On Future and Past Stability in Incompressible Systems

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1. Let $x = (x_1, \dots, x_n)$ and $f = (f_1, \dots, f_n)$ and consider the vector system of differential equations

$$(1) \quad x' = f(x) \quad \left(' = \frac{d}{dt} \right),$$

where $f(x)$ is of class C^1 on some open, invariant set of points Ω . Thus, in particular, if x_0 is in Ω , then $x(t)$, where $x_0 = x(0)$, exists and belongs to Ω for $-\infty < t < \infty$. A solution $x = x(t)$, for which $x_0 = x(0)$, will be called future stable (in the sense of LIAPOUNOFF; *cf.*, *e.g.*, [3], pp. 210–211) if for every $\epsilon > 0$ there exists a $\delta = \delta_\epsilon > 0$ such that $|x(t) - y(t)| < \epsilon$ for $t \geq 0$ whenever $|x_0 - y_0| < \delta$, where $y_0 = y(0)$ and, of course, y_0 as well as x_0 belongs to Ω . (The norm refers to the ordinary Euclidean distance.) The definition of past stable is obtained if the condition $t \geq 0$ is replaced by $t \leq 0$. A solution will be called stable if it is both future and past stable, so that $|x(t) - y(t)| < \epsilon$ holds for $-\infty < t < \infty$ whenever $|x_0 - y_0| < \delta$.

If, in addition, $f(x)$ satisfies the incompressibility condition

$$(2) \quad \operatorname{div} f \equiv \sum_{k=1}^n \frac{\partial f_k}{\partial x_k} = 0,$$

it seems likely that every stable solution must be almost periodic. In fact, as was pointed out in [2], the only known stable solutions of (1) when the latter are the Hamiltonian equations belonging to a conservative dynamical system, in which case (2) surely holds, are almost periodic. Further evidence when (1) is subject to (2) was obtained in [6]. Whether or not stability actually does imply almost periodicity in case (2) (or even, for that matter, in the dynamical case) will still remain open, although, as can be shown by a simple example (*cf.* [6], p. 353), the implication surely does not exist without some restriction, *e.g.*, that of (2), on f . Assumption (2) can be interpreted as a measure (volume) preserving condition and, when Ω is subject to certain restrictions (including the important one that it be of finite positive measure), implies certain ergodic and recurrent properties of almost all solutions of (1).

The object of the present note is to show the equivalence of the three notions of future stability, past stability, and stability in the case of an almost periodic solution of an incompressible system (1). In fact, the following will be proved:

(*) *Let Ω be an open invariant set (not necessarily of finite measure) of the system (1) in which f is of class C^1 and satisfies (2). Let the solution $x = x(t)$ of (1) in Ω be a Bohr almost periodic (vector) function of t (for $-\infty < t < \infty$) which is either future stable or past stable. Then necessarily $x = x(t)$ is stable; that is, x is both future stable and past stable.*

Whether or not the assumption in (*) concerning the almost periodicity of $x(t)$ can be omitted will remain undecided. That the assertion of (*) can be false if the incompressibility condition (2) is dropped, however, can be seen easily. In fact, $x = 0$ is a constant, hence almost periodic, future stable but not past stable, solution of the equation $x' = -x$ ($n = 1$), all solutions of which are given by $x = x_0 e^{-t}$.

A consequence of (*) is that an equilibrium solution of (1), when (2) holds, is stable if (and only if) it is either future or past stable. Incidentally, when stability refers to $-\infty < t < \infty$, the Poincaré-Birkhoff criterion (involving a sequence of invariant sets) for the stability of such a solution holds even if (2) is not assumed; cf. [8], pp. 98–99 and p. 415. But, as the simple example of the preceding paragraph shows, when (2) is not assumed, a system (1) may have an equilibrium solution which is stable from one side only, say for $t \geq 0$, without being stable on $-\infty < t < \infty$.

The idea of the proof given below was suggested by a recent result of LITTLEWOOD ([4], discussed informally in [5], pp. 132–134; cf. also [7], pp. 203–209) relating to the many-body problem of celestial mechanics and connecting the future and past courses of points in the dynamical phase space.

2. *Proof of (*)*. Clearly it is sufficient to prove (*) for the case that $x(t)$ is future stable. Let $\epsilon > 0$ be arbitrary and then choose a number $\delta = \delta_\epsilon > 0$ such that

$$(3) \quad |x(t) - y(t)| < \epsilon \quad \text{for } t \geq 0 \quad \text{whenever } |x_0 - y_0| < \delta,$$

where $x_0 = x(0)$ and $y_0 = y(0)$ belong to Ω . Let A denote the open sphere of center x_0 and radius δ (so small that A belongs to Ω) and let $B = \sum_{t \geq 0} A_t$. Here, $A_t = T_t A$ where T_t denotes the transformation defined by the solution paths of (1); thus $T_t z_0 = z(t)$ where $z(0) = z_0$ denotes an arbitrary point of Ω . Since $x(t)$ is almost periodic, $|x(t)| < \text{const.}$, and (3) implies, in particular, that $0 < \text{meas } B < \infty$.

The set B consists of those points $y = T_t y_0$ which are non-negative images (so that $t \geq 0$) of points y_0 of A . Clearly $B_t \subset B_s$ whenever $t > s$. The measure preserving condition (2) now implies that $\text{meas } B_t = \text{meas } B$ for all $t \geq 0$ and hence $\text{meas } C = \text{meas } B$ where $C = \prod_{t \geq 0} B_t$; cf., e.g., [7], p. 204. Thus, one now has

$$(4) \quad B = C + Z,$$

where Z denotes a set of measure 0. Since A is contained in B it is clear from (4) that almost all points of A also belong to C .

Now let z_0 denote any point of the product set $A \cdot C$. Then, since $z_0 = z(0)$ is in A ,

$$(5) \quad |z(t) - x(t)| < \epsilon \quad \text{for } t \geq 0.$$

Since z_0 is in C it belongs to B_t for every $t \geq 0$. But every point y of B_t can be expressed as $y = T_s y_0$ where y_0 is a point of A and $s \geq t$. Hence, for every positive integer n , there exists a point w_0 in A such that $z_0 = T_r w_0$ for some $r \geq n$. Since w_0 is in A then $|w(t) - x(t)| < \epsilon$, where $w(0) = w_0$, holds for $t \geq 0$; hence,

$$(6) \quad |z(t) - x^{(n)}(t)| < \epsilon \quad \text{for } -n \leq t < \infty,$$

where $z(0) = z_0$ and $x^{(n)}(0) = T_r x_0 = x(r)$.

Next, let α denote a small positive number whose magnitude will be determined later. Since $x(t)$ is almost periodic there exists some positive number $L = L_\alpha$ having the following property: If $t = t_1$ is an arbitrary real number, then there exists a (translation) number τ for which $t_2 = \tau + t_1$ satisfies $0 \leq t_2 \leq L$ and $|x(t_2) - x(t_1)| < \alpha$. Now, for α fixed, choose ϵ and δ_* so small that

$$(7) \quad |y(t) - x(t)| < \alpha \quad \text{for } 0 \leq t \leq L,$$

whenever $|y_0 - x_0| < \delta + \epsilon$. (Here, as before, x_0 is the center of the sphere A .) That this can be done is an obvious consequence of the continuity properties of the solutions of the differential equations (1).

In view of (6), $|z_0 - x^{(n)}(0)| < \epsilon$ and, since z_0 is in A , $|x_0 - x^{(n)}(0)| < \epsilon + \delta$. Hence, by (7), $|x(t) - x^{(n)}(t)| < \alpha$ for $0 \leq t \leq L$ ($L = L_\alpha$). However, $x^{(n)}(t) = x(t + r)$, r fixed, so that if t_1 is arbitrary, one has $|x(t_2) - x(t_1)| < \alpha$ and $|x^{(n)}(t_2) - x^{(n)}(t_1)| < \alpha$, where $0 \leq t_2 \leq L$. Consequently,

$$(8) \quad |x(t) - x^{(n)}(t)| < 3\alpha \quad \text{for } -\infty < t < \infty.$$

Relations (6) and (8) now imply that, for $-n \leq t < \infty$, $|x(t) - z(t)| < \epsilon + 3\alpha$. Since the left side of this inequality is independent of n it follows that $|x(t) - z(t)| < \epsilon + 3\alpha$ for $-\infty < t < \infty$. Since α can be chosen arbitrarily small, it is now clear that for an arbitrary $\epsilon > 0$ (corresponding, in this new notation, to the above $\epsilon + 3\alpha$) one can choose a $\delta = \delta_* > 0$ so small that

$$(9) \quad |x(t) - z(t)| \leq \epsilon \quad \text{for } -\infty < t < \infty,$$

whenever $z_0 = z(0)$ belongs to the set $A \cdot C$. (The sign $\leq$ rather than $<$ is chosen for convenience in the argument of the following paragraph.)

In order to complete the proof of (*) it will be necessary to consider those points z_0 of A which do not lie in C . But any such point z_0 must also satisfy (9); for otherwise there would exist a value $t = t_1$ for which $|x(t_1) - z(t_1)| > \epsilon$. Hence, by the continuity of the solutions of (1), $|x(t_1) - y(t_1)| > \epsilon$ would hold for all points $y(t_1)$ arising from points $y_0 = y(0)$ sufficiently close to z_0 . Thus a whole sphere about z_0 lying in A would fail to belong to C , in contradiction

with the fact that almost all points of A lie in C . Hence (9) holds for all points z_0 ($= z(0)$) in the sphere A (of sufficiently small radius δ) and the proof of (*) is now complete.

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