Energy Integrals for the Mixed Problem in Hyperbolic Partial Differential Equations of Higher Order

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Introduction. In this paper we shall discuss the mixed initial and boundary value problem of the linear hyperbolic partial differential equation of higher order in two dimensions: x — space dimension, y — time dimension. We approach the problem through a basic energy integral inequality, developed in Section 2, which provides an *a priori* estimate of the solution of the given equation.

Any solution of a hyperbolic differential equation necessarily consists of three distinct parts: Uniqueness, existence and continuability of the initial conditions. Initial conditions are called continuable, see FRIEDRICHS [5], if a solution of the equation exists such that the type of conditions imposed on the initial data for $y = 0$ holds also for appropriate space-like curves, *e.g.* for $y = y_0 > 0$. In our case of the mixed initial and boundary data we consider also another part, the continuability of the boundary conditions. By continuability of the boundary conditions we mean that the type of conditions imposed on the boundary data for $x = 0$ holds also on appropriate time-like curves, *e.g.* $x = x_0 > 0$. L. CAMPBELL & A. ROBINSON [1] treated the uniqueness and existence of the mixed hyperbolic problem by reduction to systems of ordinary differential equations.

Our use of energy integrals requires work in certain L_2 spaces and results in generalized “strong” solutions of the given equations. THOMÉE [14] developed energy integral estimates for the special case of the mixed problem for the hyperbolic equation in two dimensions of third order with constant coefficients. Energy integrals were introduced by FRIEDRICHS & LEWY [6] (also ZAREMBA [15]). LERAY [9] and GÅRDING [7] arrive at energy estimates for the Cauchy problem of the hyperbolic equation of higher order in several variables by multiplying the given hyperbolic differential operator of order $m + 1$ by another operator

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of order m whose characteristic surfaces separate those of the given operator. In our case of the mixed problem we use the same idea; however we multiply the given operator by a special m -th order operator, namely by an operator whose characteristics still separate those of the given operator but are near enough to those characteristics which on the boundary lead outside the domain under investigation.

The derivation of the strict solution from the strong solution in section 5 is based on SOBOLEV'S lemma [13] (also SCHAUDER [12]) which assures us that a function which has (as in our case of two dimensions) $k + 2$ strong derivatives in a suitable domain has in this domain k continuous derivatives.

Section 1: Definitions and Initial and Boundary Data. We shall deal with the linear hyperbolic equation in two dimensions x and y

$$Eu \equiv (L + M)u = f. \quad (1.1)$$

L is a hyperbolic linear homogeneous differential operator of order $m + 1$. M is a linear differential operator involving derivatives up to the m -th order only. f is in L_2 (in the domain described below) and the coefficients of the operators have $m + 1$ continuous derivatives. The characteristic polynomial of E is

$$(a_1\eta + b_1\xi) \cdots (a_{m+1}\eta + b_{m+1}\xi)$$

with

$$a_i = a_i(x, y) > 0, \quad b_i = b_i(x, y) \neq 0;$$

also

$$a_i^2 + b_i^2 = 1$$

can be assumed without loss of generality. It is also assumed that

$$b_1 > b_2 > \cdots > b_{j_0} > \alpha > 0 > \beta > b_{j_0+1} > \cdots > b_{m+1} \quad (1.2)$$

everywhere in the domain with α, β constants.

We note that the operator E can also be written in the form

$$Eu \equiv \left(a_1 \frac{\delta}{\delta y} + b_1 \frac{\delta}{\delta x}\right) \cdots \left(a_{m+1} \frac{\delta}{\delta y} + b_{m+1} \frac{\delta}{\delta x}\right) u + Mu \quad (1.3)$$

where we denote by $\delta/\delta x$ and $\delta/\delta y$ partial derivatives that operate on u only and treat the coefficients a_i and b_i as constants. Furthermore we will also make use of the operator E written in the form

$$Eu \equiv \left(a_{m+1} \frac{\partial}{\partial y} + b_{m+1} \frac{\partial}{\partial x}\right) \cdots \left(a_1 \frac{\partial}{\partial y} + b_1 \frac{\partial}{\partial x}\right) u + \tilde{M}u \equiv \mathcal{L}u + \tilde{M}u. \quad (1.4)$$

We denote

$$a_i \frac{\partial}{\partial y} + b_i \frac{\partial}{\partial x} = \partial_i.$$

Therefore

$$Eu \equiv \partial_{m+1} \cdots \partial_1 u + \tilde{M}u. \quad (1.5)$$

We denote by R the closed domain bounded by OMP where OM is the initial segment which we denote by I and OP is the boundary segment which we denote by B . The curve MP is the characteristic

$$\frac{dy}{dx} = \frac{a_{m+1}}{b_{m+1}} \quad \text{passing through } M. \quad (1.6)$$

R_h is the subdomain of R with $y \leq h$. On the boundary B the first j_0 characteristics lead inside R with increasing y and the last $m+1-j_0$ characteristics lead outside R .

Initial Data. On I we give the data

$$u_{y^i}(x, 0) = \varphi_i, \quad i = 0, 1, \dots, m. \quad (1.7)$$

φ_i has $m-i$ strong x derivatives* and we denote

$$u_{y^i x^{m-i}}(x, 0) = \Phi_i, \quad i = 0, 1, \dots, m. \quad (1.8)$$

Boundary Data. On B we give the data $\Lambda_i u(0, y) = \psi_i$ with

$$\Lambda_i u(0, y) \equiv \sum_{\gamma=0}^{\beta(i)} \delta_{i\gamma} u_{x^\gamma y^{\beta(i)-\gamma}}(0, y) + D_i u(0, y), \quad (1.9)$$

$$0 \leq \beta(i) \leq m, \quad i = 1, 2, \dots, j_0$$

*For the definition and properties of strong derivatives see the section "Spaces and Norms" in this paper and also FRIEDRICHS [5] and SOBOLEV [13].

where D_i is a linear expression in the derivatives of u on B up to the order $\beta(i) - 1$. ψ_i has $m - \beta(i)$ strong y derivatives on B and we denote

$$(\psi_i)_{y^{m-\beta(i)}} = \Psi_i, \quad i = 1, 2, \dots, j_0. \quad (1.10)$$

Further assume that the coefficients of Δ_i have $2m + 1 - \beta(i)$ continuous y derivatives. Differentiating (1.9) $m - \beta(i)$ times we have for $i = 1, 2, \dots, j_0$

$$\Gamma_i u(0, y) \equiv (\Delta_i + C_i)u(0, y) \equiv \sum_{\gamma=0}^m \delta_{i\gamma} u_{x^\gamma y^{m-\gamma}}(0, y) + C_i u(0, y) = \Psi_i \quad (1.11)$$

where C_i is an operator on B involving only lower order derivatives and the additional $\delta_{i\gamma}$ are all 0.

Denote further the i -th *characteristic operator*

$$P_i = \prod_{\substack{\gamma=1 \\ \gamma \neq i}}^{m+1} \left(\alpha_\gamma \frac{\partial}{\partial y} + b_\gamma \frac{\partial}{\partial x} \right) = d_{i0} \frac{\partial^m}{\partial y^m} + \dots + d_{im} \frac{\partial^m}{\partial x^m}, \quad (1.12)$$

$$i = 1, 2, \dots, m + 1.$$

Our basic assumption, which we shall refer to as *boundary data restriction*, is that on B

$$\begin{vmatrix} \delta_{10} & \dots & \delta_{1\beta(1)} & 0 & \dots & 0 \\ \vdots & & & & & \\ \delta_{j_0 0} & \dots & \delta_{j_0 \beta(j_0)} & 0 & \dots & 0 \\ d_{j_0+1,0} & \cdot & \cdot & \cdot & & d_{j_0+1,m} \\ \vdots & & & & & \\ d_{m+1,0} & \cdot & \cdot & \cdot & & d_{m+1,m} \end{vmatrix} \neq 0; \quad (1.13)$$

if $\beta(i) = m$ then in the i -th row the zeros on the right hand side need not appear. Condition (1.13) implies that on B the m -th order derivatives of u can be expressed as linear combinations of the characteristic operators $P_{i_0+1}, \dots, P_{m+1}$ and the main parts of the boundary operators, namely $\Delta_1, \dots, \Delta_{i_0}$.*

Spaces and Norms. We denote by S the class of smooth functions in the underlying domain, namely functions having a sufficient number of continuous derivatives for the purpose at hand. Further let S^0 be the class of smooth functions vanishing in the neighborhood of the origin. C_γ is the class of functions having γ continuous derivatives. C_γ^0 is the class of functions in C_γ that vanish in a neighborhood of the origin. We introduce the following spaces: H_γ is the space obtained by completion of the set of smooth functions in R under the norm $|v|_{H_\gamma}$ which is the square root of the γ -fold Dirichlet integral of all derivatives up to the γ -th order,

*CAMPBELL & ROBINSON [1] arrive at a similar restriction for the boundary data.

$$|v|_{H_\gamma}^2 = \sum_{\mu \leq \gamma} \iint_R (D^\mu v)^2 dx dy. \quad (1.14)$$

I_γ is the space obtained by completion of the set of smooth functions on I under the norm $|v|_{I_\gamma}$

$$|v|_{I_\gamma}^2 = \sum_{\mu \leq \gamma} \int_I (D_x^\mu v)^2 dx. \quad (1.15)$$

B_γ is the similar space on B with the norm $|v|_{B_\gamma}$,

$$|v|_{B_\gamma}^2 = \sum_{\mu \leq \gamma} \int_B (D_y^\mu v)^2 dy. \quad (1.16)$$

We say that a function u has strong derivatives up to the γ -th order in R if it belongs to H_γ . There exists then a sequence of approximating functions $u^{(n)}$ such that $|u^{(n)} - u|_{H_\gamma} \rightarrow 0$. The limits in H_0 of $D^\mu u^{(n)}$, $\mu \leq \gamma$, are called the strong μ derivatives of u in R . In a similar way we have strong x derivatives on I and strong y derivatives on B . For uniqueness, compatibility with the classic concept of the derivative, and basic properties of the above definitions see FRIEDRICHS [3], SOBOLEV [13] (also LAX [8], NIRENBERG [10]).

Further we define: H_γ^0 is the subspace of H_γ obtained by the completion under the norm of H_γ of the set of functions in S^0 . I_γ^0 is the subspace of I_γ obtained by the completion under the norm of I_γ of the functions in S^0 on I . B_γ^0 is the similar subspace of B_γ on B .

We note that following SOBOLEV's Lemma [13] (also SCHAUDER [12]) any function in H_{k+2} has k continuous derivatives in R and any function in I_{k+1} or B_{k+1} has k continuous x derivatives or y derivatives on I or B respectively. Specifically it follows that φ_i has $m - 1 - i$ continuous x derivatives on I and ψ_i has $m - 1 - \beta(i)$ continuous y derivatives on B .

Throughout this paper K and C will denote constants depending only on the coefficients of the given operators and the diameter of R . We shall denote by w any smooth function in the underlying domain.

Strong Solution. We define u to be a strong solution* of equation (1.1) with initial and boundary data (1.7) and (1.9) if there exists a sequence of functions $u^{(n)} \in C_{m+1}$ in R such that

$$|u - u^{(n)}|_{H_0} \rightarrow 0, \quad (1.17)$$

$$|Eu^{(n)} - f|_{H_0} \rightarrow 0, \quad (1.18)$$

$$|u_{y^i}^{(n)}(x, 0) - \varphi_i|_{I_{m-i}} \rightarrow 0, \quad i = 0, 1, \dots, m, \quad (1.19)$$

$$|\Lambda_i u^{(n)}(0, y) - \psi_i|_{B_{m-\beta(i)}} \rightarrow 0, \quad i = 1, 2, \dots, j_0. \quad (1.20)$$

*Compare FRIEDRICHS [5] (also SOBOLEV [13]).

A priori Estimates on the Boundary. Recalling (1.11) we have the j_0 boundary operators

$$\Gamma_i u(0, y) \equiv (\Delta_i + C_i)u(0, y) \equiv \sum_{\gamma=0}^m \delta_{i\gamma} u_{x^\gamma y^{m-\gamma}}(0, y) + C_i u(0, y).$$

If all Γ_i are homogeneous operators $\Gamma_i = \Delta_i$, *i.e.* involving only derivatives of the m -th order, we can express $P_1 u(0, y), \dots, P_{j_0} u(0, y)$ as a linear combination of $P_{j_0+1} u(0, y), \dots, P_{m+1} u(0, y)$ and $\Gamma_1, \dots, \Gamma_{j_0}$. However if $\Gamma_i = \Delta_i + C_i$ where C_i is an operator involving lower order derivatives we use a different method to obtain boundary estimates. Let

$$|w|_{0_{m-1}}^2 = \sum_{i+r \leq m-1} w_{x^i y^r}^2(0, 0). \quad (1.21)$$

Then for $0 \leq \bar{y} \leq y$

$$\begin{aligned} \sum_{i=1}^{j_0} [(\Delta_i - \Gamma_i)w(0, \bar{y})]^2 &\leq K \left\{ \sum_{i=1}^{j_0} \int_0^{\bar{y}} [(\Delta_i - \Gamma_i)w]^2 d\eta \right. \\ &\quad \left. + \sum_{i=1}^{j_0} |\Gamma_i w|_{B_0}^2 + \sum_{i=j_0+1}^{m+1} \int_0^y (P_i w)^2 d\eta + |w|_{0_{m-1}}^2 \right\}. \end{aligned} \quad (1.22)$$

Keeping y fixed we have an integral inequality in $\sum_{i=1}^{j_0} [(\Delta_i - \Gamma_i)w(0, \bar{y})]^2$ from which we get in a well known manner

$$\begin{aligned} \sum_{i=1}^{j_0} [(\Delta_i - \Gamma_i)w(0, \eta)]^2 &\leq K \left\{ \sum_{i=j_0+1}^{m+1} \int_0^y [P_i w(0, \eta)]^2 d\eta \right. \\ &\quad \left. + \sum_{i=1}^{j_0} |\Gamma_i w|_{B_0}^2 + |w|_{0_{m-1}}^2 \right\} e^{K\eta}. \end{aligned}$$

Absorbing $e^{K\eta}$ in the coefficient K and integrating we get after a simple calculation the basic boundary estimate

$$\begin{aligned} \sum_{i=1}^{j_0} \int_0^y [P_i w(0, \eta)]^2 d\eta &\leq K \left\{ \sum_{i=j_0+1}^{m+1} \int_0^y [P_i w(0, \eta)]^2 d\eta \right. \\ &\quad \left. + \sum_{i=1}^{j_0} |\Gamma_i w|_{B_0}^2 + |w|_{0_{m-1}}^2 \right\}. \end{aligned} \quad (1.23)$$

Section 2: Energy Inequality. We define

$$|\varphi|_{0_{m-1}}^2 = \sum_{i=0}^{m-1} \sum_{v=0}^{m-1-i} [D_x^v \varphi_i(0)]^2.$$

Theorem 1. *If u is a strong solution of (1.1) with initial and boundary data (1.7) and (1.9) then $u \in H_m$ and*

$$\begin{aligned} \int_{y=h} \sum_{\mu \leq m} (D^\mu u)^2 dS &\leq K \left\{ \iint_{R_h} f^2 dx dy + \sum_{i=0}^m |\Phi_i|_I^2 \right. \\ &\quad \left. + \sum_{i=1}^{j_0} |\Psi_i|_{B_0}^2 + |\varphi|_{0_{m-1}}^2 \right\}. \end{aligned} \quad (2.1)$$

Proof.

$$2EwP_iw = \frac{\partial}{\partial y} a_i(P_iw)^2 + \frac{\partial}{\partial x} b_i(P_iw)^2 + Q_iwP_iw, \quad i = 1, 2, \dots, m+1,$$

where Q_i is a linear operator involving derivatives up to the m -th order only. Therefore

$$2 \iint_{R_h} EwP_iw \, dx \, dy = \int_{OM+MS+ST+TO} (P_iw)^2 n_i \, dS + 2 \iint_{R_h} Q_iwP_iw \, dx \, dy \quad (2.2)$$

where n_i is the cosine of the angle between the outgoing normal of the boundaries and the i -th characteristic. All n_i are positive except that $n_1, \dots, n_{i_0}$ are negative on B (and n_{m+1} is zero on MP). Therefore $\sum_{i=1}^{i_0} \int_{TO} (P_iw)^2 n_i \, dS$ is negative. But this sum can be estimated, see (1.23), by $\sum_{i=i_0+1}^{m+1} \int_{TO} (P_iw)^2 \, dS$ and the given data. Therefore we choose a constant C large enough so that $C \sum_{i=i_0+1}^{m+1} \int_{TO} (P_iw)^2 n_i \, dS$ will be larger than the corresponding term in the estimate of $\sum_{i=1}^{i_0} \int_{TO} (P_iw)^2 n_i \, dS$. We have then

$$\begin{aligned} 2 \iint_{R_h} Ew \left(\sum_{i=1}^{i_0} P_iw + C \sum_{i=i_0+1}^{m+1} P_iw \right) dx \, dy \\ \geq \tau \left\{ \sum_{y=h} \int_{y=h} (D^m w)^2 \, dS + \sum_{i=1}^m \int_{MS} (P_iw)^2 \, dS \right\} \\ - K \left\{ \iint_{R_h} \sum (D^m w)^2 \, dx \, dy + \sum_{i=1}^{i_0} |\Gamma_i w|_{B_0}^2 + \sum |D^m w|_{I_0}^2 + |w|_{0_{m-1}}^2 \right\} \end{aligned} \quad (2.3)$$

with τ a positive constant independent of w . From this we get for $0 \leq \bar{h} \leq h$

$$\begin{aligned} \sum \int_{y=\bar{h}} (D^m w)^2 \, dS + \sum_{i=1}^m \int_{MS} (P_iw)^2 \, dS \\ \leq K \left\{ \iint_{R_{\bar{h}}} \sum (D^m w)^2 \, dx \, dy + \iint_{R_h} (Ew)^2 \, dx \, dy \right. \\ \left. + \sum_{i=1}^{i_0} |\Gamma_i w|_{B_0}^2 + \sum |D^m w|_{I_0}^2 + |w|_{0_{m-1}}^2 \right\}. \end{aligned} \quad (2.4)$$

Keeping h fixed and dropping the term $\sum_{i=1}^m \int_{MS} (P_iw)^2 \, dS$ from the left hand side of (2.4)* we get an integral inequality in $\sum \int_{y=\bar{h}} (D^m w)^2 \, dS$ from which we get in a well known way

$$\begin{aligned} \sum \int_{y=\bar{h}} (D^m w)^2 \, dS \leq K \left\{ \iint_{R_h} (Ew)^2 \, dx \, dy + \sum |D^m w|_{I_0}^2 \right. \\ \left. + \sum_{i=1}^{i_0} |\Gamma_i w|_{B_0}^2 + |w|_{0_{m-1}}^2 \right\} e^{K\bar{h}}. \end{aligned} \quad (2.5)$$

*We have kept this term so far for later reference.

Absorbing e^{K_h} and integrating we have

$$\sum \iint_{R_h} (D^m w)^2 dx dy \leq K \left\{ \iint_{R_h} (Ew)^2 dx dy + \sum |D^m w|_{I_0}^2 + \sum_{i=1}^{j_0} |\Gamma_i w|_{B_0}^2 + |w|_{0_{m-1}}^2 \right\}. \quad (2.6)$$

Clearly the lower order derivatives of w can also be estimated by the right hand side of (2.6). Substituting $w = u^{(n+k)} - u^{(n)}$ it follows that $|u^{(n)}|_{H_m}$ forms a Cauchy sequence. Therefore $u \in H_m$. Letting $w = u^{(n)}$ and $n \rightarrow \infty$ we get from (2.5)

$$\sum_{\mu \leq m} \int_{y=h} (D^\mu u)^2 dS \leq K \left\{ \iint_{R_h} f^2 dx dy + \sum_{i=0}^m |\Phi_i|_{I_0}^2 + \sum_{i=1}^{j_0} |\Psi_i|_{B_0}^2 + |\varphi|_{0_{m-1}}^2 \right\}. \quad (2.1)$$

Also from (2.6) we get

$$|u|_{H_m}^2 \leq K \left\{ |f|_{H_0}^2 + \sum_{i=0}^m |\Phi_i|_{I_0}^2 + \sum_{i=1}^{j_0} |\Psi_i|_{B_0}^2 + |\varphi|_{0_{m-1}}^2 \right\}. \quad (2.7)$$

Section 3: Uniqueness Theorem and Continuability of Initial and Boundary Data. From (2.7) we have the immediate result

Theorem 2. (*Uniqueness Theorem*). If u is a strong solution of the homogeneous equation $Eu = 0$ with homogeneous initial and boundary data $\varphi_i = 0, i = 0, \dots, m, \psi_i = 0, i = 1, \dots, j_0$, then $u = 0$.

From (2.1) we also have

Theorem 3. (*Continuability of Initial Data*). All derivatives $u_{x^i y^r}(x, h), i + r \leq m$, are L_2 -integrable on the line $y = h \geq 0$.

Theorem 4. (*Continuability of Boundary Data*). All derivatives $u_{x^i y^r}(\xi, y), i + r \leq m$, are L_2 -integrable on the line $x = \xi \geq 0$.

Proof. Let $\bar{R}_\xi$ be the closed subdomain of R bounded by MSQ . Estimating the expression

$$2 \iint_{\bar{R}_\xi} Eu^{(n)} \left(- \sum_{i=1}^{j_0} P_i u^{(n)} + \sum_{i=j_0+1}^{m+1} P_i u^{(n)} \right) dx dy \quad (3.1)$$

in a way similar to (2.4) and (2.5) we get

$$\sum \int_{x=\xi} [D^m u^{(n)}]^2 dS \leq K \left\{ \sum_{i=1}^{j_0} \int_{MS} (P_i u^{(n)})^2 dS + \sum |D^m u^{(n)}|_{H_0}^2 + |Eu^{(n)}|_{H_0}^2 + \sum |D^m u^{(n)}|_{I_0}^2 + |u^{(n)}|_{0_{m-1}}^2 \right\}. \quad (3.2)$$

From (2.4) also follows the convergence of $\sum_{i=1}^{j_0} \int_{MS} (P_i u^{(n)})^2 dS$. From (2.7) follows then the L_2 convergence on $x = \xi \geq 0$ of $[\sum (D^m u^{(n)}(\xi, y))^2]^{\frac{1}{2}}$. Furthermore, lower order derivatives can be estimated in a similar way from which follows the L_2 convergence on $x = \xi$ of $[\sum_{\mu \leq m} (D^\mu u^{(n)}(\xi, y))^2]^{\frac{1}{2}}$ which proves the theorem.

Remark. It is clear that the proofs of Theorems 3 and 4 can also be used to show the continuability of the initial conditions to any regular space-like curve and the continuability of the boundary conditions to any regular time-like curve.

Section 4: Existence of the Strong Solution. For the construction of the solution we assume a further condition of compatibility of the initial and boundary conditions and the given equation. We assume the existence of a function v which admits E in the strong sense such that equation (1.1) transforms into $E(u - v) = \bar{f}$ where $\bar{f} \in H_0^0$ and initial and boundary data are transformed into $\bar{\varphi}_i \in I_{m-i}^0$, $i = 0, 1, \dots, m$, and $\bar{\psi}_i \in B_{m-\beta(i)}^0$, $i = 1, 2, \dots, j_0$.

We define $\partial_i^{-1}g = \int g dS_i$ where the integration is taken along the i -th characteristic ending at the point (x, y) and starting either on I or B . In the operator $\partial_i = a_i \partial/\partial y + b_i \partial/\partial x$, $a_i, b_i \in C_{m+1}$ from which we can deduce by elementary considerations that if $g \in C_{m+1}^0$ then $\partial_i^{-1}g$ belongs to C_{m+1}^0 .

In this section we introduce functions $\omega_1, \dots, \omega_{m+1}$ such that ω_i is constant along the i -th characteristic. $\omega_1, \dots, \omega_{j_0}$ have to be defined on I and B . $\omega_{j_0+1}, \dots, \omega_{m+1}$ have to be defined on I alone.

Lemma 1. *There exists a solution $u \in C_{m+1}^0$ of*

$$\partial_{m+1} \cdots \partial_1 u = g \quad \text{with } g \in S^0 \quad (4.1)$$

and initial and boundary data

$$\varphi_i \in S^0, \quad i = 0, 1, \dots, m, \quad \text{and} \quad \psi_i \in S^0, \quad i = 1, 2, \dots, j_0.$$

Proof. We construct

$$u = \omega_1 + \partial_1^{-1}(\cdots(\omega_m + \partial_m^{-1}(\omega_{m+1} + \partial_{m+1}^{-1}g))\cdots). \quad (4.2)$$

Then u satisfies (4.1). $\omega_{j_0+1}, \dots, \omega_{m+1}$ are determined from the initial data. From (4.2) we have then

$$\partial_{j_0} \cdots \partial_1 u = \omega_{j_0+1} + \partial_{j_0+1}^{-1}(\cdots(\omega_{m+1} + \partial_{m+1}^{-1}g)\cdots). \quad (4.3)$$

From (4.3) we can compute on B

$$\prod_i u(0, y) = \partial_{m+1} \cdots \partial_{i+1} \partial_{i-1} \cdots \partial_1 u(0, y), \quad (4.4)$$

$$i = j_0 + 1, \dots, m + 1.$$

(4.4) together with

$$\Gamma_i u(0, y) = (\psi_i)_{y^{m-\beta(i)}}, \quad i = 1, \dots, j_0, \quad (4.5)$$

and the conditions $u_{x^i y^r}(0, 0) = 0, i + r \leq m - 1$, produce on B a set of $m + 1$ ordinary differential equations in the $m + 1$ functions $u(0, y), u_x(0, y), \dots, u_{x^m}(0, y)$. From the boundary data restriction (1.13) it follows that this set of equations has a unique solution. From this we can compute $\omega_1, \dots, \omega_{i_0}$ on B . On I we can compute these functions from the initial data. Clearly $\omega_1, \dots, \omega_{m+1}$ belong to C_{m+1}^0 . Therefore also $u \in C_{m+1}^0$.

Lemma 2. *There exists a strong solution of (1.1)*

$$Eu = f \quad \text{with} \quad f \in S^0$$

and initial and boundary conditions

$$\varphi_i \in S^0, \quad i = 0, 1, \dots, m, \quad \text{and} \quad \psi_i \in S^0, \quad i = 1, 2, \dots, j_0.$$

Proof. We introduce the space T_m which is the completion of smooth functions in R under the norm $|v|_{T_m}^*$

$$|v|_{T_m}^2 = |v|_{H_m}^2 + |v|_{I_m}^2 + \dots + |v_{y^m}|_{I_0}^2 + |\Gamma_1 v|_{B_{m-\beta(i_1)}}^2 + \dots + |\Gamma_{i_0} v|_{B_{m-\beta(i_0)}}^2.$$

T_m^0 is the subspace of T_m obtained by the completion under the norm of T_m of functions in S^0 . Clearly $T_m \subset H_m$. Let Z^0 be the subspace of T_m^0 consisting of those functions which satisfy the given initial and boundary data φ_i and ψ_i .

For smooth functions $w \in Z^0$ we define a linear transformation $\mathfrak{I} : \mathfrak{I}w = v$ where v is the solution of $\partial_{m+1} \dots \partial_1 v = -\tilde{M}w + f$ with the given initial and boundary data φ_i and ψ_i . $v \in C_{m+1}^0$ and therefore $v \in Z^0$. By completion under the norm of T_m we extend this transformation to one of Z^0 into itself. We intend to show that $\mathfrak{I}$ has a fixed point in Z^0 .

We introduce yet another space $H_{m,\lambda}$ which is the completion of the set of smooth functions in R under the norm $|v|_{H_{m,\lambda}}$,

$$|v|_{H_{m,\lambda}}^2 = \iint_R e^{-\lambda\eta} \sum_{\mu \leq m} (D^\mu v)^2 d\xi d\eta$$

where $\lambda > 0$ is a constant at our disposal. Clearly

$$|v|_{H_{m,\lambda}} \leq |v|_{H_m} \leq e^{\lambda K} |v|_{H_{m,\lambda}}.$$

Therefore $H_{m,\lambda} = H_m$. Let $w_1, w_2 \in Z^0$ and $\mathfrak{I}w_1 = v_1, \mathfrak{I}w_2 = v_2$. Then from (2.1)

$$e^{-\lambda h} \int_{y=h} \sum_{\mu \leq m} (D^\mu (v_2 - v_1))^2 d\xi \leq K e^{-\lambda h} \iint_{R_h} \sum_{\mu \leq m} (D^\mu (w_2 - w_1))^2 d\xi d\eta.$$

Integrating both sides with respect to h and using integration by parts on the right we get

$$|v_2 - v_1|_{H_{m,\lambda}}^2 \leq \frac{K}{\lambda} |w_2 - w_1|_{H_{m,\lambda}}^2.$$

*The space T_m is actually the space of functions in R that possess m strong derivatives and have proper initial and boundary values.

Choosing $\lambda > K$ we have a contracting linear transformation of Z^0 into itself, which has therefore a fixed point u . u satisfies the initial and boundary data and $\mathcal{L}u = -\tilde{M}u + f$.

Theorem 5. (*Existence of the Strong Solution*). *There exists a strong solution of (1.1)*

$$Eu = f \quad \text{with} \quad f \in H_0^0$$

and initial and boundary data

$$\varphi_i \in I_{m-i}^0, \quad i = 0, 1, \dots, m, \quad \text{and} \quad \psi_i \in B_{m-\beta(i)}^0, \quad i = 1, 2, \dots, j_0.$$

Proof. We choose a sequence of functions $f^{(n)} \in S^0$ such that $|f - f^{(n)}|_{H_0} \rightarrow 0$. Also choose $\varphi_i^{(n)} \in S^0$ such that $|\varphi_i - \varphi_i^{(n)}|_{I_{m-i}} \rightarrow 0$ and $\psi_i^{(n)} \in S^0$ such that $|\psi_i - \psi_i^{(n)}|_{B_{m-\beta(i)}} \rightarrow 0$. Let $u^{(n)}$ be the solution of $Eu^{(n)} = f^{(n)}$ with initial and boundary data $\varphi_i^{(n)}$ and $\psi_i^{(n)}$. From (2.7) it follows that $u^{(n)}$ forms a Cauchy sequence in H_m^0 which approaches a function $u \in H_m^0$ which solves $Eu = f$ with the data φ_i and ψ_i .

Section 5: Strict Solution. We define u to be a strict solution in the classical sense* of $Eu \equiv (\tilde{L} + \tilde{M})u = f$ with initial and boundary data φ_i and ψ_i in R if $u \in C_m$ in R , satisfying the given data and admitting the operator $\tilde{L}u \equiv \partial_{m+1} \cdots \partial_1 u$ in the strict sense of successive directional derivatives, such that $Eu = f$.

Remark. As u need not have all $m + 1$ continuous derivatives this definition might be meaningless at the origin. Therefore we require for the origin only $Eu(x, y) \rightarrow f(0, 0)$ for $(x, y) \rightarrow (0, 0)$ which needs no special verification if f is continuous. Moreover, it is readily verified that if f is continuous in R then u will also admit the operator $\prod [a_i(\partial/\partial y) + b_i(\partial/\partial x)]$ where the order of the factors is arbitrary.

Theorem 6. *There exists a strict solution of equation (1.1) $Eu = f$ if $f \in H_2^0$ with the initial and boundary data $\varphi_i \in I_{m+2-i}^0$, $i = 0, \dots, m$, and $\psi_i \in B_{m+2-\beta(i)}^0$, $i = 1, \dots, j_0$, and if the coefficients of the given operators have $m + 3$ continuous derivatives.*

Proof. We choose two additional directional derivatives

$$\partial_{m+2} = a_{m+2} \frac{\partial}{\partial y} + b_{m+2} \frac{\partial}{\partial x}, \quad \partial_{m+3} = a_{m+3} \frac{\partial}{\partial y} + b_{m+3} \frac{\partial}{\partial x}$$

with

$$a_{m+2}^2 + b_{m+2}^2 = a_{m+3}^2 + b_{m+3}^2 = 1; \quad a_{m+2}, a_{m+3} > 0; \quad b_{m+2}, b_{m+3} < 0$$

and we get a new operator $E_1 = \partial_{m+3}\partial_{m+2}E$.

*Compare COURANT & HILBERT [2], Vol. 2, p. 318.

We choose ∂_{m+2} , ∂_{m+3} such that the boundary data restriction (1.13) corresponding to E_1 holds. To show that this is possible take

$$\partial_{m+2} = \frac{1}{\sqrt{1+\epsilon^2}} \frac{\partial}{\partial y} - \frac{\epsilon}{\sqrt{1+\epsilon^2}} \frac{\partial}{\partial x}.$$

Now for the operator $\partial_{m+2}E$ the restriction corresponding to (1.13) will hold if $\epsilon = 0$. Therefore choose $\epsilon > 0$ such that the restriction is satisfied and such that $\epsilon < \min_x (-b_{j_0+1})$. In a similar way we choose ∂_{m+3} . We determine $\varphi_{m+1} = u_{y^{m+1}}(x, 0)$ and $\varphi_{m+2} = u_{y^{m+2}}(x, 0)$ from the relationship $Eu(x, 0) = f(x, 0)$. Now solve the equation

$$\partial_{m+3}\partial_{m+2}Eu = \partial_{m+3}\partial_{m+2}f \quad (5.1)$$

with initial and boundary data

$$\varphi_i, \quad i = 0, 1, \dots, m+2, \quad \text{and} \quad \psi_i, \quad i = 1, 2, \dots, j_0.$$

From Theorem 1 it follows that $u \in H_{m+2}$. Therefore u has m continuous derivatives in R . From (5.1) it follows that $Eu - f$ is a strong solution of

$$\partial_{m+3}\partial_{m+2}(Eu - f) = 0 \quad (5.2)$$

with initial conditions on I : $Eu - f = 0$ and $(Eu - f)_y = 0$. From Uniqueness Theorem 2 it follows that $Eu - f = 0$ in R . Therefore u solves $Eu = f$ in the strong sense, satisfying the initial and boundary data strictly, as u has m continuous derivatives in R . We still have to show that u admits $\tilde{L}$ in the strict sense. From (5.2) it follows that $\tilde{L}u \in H_1^0$, therefore $\tilde{L}u$ admits ∂_{m+2} and ∂_{m+3} in the strong sense. $\tilde{L}u$ also admits $\partial_{m+3}\partial_{m+2}$ in the strong sense. From SOBOLEV's Lemma it follows that $\tilde{L}u$ is continuous in R except possibly at the points O and M .^{*} If $u^{(n)}$ is the sequence of functions that approximates the solution of (5.1) then $\tilde{L}u^{(n)} \rightarrow \tilde{L}u$ uniformly in any subdomain of R that excludes O and M . Now

$$\partial_{m+1}^{-1}\tilde{L}u^{(n)} = -\omega_{m+1}^{(n)} + \partial_m \cdots \partial_1 u^{(n)} \quad (5.3)$$

where $\omega_{m+1}^{(n)}$ is constant along the $(m+1)$ -th characteristic such that on I

$$\omega_{m+1}^{(n)} = \partial_m \cdots \partial_1 u^{(n)}(x, 0).$$

The right hand side of (5.3) converges uniformly in R . For $n \rightarrow \infty$ we have therefore

$$\partial_{m+1}^{-1}\tilde{L}u = -\omega_{m+1} + \partial_m \cdots \partial_1 u$$

from which it follows that u admits $\tilde{L}$ in the strict sense in R except possibly at O and M . Continuity considerations extend this result also for the point M .

^{*}For a simple proof in this case, ∂_{m+2} , ∂_{m+3} being directional derivatives along constant directions, see SCHAUDER [12].

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