On Stability of Periodic Solutions of Weakly Nonlinear Differential Systems

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1. In previous papers R. A. GAMBILL & J. K. HALE [9] and L. CESARI & J. K. HALE [5] have given general existence theorems for periodic solutions (harmonics, subharmonics) of periodic weakly nonlinear differential systems. The method used in these papers is the convergent method of successive approximations which has been successively developed by L. CESARI, R. A. GAMBILL and J. K. HALE in a series of papers. In particular, in [12] general existence theorems have been given for periodic solutions (cycles) of autonomous weakly nonlinear differential systems; in [3, 4, 7, 8, 11, 13] convergent series expansions have been studied for the characteristic exponents of periodic linear differential systems, and, consequently, criteria have been given for the boundedness or unboundedness of their solutions.

In the present paper the same method, together with a theorem of LIAPOUNOFF, is applied to derive a criterion for asymptotic stability of periodic solutions of periodic weakly nonlinear differential systems as obtained in [5] and [9]. The systems discussed are introduced in §3. The final result is given in §5 and extends, to such systems, a criterion given by L. MANDELSTAM & N. PAPALEXI [15, 16] for a second order differential equation. The criterion is formally similar to the one given by a different approach in [16, p. 153 ff.] for autonomous systems.

The consistent use of the method has advantages. On the one hand the criterion for asymptotic stability of a periodic solution is cast into expressions some of which already appear in the approximation of the periodic solution; on the other hand the two aspects of the method mentioned (determination of periodic solutions of the given nonlinear system and determination of characteristic exponents of the ensuing periodic linear variational system) are taken into consideration.

Examples are discussed in §6: The well known van der Pol equation with forcing term and the rather difficult coupling of two such van der Pol equations. Other examples will be discussed in other papers.

For references on this vast subject we refer to the papers listed in the bibliography.

2. Let us consider a system of differential equations

$$(1) \quad \dot{\mathbf{x}} = \mathbf{q}(\mathbf{x}, t)$$

where $\mathbf{q} = (q_1, \dots, q_n)$ is a continuous vector function of the vector $\mathbf{x} = (x_1, \dots, x_n)$ and the time t , and suppose that (1) satisfies a uniqueness condition in a set R of the $(n+1)$ -dimensional $(\mathbf{x}, t)$ space E . A solution $\mathbf{x}^*(t) = \mathbf{x}(t; \mathbf{x}_0, t_0)$, $t_0 \leq t < +\infty$, of (1) with $\mathbf{x}^*(t_0) = \mathbf{x}_0$ is said to belong to R if $[\mathbf{x}^*(t), t]$ is an interior point of R for every $t \geq t_0$. A solution $\mathbf{x}^*(t) = \mathbf{x}(t; \mathbf{x}_0, t_0)$ of (1) is said to be asymptotically stable if there exists a $\delta > 0$ such that

- (a) every solution $\mathbf{x}(t; \mathbf{x}_1, t_0)$ exists for all $t \geq t_0$ and belongs to R if $|\mathbf{x}_1 - \mathbf{x}_0| < \delta$;
- (b) $\mathbf{x}(t; \mathbf{x}_1, t_0) - \mathbf{x}(t; \mathbf{x}_0, t_0) \rightarrow 0$ as $t \rightarrow +\infty$ if $|\mathbf{x}_1 - \mathbf{x}_0| < \delta$.

Let us suppose now that $\mathbf{q}(\mathbf{x}, t)$ in (1) is periodic in t of a given period T , and that the solution $\mathbf{x}^*(t)$ is periodic. Let $\mathbf{x} = \mathbf{x}^*(t) + \mathbf{y}(t)$, and assume that $q[\mathbf{x}^*(t) + \mathbf{y}(t), t]$ can be developed in power series of $y_1, \dots, y_n$ for all $t \geq t_0$, $|y_j| < c$, $j = 1, \dots, n$, for some $c > 0$ independent of t . Then system (1) can be transformed into the system

$$(2) \quad \dot{\mathbf{y}} = Q(t)\mathbf{y} + \mathbf{p}(\mathbf{y}, t),$$

where the periodic matrix $Q(t)$ is the Jacobian matrix

$$Q(t) = \left\| \frac{\partial q_j}{\partial x_k} \right\|_{\mathbf{x}=\mathbf{x}^*} \quad (j, k = 1, \dots, n).$$

The theorem of LIAPOUNOFF mentioned in §1 states that if the characteristic exponents of the linear system

$$(3) \quad \dot{\mathbf{z}} = Q(t)\mathbf{z}$$

have negative real parts, then the solution $\mathbf{y} \equiv \mathbf{0}$ of (2) (and hence the solution $\mathbf{x} = \mathbf{x}^*(t)$ of (1)) is asymptotically stable.

3. We shall discuss asymptotic stability of periodic solutions of the real system of differential equations

$$(5) \quad \ddot{y}_j + \sigma_j^2 y_j = \epsilon f_j(y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n, t, \epsilon), \quad j = 1, \dots, n,$$

where

(A) each f_j is a holomorphic function of $y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n$, and the parameter $\epsilon > 0$ for $|y_j|, |\dot{y}_j| < A$, $\epsilon < \epsilon_0$, $t \geq 0$, with the expansions of f_j in power series of $y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n, \epsilon$ converging for the same $y_j, \dot{y}_j, \epsilon$;

(B) each f_j is a periodic function of t with period $T = 2\pi/\omega$;

(C) each σ_j is a positive number.

4. **A review of the method.** The convergent method of successive approximations developed by CESARI, HALE and GAMBILL (see §1) will be applied here

to obtain periodic solutions of system (5) and the characteristic exponents of the ensuing periodic linear variational system.

Let us make in (5) the transformation of variables

$$(6) \quad \begin{cases} 2i\sigma_i y_i = x_{2i-1} + x_{2i} \\ 2\dot{y}_i = x_{2i-1} - x_{2i} \end{cases}, \quad j = 1, 2, \dots, n.$$

We shall obtain the system

$$(7) \quad \begin{cases} \dot{x}_{2i-1} = i\sigma_i x_{2i-1} + \epsilon f_i \left[\frac{x_1 + x_2}{2i\sigma_1}, \dots, \frac{x_1 - x_2}{2}, \dots, t, \epsilon \right] \\ \dot{x}_{2i} = -i\sigma_i x_{2i} - \epsilon f_i \left[\frac{x_1 + x_2}{2i\sigma_1}, \dots, \frac{x_1 - x_2}{2}, \dots, t, \epsilon \right], \end{cases} \quad j = 1, 2, \dots, n.$$

Let us now introduce the auxiliary system

$$(8) \quad \begin{cases} \dot{x}_{2i-1} = i\tau_i x_{2i-1} + \epsilon f_i[\dots] \\ \dot{x}_{2i} = -i\tau_i x_{2i} - \epsilon f_i[\dots] \end{cases}, \quad j = 1, 2, \dots, n,$$

where the terms in $[\dots]$ are the same as in (7), and each $\tau_i = k_i\omega/m_i$ is prescribed, k_i, m_i being positive integers. We shall take as the zeroth approximation to the solution of (8) the functions

$$(9) \quad \begin{cases} x_{0,2i-1} = a_i e^{i\tau_i t} \\ x_{0,2i} = -\bar{a}_i e^{-i\tau_i t} \end{cases}, \quad j = 1, 2, \dots, n,$$

where $\bar{a}$ denotes the complex conjugate of a . Each a_i is a non-zero complex constant which we leave undetermined for the moment.

With the zeroth approximation (9), the algorithm defined in [9] yields a periodic solution of period $2\pi m/\omega$, $m = \prod_{i=1}^n m_i$, of the system

$$(10) \quad \begin{cases} \dot{x}_{2i-1} = [i\tau_i - \epsilon V_i(\tau, \mathbf{a}, \mathbf{o}, \epsilon)]x_{2i-1} + \epsilon f_i[\dots] \\ \dot{x}_{2i} = [-i\tau_i - \epsilon \bar{V}_i(\tau, \mathbf{a}, \mathbf{o}, \epsilon)]x_{2i} - \epsilon f_i[\dots] \end{cases}, \quad j = 1, 2, \dots, n,$$

where the terms in $[\dots]$ are the same as in (7), $\tau = (\tau_1, \dots, \tau_n)$, and V_i denotes a convenient analytic function of $\tau, \mathbf{a}, \mathbf{o}, \epsilon$ which has been discussed in [9].

Each component of the solution is a power series in ϵ , with periodic coefficients, of the form

$$(11) \quad \begin{cases} x_{2i-1} = a_i e^{i\tau_i t} + \sum_{n=1}^{\infty} \epsilon^n x_{n,2i-1} \\ x_{2i} = -\bar{a}_i e^{-i\tau_i t} + \sum_{n=1}^{\infty} \epsilon^n x_{n,2i} \end{cases}.$$

The functions $V_i(\tau, \mathbf{a}, \mathbf{d}, \epsilon)$ are obtained by a casting out procedure as explained in [9]. Each V_i is a power series in ϵ ,

$$(12) \quad V_i = V_{1,i} + \epsilon V_{2,i} + \cdots,$$

where we have explicitly

$$(13) \quad V_{1,i} = \frac{1}{a_i mT} \int_0^{mT} e^{-i\tau_i t} \left\{ f_i \left[\frac{x_{0,1} + x_{0,2}}{2i\sigma_1}, \dots, \frac{x_{0,1} - x_{0,2}}{2}, \dots, t, 0 \right] \right\} dt, \\ j = 1, 2, \dots, n.$$

In order that the periodic solution (11) of (10) be also a solution of (7), it is necessary to choose either $\sigma_1, \dots, \sigma_n$ or $a_1, \dots, a_n$ so that the equations

$$(14) \quad i\tau_i - \epsilon V_i(\tau, \mathbf{a}, \mathbf{d}, \epsilon) = i\sigma_i, \quad j = 1, 2, \dots, n,$$

are satisfied. We have called equations (14) the *determining equations*.

Sufficient conditions have been given in [9] in order that equations (14) have a solution.

5. Asymptotic stability. Let us assume we have obtained a periodic solution of (7), $x_i = x_i^*(t, \mathbf{a}, \epsilon)$, $j = 1, 2, \dots, 2n$, of period mT . Let us put

$$(15) \quad x_i = x_i^* + \xi_i, \quad j = 1, 2, \dots, 2n,$$

and, as mentioned in §2, let us form the linear variational equations

$$(16) \quad \begin{cases} \dot{\xi}_{2i-1} = i\sigma_i \xi_{2i-1} + \frac{\epsilon}{2} \left\{ \left[\frac{f_{iy_1}}{i\sigma_1} + f_{i\vartheta_1} \right] \xi_1 + \left[\frac{f_{iy_1}}{i\sigma_1} - f_{i\vartheta_1} \right] \xi_2 \right. \\ \quad \left. \dots + \left[\frac{f_{iy_n}}{i\sigma_n} + f_{i\vartheta_n} \right] \xi_{2n-1} + \left[\frac{f_{iy_n}}{i\sigma_n} - f_{i\vartheta_n} \right] \xi_{2n} \right\} \\ \dot{\xi}_{2i} = -i\sigma_i \xi_{2i} - \frac{\epsilon}{2} \{ \dots \}, \quad j = 1, 2, \dots, n, \end{cases}$$

where the arguments in $\{ \dots \}$ are the same as in the previous equation and $f_{iy_k} = \partial f_i / \partial y_k$, $f_{i\vartheta_k} = \partial f_i / \partial \dot{y}_k$, evaluated along the trajectory $x_i = x_i^*$.

From the Floquet theory there is at least one solution of (16) of the form $\xi_i = e^{\beta^* t} \xi_i^*$, $j = 1, 2, \dots, 2n$, where β^* is a complex constant and each ξ_i^* is periodic of period mT . Let us then put

$$(17) \quad \xi_i = e^{i\beta t} p_i, \quad j = 1, 2, \dots, 2n,$$

in (16), obtaining the system

$$(18) \quad \begin{cases} \dot{p}_{2i-1} = (-\epsilon\beta + i\sigma_i) p_{2i-1} + \frac{\epsilon}{2} \{ \dots \} \\ \dot{p}_{2i} = (-\epsilon\beta - i\sigma_i) p_{2i} - \frac{\epsilon}{2} \{ \dots \}, \quad j = 1, 2, \dots, n, \end{cases}$$

where the terms in $\{ \dots \}$ are the same as in (16), with ξ_i replaced by p_i .

We now seek a periodic solution, of period mT , of system (18). To this end let us replace (18) by the auxiliary linear system

$$(19) \quad \begin{cases} \dot{p}_{2i-1} = i\tau_i p_{2i-1} + \frac{\epsilon}{2} \{\dots\} \\ \dot{p}_{2i} = -i\tau_i p_{2i} - \frac{\epsilon}{2} \{\dots\}, \quad j = 1, 2, \dots, n, \end{cases}$$

where the terms in $\{\dots\}$ are the same as in (18). The numbers τ_i are the same as in system (8). In harmony with the method described in [9], we may now consider

$$(20) \quad \begin{cases} p_{2i-1} = \sum_{k=0}^{\infty} \epsilon^k p_{k,2i-1} \\ p_{2i} = \sum_{k=0}^{\infty} \epsilon^k p_{k,2i} \end{cases}$$

where the coefficients $p_{k,2i-1}$, $p_{k,2i}$ are defined recursively by the formulas

$$(21a) \quad \begin{cases} p_{0,2i-1} = b_{2i-1} e^{i\tau_i t} \\ p_{0,2i} = b_{2i} e^{-i\tau_i t} \end{cases}$$

$$(21b) \quad \begin{cases} p_{k+1,2i-1} = \frac{1}{2} e^{i\tau_i t} \int e^{-i\tau_i t} \{ F_{i,1} p_{k,1} + F_{i,1} p_{k,2} + \dots + F_{i,n} p_{k,2n-1} \\ \quad + F_{i,n} p_{k,2n} - 2S_{k+1,2i-1} p_{0,2i-1} - 2S_{k,2i-1} p_{1,2i-1} \\ \quad - \dots - 2S_{1,2i-1} p_{k,2i-1} \} dt \\ - p_{k+1,2i} = \frac{1}{2} e^{-i\tau_i t} \int e^{i\tau_i t} \{ F_{i,1} p_{k,1} + F_{i,1} p_{k,2} + \dots + F_{i,n} p_{k,2n-1} \\ \quad + F_{i,n} p_{k,2n} + 2S_{k+1,2i} p_{0,2i} + 2S_{k,2i} p_{1,2i} + \dots + 2S_{1,2i} p_{k,2i} \} dt \\ \quad j = 1, 2, \dots, n, \quad k = 0, 1, 2, \dots, \end{cases}$$

where

$$F_{i,k+} = (i\sigma_k)^{-1} f_{i\psi_k} + f_{i\psi_k}, \quad F_{i,k-} = (i\sigma_k)^{-1} f_{i\psi_k} - f_{i\psi_k},$$

b_{2i-1} , b_{2i} are complex constants and

$$(22) \quad 2S_{k+1,2i-1} = \frac{1}{b_{2i-1} mT} \int_0^{mT} e^{-i\tau_i t} \{ F_{i,1} p_{k,1} + F_{i,1} p_{k,2} \\ + \dots + F_{i,n} p_{k,2n-1} + F_{i,n} p_{k,2n} \} dt,$$

$$(23) \quad 2S_{k+1,2i} = \frac{1}{b_{2i} mT} \int_0^{mT} (-1) e^{i\tau_i t} \{\dots\} dt, \quad j = 1, 2, \dots, n,$$

$$k = 0, 1, 2, \dots$$

The terms in $\{\dots\}$ in (23) are the same as in (22). The numbers $S_{k,2i-1}$, $S_{k,2i}$ are the so-called "secular terms" associated with the differential system (19) and the zeroth approximation (21a). As mentioned in [9], all integrands in (21b) have mean value zero and the integrals are univocally determined by the conditions to have mean value zero.

It has been shown in [9] that the power series (20), with coefficients defined by (21, a, b), converge uniformly and absolutely for all t , $\epsilon < \epsilon_0$, $\epsilon_0 > 0$, to a periodic solution of period mT of the differential system

$$(24) \quad \begin{cases} \dot{p}_{2i-1} = [i\tau_i - \epsilon S_{2i-1}(\tau, \mathbf{b}, \mathbf{\delta}, \epsilon)]p_{2i-1} + \frac{\epsilon}{2} \{\dots\} \\ \dot{p}_{2i} = [-i\tau_i - \epsilon S_{2i}(\tau, \mathbf{b}, \mathbf{\delta}, \epsilon)]p_{2i} - \frac{\epsilon}{2} \{\dots\}, \quad j = 1, 2, \dots, n, \end{cases}$$

where the terms in $\{\dots\}$ are the same as in (19), and where

$$(25) \quad \begin{cases} S_{2i-1} = \sum_{k=1}^{\infty} \epsilon^{k-1} S_{k,2i-1} \\ S_{2i} = \sum_{k=1}^{\infty} \epsilon^{k-1} S_{k,2i} . \end{cases}$$

The periodic solution (20) is also a solution of (18), if the numbers $\beta, b_1, \dots, b_{2n}$ can be chosen so that the equations

$$(26) \quad \begin{cases} i\tau_i - \epsilon S_{2i-1} = -\epsilon\beta + i\sigma_i \\ -i\tau_i - \epsilon S_{2i} = -\epsilon\beta - i\sigma_i \quad j = 1, 2, \dots, n, \end{cases}$$

are satisfied. Using equations (14) we obtain from equations (26), after dividing by ϵ ,

$$(27) \quad \begin{cases} \beta = S_{2i-1} - V_i \\ \beta = S_{2i} - \bar{V}_i, \quad j = 1, \dots, n. \end{cases}$$

More explicitly, we obtain from (12), (22), (23), (25), (27) the following

$$(28) \quad \begin{cases} b_{2i-1}\beta - \frac{1}{mT} \int_0^{mT} \frac{1}{2} e^{-i\tau_i t} \{F_{i,1+p_{0,1}} + F_{i,1-p_{0,2}} + \dots + F_{i,n+p_{0,2n-1}} \\ \quad + F_{i,n-p_{0,2n}}\} dt + b_{2i-1}V_{1,i} - b_{2i-1} \sum_{k=2}^{\infty} \epsilon^{k-1} (S_{k,2i-1} - V_{k,i}) = 0 \\ b_{2i}\beta - \frac{1}{mT} \int_0^{mT} (-\frac{1}{2}) e^{i\tau_i t} \{F_{i,1+p_{0,1}} + F_{i,1-p_{0,2}} + \dots + F_{i,n+p_{0,2n-1}} \\ \quad + F_{i,n-p_{0,2n}}\} dt + b_{2i}\bar{V}_{1,i} - b_{2i} \sum_{k=2}^{\infty} \epsilon^{k-1} (S_{k,2i} - \bar{V}_{k,i}) = 0, \\ \quad j = 1, 2, \dots, n. \end{cases}$$

Let us write equations (28) in the form

$$(29) \quad (A - \beta I)\mathbf{b} + \epsilon \mathbf{F}(\mathbf{b}, \epsilon) = \mathbf{0},$$

where the elements a_{ij} of the $2n \times 2n$ matrix of coefficients $A = ||a_{ij}||$ are explicitly

$$\begin{aligned} a_{2r-1, 2s-1} &= \frac{1}{2mT} \int_0^{mT} e^{-i(\tau_r - \tau_s)t} F_{r, s+} dt, \\ a_{2r-1, 2s} &= \frac{1}{2mT} \int_0^{mT} e^{-i(\tau_r + \tau_s)t} F_{r, s-} dt, \\ a_{2r, 2s-1} &= -\frac{1}{2mT} \int_0^{mT} e^{i(\tau_r + \tau_s)t} F_{r, s+} dt, \\ a_{2r, 2s} &= -\frac{1}{2mT} \int_0^{mT} e^{i(\tau_r - \tau_s)t} F_{r, s-} dt, \quad r, s = 1, 2, \dots, n, r \neq s, \end{aligned}$$

and

$$\begin{aligned} a_{2r-1, 2r-1} &= \frac{1}{2mT} \int_0^{mT} F_{r, r+} dt - V_{1, r} \\ a_{2r, 2r} &= -\frac{1}{2mT} \int_0^{mT} F_{r, r-} dt - \bar{V}_{1, r}. \end{aligned}$$

I is the $2n \times 2n$ unit matrix, and $\mathbf{b} = (b_1, \dots, b_{2n})$, $\mathbf{F} = (F_1, \dots, F_{2n})$. We assume that the characteristic roots of the matrix A are distinct. Hence, there is a non-singular matrix P such that $P^{-1}AP = B = \text{diag}(\beta_1^{(0)}, \dots, \beta_{2n}^{(0)})$, where $\beta_1^{(0)}, \dots, \beta_{2n}^{(0)}$ are the characteristic roots of A . We apply now the linear transformation

$$(30) \quad \mathbf{b} = P\mathbf{c}.$$

Substituting (30) in (29) and multiplying by P^{-1} , we obtain

$$(31) \quad (B - \beta I)\mathbf{c} + \epsilon P^{-1}\mathbf{F}(P\mathbf{c}, \epsilon) = \mathbf{0}.$$

In this system of $2n$ equations in the $2n + 1$ unknowns $\beta, c_1, c_2, \dots, c_{2n}$, it is natural to assign a value to one of the c 's, say $c_i, c_i \neq 0$. We then use the implicit function theorem to obtain sufficient conditions for $2n$ solutions of the above $2n$ equations. Each of these $2n$ solutions has the form

$$(32) \quad \beta_i = \sum_{k=0}^{\infty} \beta_i^{(k)} \epsilon^k, \quad c_j = \sum_{k=1}^{\infty} c_j^{(k)} \epsilon^k, \quad j = 1, 2, \dots, 2n, \\ j \neq i, c_i \text{ arbitrary, } \epsilon < \epsilon_0, \quad i = 1, 2, \dots, 2n.$$

For $\epsilon = 0$ system (31) has the form

$$(33) \quad \mathbf{G} \equiv (B - \beta I)\mathbf{c} = \mathbf{0}.$$

If we choose $\beta = \beta_i^{(0)}$, the i^{th} characteristic root, then c_i is arbitrary, $c_i \neq 0$, and $c_j = 0$, $j \neq i$. Each Jacobian

$$(34) \quad J_1 = \frac{\partial(G_1, \dots, G_{2n})}{\partial(\beta, c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_{2n})}$$

$$= (-1)^i c_i \prod_{k=1}^{2n} (\beta_k^{(0)} - \beta_i^{(0)}) \neq 0, \quad k \neq i,$$

at $\beta = \beta_i^{(0)}$, $c_i = 0$, $j \neq i$, $i = 1, 2, \dots, 2n$; hence, by the implicit function theorem, there are solutions of system (31) of the form (32).

Since we have assumed the characteristic roots $\beta_i^{(0)}$ of A are distinct, it follows that for ϵ sufficiently small, the numbers β_i of (32) are distinct, and hence the characteristic exponents of the variational system (16) are distinct. We have then determined a fundamental system (17) of solutions of system (16). Thus we conclude with the following:

Theorem. *If the characteristic roots of the matrix A are distinct and if each root has real part negative, the periodic solution $x_i = x_i^*$, $j = 1, 2, \dots, 2n$, of (7) is asymptotically stable.*

With this theorem the determination of the characteristic exponents of system (16) in terms of FLOQUET's theory is replaced by the easier determination of the characteristic roots of a known matrix. This matrix is written in terms of the same elements which have been used for the determination of the periodic solution under discussion.

6. Applications to second order equations and systems. We now consider a second order equation

$$(35) \quad \ddot{y} + \sigma^2 y = \epsilon f(y, \dot{y}, t, \epsilon),$$

satisfying conditions (A), (B), (C) of the introduction, and we apply the discussion of the previous section to obtain criteria for asymptotic stability of periodic solutions of (35). With the transformation of variables (6), we obtain from (35)

$$(36) \quad \begin{cases} \dot{x}_1 = i\sigma x_1 + \epsilon f\left[\frac{x_1 + x_2}{2i\sigma}, \frac{x_1 - x_2}{2}, t, \epsilon\right] \\ \dot{x}_2 = -i\sigma x_2 - \epsilon f\left[\frac{x_1 + x_2}{2i\sigma}, \frac{x_1 - x_2}{2}, t, \epsilon\right]. \end{cases}$$

After introducing the auxiliary system

$$(37) \quad \begin{cases} \dot{x}_1 = i\tau x_1 + \epsilon f[\dots] \\ \dot{x}_2 = -i\tau x_2 - \epsilon f[\dots], \end{cases}$$

where the arguments in $[\dots]$ are the same as in (36), and the zeroth approximation

$$(38) \quad \begin{cases} x_{0,1} = ae^{i\tau t} \\ x_{0,2} = -\bar{a}e^{-i\tau t}, \end{cases}$$

we obtain a periodic solution, $x_1 = x_1^*$, $x_2 = x_2^*$, of the system

$$(39) \quad \begin{cases} \dot{x}_1 = [i\tau - \epsilon V(\tau, a, \sigma, \epsilon)]x_1 + \epsilon f[\dots] \\ \dot{x}_2 = [-i\tau - \epsilon \bar{V}(\tau, a, \sigma, \epsilon)]x_2 + \epsilon f[\dots]. \end{cases}$$

The number $V(\tau, a, \sigma, \epsilon)$ has the form (12). If we put $a = |a|e^{i\varphi}$, then, in correspondence with (13),

$$(40) \quad \begin{aligned} V_1 &= \frac{1}{|a|mT} \left\{ \int_0^{mT} f \left[\frac{|a|}{\sigma} \sin(\tau t + \varphi), |a| \cos(\tau t + \varphi), t, 0 \right] \cos(\tau t + \varphi) dt \right. \\ &\quad \left. - i \int_0^{mT} f \left[\frac{|a|}{\sigma} \sin(\tau t + \varphi), |a| \cos(\tau t + \varphi), t, 0 \right] \sin(\tau t + \varphi) dt \right\} \\ &= H - iG, \quad i = \sqrt{-1}. \end{aligned}$$

The determining equations (14) then become

$$(41) \quad \begin{cases} \tau + \epsilon G + O(\epsilon^2) = \sigma \\ H + O(\epsilon) = 0. \end{cases}$$

As in [9], the first terms in the amplitude $|a|$ and phase φ of the solution of (36) and hence (35) are given by the solution of

$$(42) \quad \begin{cases} \tau + \epsilon G = \sigma \\ H = 0. \end{cases}$$

The matrix A of equation (29) can now be written explicitly for this case

$$(43) \quad A = \left\| \begin{aligned} &-H + iG + \frac{1}{2mT} \int_0^{mT} \left\{ \frac{f_y}{i\sigma} + f_y \right\} dt, \frac{1}{2mT} \int_0^{mT} e^{-2i\tau t} \left\{ \frac{f_y}{i\sigma} - f_y \right\} dt \\ &-\frac{1}{2mT} \int_0^{mT} e^{2i\tau t} \left\{ \frac{f_y}{i\sigma} + f_y \right\} dt, -H - iG - \frac{1}{2mT} \int_0^{mT} \left\{ \frac{f_y}{i\sigma} - f_y \right\} dt \end{aligned} \right\|$$

where $f_y = \partial f / \partial y$, $f_{\dot{y}} = \partial f / \partial \dot{y}$ evaluated along $y = (|a|/\sigma) \sin(\tau t + \varphi)$, $\dot{y} = |a| \cos(\tau t + \varphi)$. The characteristic equation of A has the form

$$(44) \quad \beta^2 - 2\alpha\beta + \gamma = 0.$$

The requirement for stability states that the roots β_1 , β_2 of (44) must be distinct and have negative real parts. In other words, the coefficients α , γ must satisfy

$$(45) \quad \alpha < 0, \quad \gamma > 0, \quad \alpha^2 \neq \gamma.$$

In terms of the elements of the matrix A , conditions (45) take the form

$$(46) \quad \begin{cases} 2mT\alpha = \int_0^{mT} f_\nu dt < 0 \\ 4(mT)^2\gamma = \left(\int_0^{mT} f_\nu dt\right)^2 + \left(2mTG - \frac{1}{\sigma} \int_0^{mT} f_\nu dt\right)^2 \\ \quad - \left(\frac{1}{\sigma} \int_0^{mT} f_\nu \cos 2\tau t dt - \int_0^{mT} f_\nu \sin 2\tau t dt\right)^2 \\ \quad - \left(\frac{1}{\sigma} \int_0^{mT} f_\nu \sin 2\tau t dt + \int_0^{mT} f_\nu \cos 2\tau t dt\right)^2 > 0 \\ \alpha^2 \neq \gamma. \end{cases}$$

Conditions (46) appear in [15] and [16, p. 330] in a slightly different form and do not contain the number G from the determining equations (42) explicitly.

Van der Pol equation with a forcing term. Consider the equation

$$(47) \quad \ddot{y} + y = \epsilon(1 - y^2)\dot{y} + \epsilon p\omega \cos(\omega t + \alpha),$$

where p, ω, α are constants and $\epsilon > 0$ is a real parameter, assumed small. Existence and stability of periodic and almost periodic solutions of (47) have been studied by, among others, M. L. CARTWRIGHT [2]. Existence of periodic solutions of the equation

$$(48) \quad \ddot{y} + y = \epsilon(1 - y^{2n})\dot{y} + \epsilon p\omega \cos(\omega t + \alpha),$$

where $n \geq 1$ is an integer, has been studied in [9]. In this paragraph we shall consider only asymptotic stability of the *harmonic* solutions of (47). Hence the number τ used in equation (37) is, in this case, $\tau = \omega$. The number V_1 from equation (40) is

$$\begin{aligned} V_1 &= \frac{\omega}{2\pi a} \int_0^{2\pi/\omega} \{(1 - |a|^2 \sin^2(\omega t + \varphi))|a| \cos(\omega t + \varphi) + p\omega \cos(\omega t + \alpha)\} \\ &\quad \{\cos(\omega t + \varphi) - i \sin(\omega t + \varphi)\} dt \\ &= \frac{1}{2} \left\{ 1 - \frac{|a|^2}{4} + \frac{p\omega}{|a|} \cos(\varphi - \alpha) \right\} - \frac{ip\omega}{2|a|} \sin(\varphi - \alpha), \end{aligned}$$

hence the determining equations (42) take the form

$$(49) \quad \begin{cases} \omega + \frac{\epsilon p\omega}{2|a|} \sin(\varphi - \alpha) = 1 \\ 1 - \frac{|a|^2}{4} + \frac{p\omega}{|a|} \cos(\varphi - \alpha) = 0. \end{cases}$$

The stability conditions (46) are

$$(50) \quad \begin{cases} 2\alpha = 1 - \frac{|a|^2}{2} < 0 \\ 4\gamma = \frac{3}{16} |a|^4 - |a|^2 + 1 + \left[\frac{p\omega}{|a|} \sin(\varphi - \alpha) \right]^2 > 0. \\ \frac{|a|^4}{16} - \left[\frac{p\omega}{|a|} \sin(\varphi - \alpha) \right]^2 \neq 0. \end{cases}$$

The first two conditions, obtained by a different method, appear in [2, p. 213]. For a complete discussion of (49) and (50), see [2].

Coupling of two van der Pol equations. Consider the system

$$(51) \quad \begin{cases} \ddot{y}_1 + \sigma_1^2 y_1 = \epsilon(\alpha - \gamma y_2^2) \dot{y}_1 + p \cos t \\ \ddot{y}_2 + \sigma_2^2 y_2 = \epsilon(\delta - \rho y_1^2) \dot{y}_2 + r \cos 2t, \end{cases}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, $\alpha, \gamma, \delta, \rho, p, r$ are constants and $\epsilon > 0$ is a parameter. We shall obtain the determining equations (14) and the matrix A (29), for system (51), and examine the existence and stability conditions in a special case. Making the transformation of variables (6) and introducing the auxiliary system (8) with $\tau_1 = \tau_2 = 1$ and the zeroth approximation (9), we obtain from (13), after putting $a_1 = |a_1| e^{i\varphi_1}$, $a_2 = |a_2| e^{i\varphi_2}$,

$$(52) \quad \begin{cases} V_{1,1} = R(V_{1,1}) + iI(V_{1,1}) = \frac{\alpha}{2} + \frac{\gamma|a_2|^2}{8\sigma_2^2} [\cos 2(\varphi_1 - \varphi_2) - 2] \\ \quad + \frac{p}{2|a_1|} \cos \varphi_1 - i \left[\frac{\gamma|a_2|^2}{8\sigma_2^2} \sin 2(\varphi_1 - \varphi_2) - \frac{p}{2|a_1|} \sin \varphi_1 \right] \\ V_{1,2} = R(V_{1,2}) + iI(V_{1,2}) = \frac{\delta}{2} + \frac{\rho|a_1|^2}{8\sigma_1^2} [\cos 2(\varphi_1 - \varphi_2) - 2] \\ \quad + i \frac{\rho|a_1|^2}{8\sigma_1^2} \sin 2(\varphi_1 - \varphi_2) \end{cases}$$

Hence the first terms in the determining equations (14) are

$$(53) \quad \begin{cases} 1 + \epsilon \left\{ \frac{\gamma|a_2|^2}{8\sigma_2^2} \sin 2(\varphi_1 - \varphi_2) + \frac{p}{2|a_1|} \sin \varphi_1 \right\} = \sigma_1 \\ \alpha + \frac{\gamma|a_2|^2}{4\sigma_2^2} [\cos 2(\varphi_1 - \varphi_2) - 2] + \frac{p}{|a_1|} \cos \varphi_1 = 0 \\ 1 - \epsilon \left\{ \frac{\rho|a_1|^2}{8\sigma_1^2} \sin 2(\varphi_1 - \varphi_2) \right\} = \sigma_2 \\ \delta + \frac{\rho|a_1|^2}{4\sigma_1^2} [\cos 2(\varphi_1 - \varphi_2) - 2] = 0. \end{cases}$$

Using the zeroth approximation $y_1 = (|a_1|/\sigma_1) \sin(t + \varphi_1)$, $\dot{y}_1 = |a_1| \cos(t + \varphi_1)$, $y_2 = (|a_2|/\sigma_2) \sin(t + \varphi_2)$, $\dot{y}_2 = |a_2| \cos(t + \varphi_2)$, the matrix A of equation (29) becomes

$$(54) \quad A = \begin{vmatrix} K & C & D & E \\ \bar{C} & \bar{K} & \bar{E} & \bar{D} \\ G & H & B & F \\ \bar{H} & \bar{G} & \bar{F} & \bar{B} \end{vmatrix}$$

where

$$\begin{aligned} K &= \frac{\alpha}{2} - \frac{\gamma|a_2|^2}{4\sigma_2^2} - V_{1,1} & F &= \frac{-\rho|a_1|^2}{8\sigma_1^2} e^{2i\varphi_1} \\ B &= \frac{\delta}{2} - \frac{\rho|a_1|^2}{4\sigma_1^2} - V_{1,2} & G &= \frac{i\rho|a_1||a_2|}{2\sigma_1^2} \sin(\varphi_1 - \varphi_2) \\ C &= \frac{-\gamma|a_2|^2}{8\sigma_2^2} e^{2i\varphi_2} & H &= \frac{\rho|a_1||a_2|}{4\sigma_1^2} e^{i(\varphi_1 + \varphi_2)} \\ D &= \frac{-i\gamma|a_1||a_2|}{2\sigma_2^2} \sin(\varphi_1 - \varphi_2) & E &= \frac{\gamma|a_1||a_2|}{4\sigma_2^2} e^{i(\varphi_1 + \varphi_2)}. \end{aligned}$$

From the theorem of §5, if all the characteristic roots of the matrix A (54) have negative real parts and are distinct for some solution of the system (53), then that periodic oscillation determined by the solution of (53) is asymptotically stable. As a particular case, let us assume that $\sigma_1 = \sigma_2 = 1$. Then $V_{1,1} = V_{1,2} = 0$, and

$$(55) \quad \begin{cases} \varphi_1 = \varphi_{10} = 0 \\ \varphi_2 = \varphi_{20} = 0 \\ |a_1|^2 = |a_{10}|^2 = \frac{4\delta}{\rho} \\ |a_2|^2 = |a_{20}|^2 = \frac{4}{\gamma} \left(\alpha + \frac{p}{|a_{10}|} \right) \end{cases}$$

is of solution of system (53). The Jacobian

$$(56) \quad J = \left| \frac{\partial(R(V_{1,1}), I(V_{1,1}), R(V_{1,2}), I(V_{1,2}))}{\partial(\varphi_1, \varphi_2, |a_1|, |a_2|)} \right|_{\substack{\varphi_1 = \varphi_{10}, \varphi_2 = \varphi_{20}, \\ |a_1| = |a_{10}|, |a_2| = |a_{20}|}} \\ = \frac{-\rho p \gamma |a_{10}|^2 |a_{20}|}{128} \neq 0,$$

hence the determining equations (14) for system (51) have a solution $|a_1|$, $|a_2|$, φ_1 , φ_2 with first terms given by (55) [9]. The matrix A (54) then becomes

$$(57) \quad A = \begin{vmatrix} \frac{-\alpha}{2} - \frac{p}{|a_{10}|} & \frac{-\alpha}{2} - \frac{p}{2|a_{10}|} & 0 & \frac{\gamma|a_{10}a_{20}|}{4} \\ \frac{-\alpha}{2} - \frac{p}{2|a_{10}|} & \frac{-\alpha}{2} - \frac{p}{|a_{10}|} & \frac{\gamma|a_{10}a_{20}|}{4} & 0 \\ 0 & \frac{\rho|a_{10}a_{20}|}{4} & \frac{-\delta}{2} & \frac{-\delta}{2} \\ \frac{\rho|a_{10}a_{20}|}{4} & 0 & \frac{-\delta}{2} & \frac{-\delta}{2} \end{vmatrix}.$$

Using the Routh test functions to determine whether all the characteristic roots of A have real part negative, we have the following conditions:

$$(58) \quad \begin{cases} R_1 = \alpha + \delta + \frac{2p}{|a_{10}|} > 0 \\ R_2 = 8\alpha^2\delta p + 7\alpha\rho p^2|a_{10}| + 16\delta^2\alpha p + 6p^3\rho + 8|a_{10}|\delta p^2\rho + 16\delta^3p > 0 \\ R_3 = 16|a_{10}|p^2\delta\rho(2\delta^3 + 17\alpha^2\delta + 10\alpha\delta^2 + 12\alpha^3) + |a_{10}|p^4\rho^2(32\delta + 95\alpha) \\ \quad + 128\delta^2p\alpha(\delta^3 + 2\delta^2\alpha + 2\delta\alpha^2 + \alpha^3) + 16p^3\delta\rho(5\delta^2 + 22\delta\alpha + 26\alpha^2) \\ \quad + 30p^5\rho^2 < 0 \\ R_4 = \delta^2p\left(\frac{\alpha}{|a_{10}|} + \frac{p\rho}{4\delta}\right) < 0. \end{cases}$$

The above conditions, along with the conditions $|a_{10}| > 0$, $|a_{20}| > 0$, are consistent. For example, if we choose $\alpha, \gamma, \delta, \rho > 0$ and $p < 0$ with $\max[-\alpha|a_{10}|, -\frac{1}{2}|a_{10}|(\alpha + \delta)] < p < 0$, then $R_4 < 0$, $R_1 > 0$, $|a_{20}| > 0$. To show that $R_2 > 0$, $R_3 < 0$, let us write p in the form $p = -k|a_{10}|$, $k > 0$ a constant. If we now choose δ to be any of the numbers $\delta = 2k - \alpha + \eta$, where $\eta = \eta(\alpha)$ is a positive number, then for all k such that $\frac{1}{2}\alpha + \xi_1(\eta) < k < \frac{3}{2}\alpha - \xi_2(\eta)$, where $\xi_1(\eta)$, $\xi_2(\eta)$ are positive numbers, the conditions $R_2 > 0$, $R_3 < 0$ are satisfied. For any particular set $(\alpha, \gamma, \delta, \rho, p)$, the simplicity of the roots of the characteristic equation of matrix A (57) may be checked by the usual means.

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BIBLIOGRAPHY

- [1] BELLMAN, R., *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.
- [2] CARTWRIGHT, M. L., *Forced Oscillations in Non-linear Systems*, Ann. Math. Studies, No. 20, pp. 149-241, Princeton University Press, 1950.
- [3] CESARI, L., Sulla stabilità delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici, *Atti. Accad. Italia*, Mem. Cl. Sci. Fis. Mat. Nat. (6) 11, 633-692 (1940).

- [4] CESARI, L. & HALE, J. K., Second Order Linear Differential Systems with Periodic L-integrable Coefficients, *Rivista Mat. Univ. Parma* **5**, 55-61 (1954).
- [5] CESARI, L. & HALE, J. K., A New Sufficient Condition for Periodic Solutions of Weakly Non-linear Differential Equations, *Proc. Amer. Math. Soc.* **8**, 757-765 (1957).
- [6] CODDINGTON, E. A. & LEVINSON, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
- [7] GAMBILL, R. A., Stability Criteria for Linear Differential Systems with Periodic Coefficients, *Rivista Mat. Univ. Parma* **5**, 169-181 (1954).
- [8] GAMBILL, R. A., Criteria for Parametric Instability for Linear Differential Systems with Periodic Coefficients, *Ibid.* **6**, 37-43 (1955).
- [9] GAMBILL, R. A. & HALE, J. K., Subharmonic and Ultraharmonic Solutions for Weakly Non-linear Systems, *Journal Rat. Mech. and Analysis* **5**, 353-394 (1956).
- [10] HALE, J. K., Evaluations concerning products of Exponential and Periodic Functions, *Rivista Mat. Univ. Parma* **5**, 63-81 (1954).
- [11] HALE, J. K., On Boundedness of the Solutions of Linear Differential Systems with Periodic Coefficients, *Ibid.* **5**, 137-167 (1954).
- [12] HALE, J. K., Periodic Solutions of Non-linear Systems of Differential Equations, *Ibid.* **5**, 281-311 (1954).
- [13] HALE, J. K., On a Class of Linear Differential Equations with Periodic Coefficients, *Illinois J. Math.* **1** (1957).
- [14] LIAPOUNOFF, A., Probleme général de la stabilité du mouvement, *Comm. Soc. Mat. Kharkow* (1893); *Annales Faculté Sciences Toulouse*, (2) **9**, 203-475 (1907); *Annals of Math. Studies* No. 17, Princeton University Press, 1947.
- [15] MANDELSTAM, L. & PAPALEXI, N., Uber Resonanzerscheinungen bei Frequenzteilung, *Zeit. Physik* **73** (1932).
- [16] MINORSKY, N., *Introduction to Non-linear Mechanics*, J. W. Edwards, Ann Arbor, 1947.
- [17] MINORSKY, N., On Asynchronous Action, *Jour. Frank. Inst.* **259**, 209-219 (1955).
- [18] SANSONE, G., *Equazioni differenziali nel compo reale*, 2 vols., Zanichelli, Bologna, 1948-49.

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