

and we will write $\sigma_*(t, (x_1, x_2), (\cdot, *))$ for the ν_1 -integrable function

$$\int_{X_2} \sigma_*(t, (x_1, x_2), (\cdot, x'_2)) d\nu_2(x'_2).$$

If $E = X_1 \times A_2$, we will write $\sigma_*(t, (x_1, x_2), (*, \cdot))$ for the ν_2 -integrable function

$$\int_{X_1} \sigma_*(t, (x_1, x_2), (x'_1, \cdot)) d\nu_1(x'_1).$$

Clearly

$$\begin{aligned} \int_{X_1} \sigma_*(t, (x_1, x_2), (\cdot, *)) d\nu_1 &= \int_{X_2} \sigma_*(t, (x_1, x_2), (*, \cdot)) d\nu_2 \\ &= s_*(t, (x_1, x_2)) [\nu]. \end{aligned}$$

Theorem C. *Retaining the hypothesis of (A, §15), Let $(X, S, \nu) = (X_1 \times X_2, S_1 \times S_2, \nu_1 \times \nu_2)$ be a product carrier space. If a radiative process $\mathcal{R}_\phi$ on (X, S, ν) is X_1 -restricted on $(t_2, R_1) \in T \times S$ and X_2 -restricted on $(t_2, R_2) \in T \times S$, then, for almost every $[\nu] (x_1, x_2) \in X_1 \times X_2$,*

$$\begin{aligned} \partial^+ G(t_1, E_1; t_2, (x_1, x_2)) / \partial t_2 &= -(a_*(t_2, (x_1, x_2)) \\ &+ \chi_{(R_1 \cap R_2)'}((x_1, x_2)) s_*(t_2, (x_1, x_2))) G(t_1, E_1; t_2, (x_1, x_2)) \\ &+ \int_{X_2} \chi_{R_1 \cap R_2'}((x_1, x'_2)) G(t_1, E_1; t_2, (x_1, x'_2)) \sigma_*(t_2, (x_1, x'_2), (*, x_2)) d\nu_2(x'_2) \\ &+ \int_{X_1} \chi_{R_1' \cap R_2}((x'_1, x_2)) G(t_1, E_1; t_2, (x'_1, x_2)) \sigma_*(t_2, (x'_1, x_2), (x_1, *)) d\nu_1(x'_1) \\ &+ \int_{(R_1 \cup R_2)'} G(t_1, E_1; t_2, (x'_1, x'_2)) \sigma_*(t_2, (x'_1, x'_2), (x_1, x_2)) d\nu((x'_1, x'_2)). \end{aligned}$$

Proof: From the proof of (C, §15), for every measurable rectangle

$$\begin{aligned} A_1 \times A_2 \in S, \int_{A_1 \times A_2} G^+ d\nu &= - \int_{A_1 \times A_2} \alpha_*(t_2, \cdot) G d\nu \\ &+ \int_{X_1 \times X_2} G \bar{\sigma}_*(t_2, \cdot, A_1 \times A_2) d\nu < \infty, \end{aligned}$$

where G is an abbreviation of $G(t_1, E_1; t_2, \cdot)$.

$$\{(R_1 \cap R_2), (R_1 \cap R_2'), (R_1' \cap R_2), (R_1 \cup R_2)'\}$$

is a partition of X . We examine each of the integrals over the first three elements of the partition. First,

$$\int_{R_1 \cap R_2} G \bar{\sigma}_*(t_2, \cdot, A_1 \times A_2) d\nu = \int_{A_1 \times A_2} \chi_{R_1 \cap R_2} G s_*(t_2, \cdot) d\nu$$

by applications of (a) and (b) of the preceding lemma and the relation $\bar{\sigma}_*(t_2, \cdot, X) = s_*(t_2, \cdot) [\nu]$. Second

$$\begin{aligned} \infty &> \int_{R_1 \cap R_2'} G \bar{\sigma}_* d\nu = \int_{A_1 \times X_2} \chi_{R_1 \cap R_2'} G \bar{\sigma}_* d\nu \\ &= \int_{A_1 \times A_2} \left\{ \int_{X_2} \chi_{R_1 \cap R_2'}((x_1, x_2)) G(t_1, E_1; t_2, (x_1, x_2)) \right. \\ &\quad \cdot \left[\int_{X_1} \sigma_*(t_2, (x_1, x_2), (x'_1, x'_2)) d\nu_1(x'_1) \right] d\nu_2(x_2) \Big\} d\nu_1(x_1) d\nu_2(x'_2) \\ &= \int_{A_1 \times A_2} \left\{ \int_{X_2} \chi_{R_1 \cap R_2'}((x_1, x'_2)) G(t_1, E_1; t_2, (x_1, x'_2)) \right. \\ &\quad \cdot \sigma_*(t_2, (x_1, x'_2), (*, x_2)) d\nu_2(x'_2) \Big\} d\nu((x_1, x_2)). \end{aligned}$$

The first equality above is obtained by using (a) of the preceding lemma; the second equality by a Fubini argument; the third by using the definition of $\sigma_*(t_2, (x_1, x_2), (*, x_2))$ and a change of dummy variable notation. Finally,

$$\begin{aligned} \infty &> \int_{R_1' \cap R_2} G \bar{\sigma}_* d\nu = \int_{A_1 \times A_2} \left\{ \int_{X_1} \chi_{R_1' \cap R_2}((x'_1, x_2)) G(t_1, E_1; t_2, (x'_1, x_2)) \right. \\ &\quad \cdot \sigma_*(t_2, (x'_1, x_2), (x_1, *)) d\nu_1(x'_1) \Big\} d\nu((x_1, x_2)), \end{aligned}$$

by an argument similar to that applied to the integral over $R_1 \cap R_2'$.

We observe that the integral over the fourth partition element can be cast into an iterated integral of the same general form as those over the second and third partition elements using a simple Fubini argument. We have shown that the integrated form (over $A_1 \times A_2$) of the relation given in the conclusion of the theorem's statement holds for every measurable rectangle in $\mathbf{S}$. However, the non-empty subset $\mathbf{S}_0 \subset \mathbf{S}$ on which the ν -integrated relation holds is a σ -ring (an indefinite integral of an integrable function is countably additive) which must then contain the σ -ring generated by the measurable rectangles. An application of ([8], p. 105) completes the proof of the theorem.

If a radiative process $\mathcal{R}_\phi$ on $(X, \mathbf{S}, \nu) = (X_1 \times X_2, \mathbf{S}_1 \times \mathbf{S}_2, \nu_1 \times \nu_2)$ is X_1 -restricted on (t, X) and X_2 -restricted on $(t, N) \times T \times \mathbf{S}$ where $\nu(N) = 0$, then we will say that $\mathcal{R}_\phi$ is 1-factored on (t, X) . A similar definition may be made for a process $\mathcal{R}_\phi$ which is 2-factored on (t, X) . By (A, §17) and the subsequent discussion, an i -factored process ($i = 1$ or 2) $\mathcal{R}_\phi$ is in general unrestricted on (t, X) . As an immediate consequence of the preceding theorem we have the important

Corollary. *If a radiative process $\mathcal{R}_\phi$ on the product carrier space is 1-factored, then*

$$\partial^+ G(t_1, E_1; t_2, (x_1, x_2)) / \partial t_2 = -\alpha_*(t_2, (x_1, x_2)) G(t_1, E_1; t_2, (x_1, x_2)) \\ \int_{X_2} G(t_1, E_1; t_2, (x_1, x'_2)) \sigma_*(t_2, (x_1, x'_2), (*, x_2)) dv_2(x'_2) [\nu].$$

A similar expression holds for the case when $\mathcal{R}_\phi$ is 2-factored. i -factored radiative processes are the abstract generalizations of the classical radiative processes. We observe that if, in the preceding theorem, $R_1 = R_2 [\nu]$, then the conclusion reduces to that of (A, §16).

III. RADIATIVE TRANSFER PROCESSES

18. Transfer Process Axiom. This chapter is devoted to the introduction of the last important structure axiom. This axiom will be used in conjunction with the radiative process axiom to define what we will call the radiative transfer process. The radiative transfer process will be shown to possess sufficient structure to yield a measure-theoretic foundation for the resulting classes of attenuating functions, and finally we show that the radiative transfer process on an abstract carrier space yields the desired generalization of the classical equation of transfer.

The transfer process axiom is an assertion about the existence of a set of measure preserving transformations with domain and range on the carrier space. The initial motivation for this axiom has been sketched in [17]. By way of introduction here we will show that the radiative processes of the preceding chapter possess features resembling some of those of a transfer process but fail to have all of the requisite structure to carry the present study to its proposed goal.

As shall be shown below, to complete the present study we require a certain set of one to one transformations T from $(X, \mathbf{S}, \nu)$ onto $(X, \mathbf{S}, \nu)$ such that T and T^{-1} are measurable transformations which preserve ν -measure. Do the elements of $\mathcal{R}_\phi$ have this property? Strictly speaking, no, for the domain of the elements of $\mathcal{R}_\phi$ is $\mathfrak{M}_\nu$. Yet the elements of $\mathcal{R}_\phi$ can be made to assign certain elements of $\mathbf{S}$ to other elements of $\mathbf{S}$ in the following way: Let E be a maximal full support for $\mu \in \mathfrak{M}_{\nu, t_1}$, hence E is determined uniquely up to a set of ν -measure zero. It is easy to show that, by (i) R_3 , there exists a t_2 in T such that $t_2 > t_1$ and that $\mu R_{t_1, t_2} \in \mathfrak{M}_{\nu, t_2}$ is supported by X . Hence there exists a unique $[\nu]$ maximal full support F for $\mu R_{t_1, t_2}$. We may then say that, in this way, F is assigned to E by the element $R_{t_1, t_2} \in \mathcal{R}_\phi$. Conversely, if F is a maximal full support for $\mu_2 \in \mathfrak{M}_{\nu, t_2}$ such that $\mu_2 = \mu_1 R_{t_1, t_2}$, $\mu_1 \in \mathfrak{M}_{\nu, t_1}$, $(t_1, t_2) \in D$, then, since the elements of $\mathcal{R}_\phi$ are linear transformations on the non-negative cone $\mathfrak{M}_\nu$ of radiative measures, X must support μ_1 , from which we deduce the existence of a unique $[\nu]$ maximal full support E for μ_1 . Again, E is assigned to F by means of an element $R_{t_1, t_2} \in \mathcal{R}_\phi$. Since the important elements of $\mathbf{S}$ in this work are precisely those which fully support radiative measures, $\mathcal{R}_\phi$ can indeed supply

a set of one to one transformations from the σ -algebra $S(\nu)$ onto itself ([8], p. 167). If the carrier space is discrete (§17), the elements of $\mathcal{R}_\phi$ may then—in the above sense—have domain and range on X . However, for discrete spaces such that the singletons are assigned zero measure by ν (as is the state of affairs in the classical carrier space) there no longer is possible a unique assignation between $\{x\} = E \in S$ and $\{y\} = F \in S$ by means of the elements of $\mathcal{R}_\phi$, because singletons in this case cannot support radiative measures.

An examination of the source cited in the discussion in ([17], p. 13) would show that the idea of a radiative process (as summarized in R_1) has been abstracted from the linear properties of (and linear operations on) the classical i.d.e. for specific intensity. The latter, however, is customarily derived with implicit help from the theory of geometrical optics which supplies the natural paths (rays) for photons in a transparent optical medium. It follows that radiative processes must have this geometrical feature packed away deep in their innards. The discussion in the preceding paragraph indicates that the classical radiative process has been abstracted to such an extent that this specific feature has been inextricably snarled up in the other properties of the elements of $\mathcal{R}_\phi$. In this light, it is then remarkable that the abstract radiative process was still sufficiently detailed to allow the development of its theory up to (C, §17) and its corollary.

With these remarks in mind, we add to the preceding three structure axioms, C_1 , M_1 , and R_1 , the fourth and final structure axiom T_1 . Let (X, S, ν) be a carrier space. Let $\mathfrak{I}$ be the (non-empty) set of all measure preserving transformations T from (X, S, ν) onto (X, S, ν) . We will use the notation $T(x)$ and $T(A)$ to designate the images under $T \in \mathfrak{I}$ of $x \in X$ and $A \subset X$, respectively. I_x will denote the identity element in $\mathfrak{I}$. Henceforth, the unadorned symbol T will retain the meaning given it in §2.

T_1 : *There exists a transformation ψ from D into $\mathfrak{I}$ such that $\psi(D) \equiv \mathfrak{I}_\psi$ has the following properties. Let $\psi[(t_1, t_2)] \equiv T_{t_1, t_2}$, $(t_1, t_2) \in D$; then*

- (i) *If t_1, t_2, t_3 are elements of T and if $t_1 \leq t_2 \leq t_3$, then $T_{t_1, t_3} = T_{t_2, t_3} T_{t_1, t_2}$.*
- (ii) *For every $t \in T$, $T_{t, t} = I_\psi$.*

The set $\mathfrak{I}_\psi$ of measure preserving transformations will be called the *transfer process on the carrier space* (X, S, ν) .

19. Radiative Transfer Processes. The pair $(\mathcal{R}_\phi, \mathfrak{I}_\psi)$ will be called the *radiative transfer process on the carrier space* (X, S, ν) .¹ The axioms C_1 , M_1 , R_1 , and T_1 are the *structure axioms* for the radiative transfer process. The elements of $\mathfrak{I}_\psi$ assign a function f to every function g on X ; f is defined by $f(x) = g(T_0(x))$, $x \in X$, $T_0 \in \mathfrak{I}_\psi$. We will designate this correspondence in the usual way: $f = gT_0$. Similarly, the elements of $\mathfrak{I}_\psi$ assign a radiative measure m to every radiative measure $\mu \in \mathfrak{M}_\nu$; m is defined by $m(E) = \mu(T_0(E))$, $E \in S$,

¹ Once RT_3 and RT_4 have been formulated, it will be understood that the pair $(\mathcal{R}_\phi, \mathfrak{I}_\psi)$ satisfies the requirements of all the appropriate axioms (see the diagram below).

and we will designate this correspondence by $m = \mu T_0$. The measure preserving character of the transformations is clearly needed in the latter correspondence; and the measurable character of the transformations is needed in the former correspondence in order to assure that the images of measurable functions on the carrier space are again measurable ([8], p. 162). If $\mu \in \mathfrak{M}_{\nu, t}$, $(t, t') \in D$, consider the radiative measure $\mu R_{t, t'} T_{t, t'} \in \mathfrak{M}_{\nu, t'}$ ((ii)M₁). The radiant energy $\mu R_{t, t'} T_{t, t'}(E)$ will be called the *residue at t' of $\mu(E)$* , $E \in S$. This is a precise generalization of the notion of the Lagrangian residue of an ensemble of photons (cf. ([11], p. 4)). As an important example, consider $\mu \in \mathfrak{M}_{\nu, t_1}$, $E_1 \in S$, $(t_1, t_2) \in D$, and $\Delta \in T$. Then $\mu R_{t_1, t_2 + \Delta} T_{t_1, t_2 + \Delta} \in \mathfrak{M}_{\nu, t_2 + \Delta}$. Using (ii)R₁, we see that the radiant energy $\mu(t_1, E_1; t_2 + \Delta, T_{t_1, t_2 + \Delta}(E_2))$ is the residue at $t_2 + \Delta$ of $\mu(t_1, E_1; t_2, E_2)$. Further, if $G(t_1, E_1; t_2 + \Delta, \cdot)$ is the general radiant density function associated with $\mu(t_1, E_1; t_2 + \Delta, \cdot)$, the radiant density

$$[G(t_1, E_1; t_2 + \Delta, \cdot) T_{t_1, t_2 + \Delta}](x) = G(t_1, E_1; t_2 + \Delta, T_{t_1, t_2 + \Delta}(x))$$

will be called the *residue at $t_2 + \Delta$ of $G(t_1, E_1; t_2, x)$* , $x \in X$. Of course, this definition can be phrased generally as was done above for the case of μ . As should be expected, the following relation

$$\int_{E_2} [G(t_1, E_1; t_2 + \Delta, \cdot) T_{t_1, t_2 + \Delta}] d\nu = (\mu(t_1, E_1; t_2 + \Delta, \cdot) T_{t_1, t_2 + \Delta})(E_2)$$

holds between the two residues. That this is true may be shown by using the measure preserving properties of the transformations $T_{t_1, t_2 + \Delta}$, ([8], p. 163), and (A, §5). A similar discussion can be made for the γ -density $\gamma(t_1, x_1; t_2 + \Delta, \cdot)$.

20. Regularity Axioms for Radiative Transfer Processes. Consequences. The transfer process axiom may take its place, in the order of axioms, between R₁ and R₂. From this point on, a theory of radiative *transfer* processes may be developed along lines which very closely parallel those followed to develop the radiative process theory in Chapter II. An examination of the three regularity axioms R₂, R₃, and R₄ will show that: (a) R₂ need not be changed; (b) R₃ needs to be superficially modified to a form which makes use of the $\mathfrak{I}_\nu$ -transforms of the various functions involved; (c) R₄ requires only token notational changes to be converted into a radiative transfer context. The burden of the conversion from radiative processes to radiative transfer processes is then placed on R₃ which we now modify.

RT₃: Let $(t_1, t_2), (t_2, t_3) \in D$, $\mu \in \mathfrak{M}_{\nu, t_1}$, and $E_1, E_3 \in S$. If X supports $\mu' = \mu R_{t_1, t_2}$, let F denote the maximal full support for μ' . If $g'(t_2, \cdot; t_3, T_{t_2, t_3}(E_3))$ is the g -density with respect to μ' of $\mu'(t_2, \cdot; t_3, T_{t_2, t_3}(E_3))$ and $\{\Delta_n\}$ is any decreasing null sequence of positive elements in T , then:

(i) The sequence

$$\{h'_n(t_2, \cdot, E_3)\} = \{(\chi_{E_3 \cap F} - g'(t_2, \cdot; t_2 + \Delta_n, T_{t_2, t_2 + \Delta_n}(E_3))) / \Delta_n\}$$

converges uniformly on X to a unique $[\nu]$ essentially bounded function $h'_0(t_2, \cdot, E_3)$.

- (ii) *The corresponding function $h'_0(t_2, x_2, \cdot)$ is countably additive on S for almost every $[\nu] x_2 \in X$.*
- (iii) $\{\gamma_n\} = \{(\gamma(t_1, x_1; t_2 + \Delta_n, \cdot)T_{t_2, t_2 + \Delta_n} - \gamma(t_1, x_1; t_2, \cdot))/\Delta_n\}$
is fundamental in measure for almost every $[\nu] x_1 \in X$.
- (iv) $\{G_n\} = \{(G(t_1, E_1; t_2 + \Delta_n, \cdot)T_{t_2, t_2 + \Delta_n} - G(t_1, E_1; t_2, \cdot))/\Delta_n\}$
is fundamental in measure.

To distinguish the new families of measure attenuating functions arising from RT_3 , we replace the earlier asterisk subscript by the circle subscript. It follows that each theorem and definition between R_3 and R_4 has a precisely determinable analog which differs from the original by trivial changes in notation (asterisk to circle; E to $T_\circ(E)$; $\mathcal{R}_\phi$ references to $(\mathcal{R}_\phi, \mathfrak{I}_\psi)$ references). The μ' -differentiability Axiom R_4 is now asserted for the density attenuating functions of the radiative *transfer* process and is assigned the signature RT_4 ; nothing is changed from the original except the obvious conversions from asterisk subscripts to circle subscripts and $\mathcal{R}_\phi$ references to $(\mathcal{R}_\phi, \mathfrak{I}_\psi)$ references. Again, it follows that each theorem following R_4 has an analog for the radiative transfer process. It is asserted that "nothing is changed" in this conversion from $\mathcal{R}_\phi$ to $(\mathcal{R}_\phi, \mathfrak{I}_\psi)$. Of course this phrase refers to the mathematical content of the definitions and theorems concerned. The physical consequences—with one notable exception—are profoundly altered: the $(\mathcal{R}_\phi, \mathfrak{I}_\psi)$ analogs of the theorems (A, §12), (A, §15), (B, §15), (C, §15), (A, §16), and (C, §17) now involve total (right hand) derivatives with respect to the time t_2 , yielding the desired generalization of the classical substantial (or Lagrangian) derivative. The one notable exception is the general definition of the measure (and hence also of the density) absorption function. This fact may be checked by an examination of the definitions of A_* and $A_\circ$. The concept of an absorption function (on S or on X) for either a radiative or a radiative transfer process appears to be unavoidably considered from an Eulerian point of view (*cf.*, *e.g.*, ([11], pp. 3–5)).

We summarize below the two axiom schemes for the foundations of radiative and radiative transfer processes as developed in this paper:

21. Conclusion. The purpose of this section is to exhibit the reduction of the general dynamic transition relation for G to the classical equation of transfer for N , thereby exhibiting a solution to the long standing problem of HOPF.

(a) *The velocity function.*

Let $\phi_0 = (\mathbf{x}_0, \xi_0) \in \Phi$, $\mathbf{x}_0 \in X$, $\xi_0 \in \Xi$. The transformation $T_{t_0, t'}$ determines a unique set $P(t_0, t)$ of points in Φ : the *light ray* between ϕ_0 and $\phi = T_{t_0, t}(\phi_0)$, that is,

$$P(t_0, t) = \{\phi' : \phi' = T_{t_0, t'}(\phi_0), t \leq t' \leq t\}.$$

Using the metric and algebraic structure of Φ , let $\|P(t_0, t)\|$ denote the length of $P(t_0, t)$ defined in the usual way. The limit

$$\lim_{t \rightarrow t_0} \|P(t_0, t)\|/(t - t_0)$$

exists and is defined as the *speed* of light at $(t_0, \phi_0) \in T \times \Phi$ and is denoted by $v(t_0, \phi_0)$. The customary assumption of refractive isotropy is equivalent to the statement that, for arbitrary fixed $t \in T$, $v(t, \cdot)$ on Φ at each $(\mathbf{x}, \xi) \in \Phi$ is independent of ξ [14], and the value of v at time t at $(\mathbf{x}, \xi)$ may be written $v(t, \mathbf{x})$ to emphasize this independence. v is the *velocity function* on $T \times \Phi$. Within the present formalism, the speed of light in a “vacuum” may be defined by means of the corollary of (C, §17) in which $\alpha_* = 0$ on $T \times \Phi$, and (iv)RT₃. The resulting limit of the kind considered above is a constant function $[p]$ on $T \times \Phi$ and is denoted by c . The *index of refraction function* n on $T \times \Phi$ is then defined by the relation $n = c/v$.

(b) *The classical radiometric and attenuating functions.*

The following table exhibits the definitions of the classical radiometric and attenuating functions N , a , s , σ , and α on $T \times \Phi$ in terms of the precisely defined functions G , a_0 , s_0 , σ_0 , and α_0 respectively. The columns marked [16] and [2] collate the abstract functions with their classical counterparts in the fields of geophysical and astrophysical optics respectively.

	[16]	[2]
(i) $N \equiv Gn^2c$	radiance function N	specific intensity function I_ν
(ii) $a \equiv a_0/v$	volume absorption function μ'	mass (true) absorption function $(\kappa_\nu - 4\pi\bar{\omega}_0\kappa_\nu)$
(iii) $s \equiv s_0/v$	volume total scattering function S'	mass scattering function $4\pi\bar{\omega}_0\kappa_\nu$
(iv) $\sigma \equiv \sigma_0/v$	volume scattering function σ	phase function p ($\int_{4\pi} p d\omega = 4\pi\bar{\omega}_0$)
(v) $\alpha \equiv \alpha_0/v$	volume attenuation function β'	mass absorption function κ_ν

The domains of these functions have already been defined in the introduction. We observe the following additional relations: from (§13), $\alpha = a + s$, and, from (§14), $s = \int_{\Xi} \sigma \, d\Omega$. Thus the basic pair of attenuating functions may be taken as (α, σ) . An alternative pair would be (a, σ) . The basic radiometric function is N in terms of which all other functions may be defined, as is well known.

(c) *The classical equation of transfer.*

From the corollary of (C, §17) now applied to the classical carrier space (Φ, Φ, ν) we have

$$d^+G(t_1, E_1; t_2, (\mathbf{x}, \xi))/dt_2 = -\alpha_0(t_2, (\mathbf{x}, \xi))G(t_1, E_1; t_2, (\mathbf{x}, \xi)) \quad (1)$$

$$+ \int_{\Xi} G(t_1, E_1; t_2, (\mathbf{x}, \xi')) \sigma_0(t_2, \mathbf{x}, \xi', \xi) \, d\Omega(\xi'),$$

where the notational contraction $\sigma_0(t_2, (\mathbf{x}, \xi'), (*, \xi)) = \sigma_0(t_2, \mathbf{x}, \xi', \xi)$ has been made. (1) is understood to be the integro-differential representation of a radiative transfer process on (Φ, Φ, ν) . Using the identities $N \equiv Gn^2c$, $\alpha \equiv \alpha_0/\nu$, and $\sigma \equiv \sigma_0/\nu$, (1) may be written as

$$(1/\nu) \, D(N/n^2)/Dt = -\alpha(N/n^2) + \int_{\Xi} (N/n^2)\sigma \, d\Omega, \quad (2)$$

or

$$(n^2/\nu) \, D(N/n^2)/Dt = -\alpha N + \int_{\Xi} N\sigma \, d\Omega, \quad (3)$$

either of which is the correct homogeneous form of the classical equation of transfer. The notations d^+/dt and D/Dt are interchangeable symbols for the Lagrangian derivative operator. The function $\int_{\Xi} N\sigma \, d\Omega$ is known as the *path function* and is given the symbol N_* .

Continuing to descend the scale of generality by now supposing n to be locally constant on $T \times \Phi$, (3) may be written

$$(1/\nu) \, DN/Dt = -\alpha N + N_*, \quad (4)$$

which is the homogeneous form of equation (1) of the introductory chapter. Equation (4) and its minor variants are the usual starting points for most applications in the two main branches of radiative transfer: geophysical and astrophysical optics.

(d) *The local source axiom.*

The additive adjunction of an emission term to (1) or (3) can be justified within the present formalism by the inclusion of a properly phrased *local source*

axiom S_1 . This procedure is in keeping with the predominantly phenomenological outlook of the tenets of modern radiative transfer theory [20]. If N_η represents the *emission function* on $T \times \Phi$, then, under S_1 , (3) would take the form:

$$(n^2/v) D(N/n^2)/Dt = -\alpha N + N_* + N_\eta \quad (5)$$

in which the duality of the pairs $(\alpha N, N_\eta)$ and (sN, N_*) may be disclosed by writing

$$(n^2/v) D(N/n^2)/Dt = (N_* - sN) + (N_\eta - \alpha N). \quad (6)$$

The term $(N_* + N_\eta)/\alpha$ is the *Ergiebigkeit* (the productivity) of the radiation field, a term due to SCHWARZSCHILD and used by HOPF. CHANDRASEKHAR has called the term the *source function*. The equation of transfer as written in (6) is of great generality. While we have in this discussion limited it to the radiative transfer context, our earlier observations have pointed out its roles in other branches of physics and mathematics. The form of (6) underscores its applicability to the study of any ensemble of entities whose space-time rate of change in numbers is subject to the generalized linear processes of immigration, emigration, birth, and death.

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