The Green's Function for Periodic Structures in Diffraction Theory with an Application to Parallel Plate Media, II

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1. Introduction. In our first paper [1] in this series, we discussed the effect of a plane wave incident upon an infinite stack of semi-infinite parallel plates with equal spacing and constant angle of stagger. We found there that this problem could be reduced to the solution of the two-dimensional wave equation in a duct because of the periodic character of the structure and the mode of excitation. With the appropriate Green's function, which we derived in [1], the boundary value problem was reduced to an integral equation of the Wiener-Hopf type and could, in theory, be solved for any range of the parameter bk , where k is the wave number and b the spacing between the parallel plates. However, we only treated the case $\pi < bk < 2\pi$ with the wave function subject to Dirichlet conditions, that is, when the wave function vanished on the parallel plates.

There is a second problem which can be solved with the methods we employed in [1]. This problem deals with the same geometry and excitation as we employed in our first paper, but now we impose Neumann conditions on the plates, that is, the normal derivative of the wave function vanishes on the parallel plates. A number of details require modification by this change of boundary condition, one being that the parallel plate region can now sustain a parallel plate mode of the form e^{*ikz} for $0 \leq bk \leq \pi$.^{*} We shall not attempt to solve this problem for a larger wave-length band, although this is possible. A second point deals with the application of the boundary condition. If we carry out the operations formally, we encounter a non-integrable kernel for the integral equation. This

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^{*} We shall use the same notation that we employed in [1] so that we can take over any similar details.

difficulty may be avoided by a simple device which will cast the integral equation into one which is almost identical with the one we discussed in [1], save for the different range for bk . It will turn out that all reflection and transmission coefficients can be expressed in a manner similar to those which we found for the boundary value problem involving Dirichlet conditions.

2. Formulation of the problem. We have already seen that a representation of the solution of the partial differential equation

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0$$

inside the duct

$$-\infty < x < \infty, \quad 0 \leq y \leq b$$

may be expressed by Green's theorem in the form

$$\begin{aligned} \phi(x, y) &= A \exp \{ik(x \cos \theta + y \sin \theta)\} \\ &= B \exp \{ik[x \cos (2\alpha - \theta) + y \sin (2\alpha - \theta)]\} \\ &\quad + \sum F_n \exp \{i(x\gamma + y\delta)\} \\ (2.1) \quad &- \int_{-\infty}^{\infty} [G^1\phi_{y'} - \phi G_{y'}^1] dx' + \int_{-\infty}^{\infty} [G^1\phi_{y'} - \phi G_{y'}^1] dx' \end{aligned}$$

if we assume the same type of excitation which we did in [1]. That is, we assume plane wave excitation for $\xi < 0$ and the particular form of parallel plate mode excitation which we described in Section 1 of the present paper. The integrals are evaluated at $y' = 0^+$ and b^- respectively. G^1 , $G_{y'}^1$, ϕ and $\phi_{y'}$ satisfy the same periodic boundary conditions which they did in [1], so that the integrals in (2.1) reduce to

$$- \int_0^{\infty} (G^1\phi_{y'} - \phi G_{y'}^1) dx' + \int_a^{\infty} [G^1\phi_{y'} - \phi G_{y'}^1] dx',$$

the integrals being evaluated at $y' = 0^+$ and b^- respectively. Furthermore, $\phi_{y'} = 0$ on any plate, so that these integrals may be simplified further and we have

$$(2.2) \quad \int_0^{\infty} \phi G_{y'}^1 dx' - \int_a^{\infty} \phi G_{y'}^1 dx'.$$

But

$$\phi(x + a, b^-) = \exp \{ik(a \cos \theta + b \sin \theta)\} \phi(x, 0^-),$$

so that (2.2) now becomes

$$\int_0^{\infty} \phi(x', 0^+) \frac{\partial G^1(x, y, x', y')}{\partial y'} dx' - \int_0^{\infty} \phi(x', 0^-) e^{ik\rho} \frac{\partial G^1(x, y, x', y')}{\partial y'} dx'.$$

Here $\rho = a \cos \theta + b \sin \theta$ and $\partial G^1/\partial y'$ is evaluated at $y' = 0$ in the first integral and $y' = b$ in the second. But

$$\left. \frac{\partial G^1}{\partial y'} \right|_{y'=0} = e^{ik\rho} \left. \frac{\partial G^1}{\partial y'} \right|_{y'=b},$$

and hence the integral reduces to

$$\int_0^\infty [\phi(x', 0^+) - \phi(x', 0^-)] \frac{\partial G^1}{\partial y'} dx', y' = 0.$$

In acoustical applications, the square bracket $[\phi(x', 0^+) - \phi(x', 0^-)] \equiv [\phi]$ is the discontinuity in the velocity potential across a rigid barrier. In electromagnetic applications, $[\phi]$ would be proportional to the surface current density on one of the barriers. Hence, if we use Neumann rather than Dirichlet conditions on the barriers, we have the representation

$$(2.3) \quad \begin{aligned} \phi(x, y) = & A \exp \{ik(x \cos \theta + y \sin \theta)\} \\ & + B \exp \{ik[x \cos (2\alpha - \theta) + y \sin (2\alpha - \theta)]\} \\ & + \sum F_n \exp \{ix\gamma + iy\delta\} + \int_0^\infty [\phi] \frac{\partial G^1}{\partial y'} dx', y' = 0. \end{aligned}$$

An integral equation incorporating the condition that $\phi_y = 0$ on any plate may be had by differentiating (2.3) with respect to y and putting $y = 0$. As it has been noted in the past, to bring the operation $\partial/\partial y$ under the integral sign in (2.3) and to evaluate the resulting expression at $y = 0$ serves only to supply us with a kernel of the integral equation which is non-integrable. That is, since G^1 has a logarithmic singularity in the neighborhood of $x = x', y = y'$, the kernel will behave like $1/(x - x')^2$ in the neighborhood of this point. We can overcome this difficulty by noting that

$$\frac{\partial G^1}{\partial y'} = -\frac{\partial G^1}{\partial y}, \quad y' = 0,$$

as a result of the particular y and y' dependence of G^1 , and therefore (2.3) can be rewritten

$$\begin{aligned} \phi(x, y) = & A \exp \{ik(x \cos \theta + y \sin \theta)\} \\ & + B \exp \{ik[x \cos (2\alpha - \theta) + y \sin (2\alpha - \theta)]\} \\ & + \sum F_n \exp \{ix\gamma + iy\delta\} - \int_0^\infty [\phi] \frac{\partial}{\partial y} G^1(x, y, x', 0) dx'. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial \phi}{\partial y} = & Aik \sin \theta \exp \{ik(x \cos \theta + y \sin \theta)\} \\ & + Bik \sin (2\alpha - \theta) \exp \{ikx \cos (2\alpha - \theta) + iky \sin (2\alpha - \theta)\} \\ & + \sum F_n i\delta \exp \{ix\gamma + iy\delta\} - \int_0^\infty [\phi] \frac{\partial^2}{\partial y^2} G^1(x, y, x', 0) dx' \end{aligned}$$

for any point $P(x, y)$ not on the plates. But for any such point

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right\} G^1 = 0$$

and therefore

$$\begin{aligned} \frac{\partial \phi}{\partial y} = & Aik \sin \theta \exp \{ik(x \cos \theta + y \sin \theta)\} \\ & + Bik \sin (2\alpha - \theta) \exp \{ikx \cos (2\alpha - \theta) + iky \sin (2\alpha - \theta)\} \\ & + \sum F_n i \delta \exp \{ix\gamma + iy\delta\} + \left\{ \frac{\partial^2}{\partial x^2} + k^2 \right\} \int_0^\infty G^1(x, y, x', 0) [\phi] dx'. \end{aligned}$$

In this form, we may now apply the limit operation $y = 0$, provided we recognize that the operation $(\partial^2/\partial x^2 + k^2)$ may not be brought under the integral sign. We have then for $x > 0, y = 0$

$$\begin{aligned} (2.5) \quad & Aik \sin \theta \exp \{ikx \cos \theta\} + ikB \sin (2\alpha - \theta) \exp \{ikx \cos (2\alpha - \theta)\} \\ & + \sum F_n i \delta \exp \{ix\gamma\} + \left\{ \frac{\partial^2}{\partial x^2} + k^2 \right\} \int_0^\infty G^1(x, 0, x', 0) [\phi] dx' = 0. \end{aligned}$$

Upon evaluating the differential operator before the integral sign, we have for $x > 0, y = 0$

$$\begin{aligned} (2.6) \quad 0 = & \frac{Ai \exp \{ikx \cos \theta\}}{k \sin \theta} + \frac{Bi \exp \{ikx \cos (2\alpha - \theta)\}}{k \sin (2\alpha - \theta)} \\ & + c_1 \exp \{ikx\} + c_2 \exp \{-ikx\} + \sum iF_n \delta \frac{\exp \{ix\gamma\}}{(k^2 - \gamma^2)} \\ & + \int_0^\infty [\phi] G^1(x, 0, x', 0) dx' \end{aligned}$$

where now c_1 and c_2 are arbitrary constants. For the time being the possibility of $\sin \theta, \sin (2\alpha - \theta)$ or the δ 's vanishing is excluded. Let us observe that if we write

$$\frac{Ai}{k \sin \theta} = \bar{A}, \quad \frac{Bi}{k \sin (2\alpha - \theta)} = \bar{B}, \quad \frac{i\delta F_n}{(k^2 - \gamma^2)} = \bar{F}_n,$$

Eq. (2.6) takes the form

$$\begin{aligned} 0 = & \bar{A} \exp \{ikx \cos \theta\} + \bar{B} \exp \{ikx \cos (2\alpha - \theta)\} \\ & + \sum \bar{F}_n \exp \{ix\gamma\} + c_1 \exp \{ikx\} + c_2 \exp \{-ikx\} \\ & + \int_0^\infty [\phi] G^1(x, 0, x', 0) dx'. \end{aligned}$$

Save for the two additional terms, this last equation has the same general form that we encountered in the problem involving Dirichlet conditions in [1] and is one of the Wiener-Hopf type.

3. The Fourier transform solution of Equation (2.6). In order to solve Eq. (2.6) we perform the same mutilation of functions which we did in [1]. We write Eq. (2.6) as

$$(3.1) \quad \bar{\phi}_2(x) = \bar{\phi}_1(x) + \int_0^\infty [\phi] G^1(x, 0, x', 0) dx'$$

where now

$$\begin{aligned} \bar{\phi}_2(x) &\equiv 0, & x > 0 \\ [\phi] &\equiv 0, & x < 0 \end{aligned}$$

and

$$\begin{aligned} \bar{\phi}_1(x) &\equiv \bar{A} \exp \{ikx \cos \theta\} + \bar{B} \exp \{ikx \cos (2\alpha - \theta)\} \\ &\quad + \sum \bar{F}_n \exp \{ix\gamma\} + c_1 \exp \{ikx\} + c_2 \exp \{-ikx\}, & x > 0 \\ &\equiv 0, & x < 0. \end{aligned}$$

The integrability of $\bar{\phi}_2$ and $[\phi]$ for $x \rightarrow 0$ is guaranteed by the results of [2]. Furthermore, for $x \rightarrow \infty$, $[\phi]$ is asymptotic to $e^{-\epsilon kx}$, while $\bar{\phi}_2(x) = O(e^{\epsilon x})$, $\epsilon > 0$, for $x \rightarrow -\infty$. Hence if we write

$$\Phi(w) = \int_0^\infty e^{-iwx} [\phi] dx,$$

$\Phi(w)$ will be analytic in the lower half plane $\text{Im } w < 0$. Similarly

$$\bar{\Phi}_1(w) = \int_0^\infty e^{-iwx} \bar{\phi}_1(x) dx$$

will be analytic in the lower half plane $\text{Im } w < 0$, while

$$\Phi_2(w) = \int_{-\infty}^0 e^{-iwx} \phi_2(x) dx$$

will be analytic in the upper half plane $\text{Im } w > -\epsilon$. The bilateral Fourier transform of $G^1(x, 0, 0, 0)$ which we denoted by $K(w)$ in [1] is analytic in the strip $-\epsilon < \text{Im } w < 0$. Hence, as we saw there, it is now possible to write Eq. (3.1) equivalently as a function of w in the following form:

$$(3.2) \quad \bar{\Phi}_1(w) = \bar{\Phi}_2(w) - K(w)\Phi(w).$$

The rearrangement of Eq. (3.2) so that we may determine $\Phi(w)$ is almost identical to the procedure we employed in [1]. The major point of difference occurs in the factoring of

$$\begin{aligned} \frac{\sin(b\sigma)}{b\sigma} &= \prod_1^\infty \left[1 - \frac{(k^2 - w^2)b^2}{n^2\pi^2} \right] \\ &= \prod_1^\infty \left[\sqrt{1 - \frac{k^2b^2}{n^2\pi^2}} + \frac{iwb}{n\pi} \right] e^{-iwb/n\pi} \prod_1^\infty \left[\sqrt{1 - \frac{k^2b^2}{n^2\pi^2}} - \frac{iwb}{n\pi} \right] e^{iwb/n\pi}. \end{aligned}$$

This factoring differs from the situation in [1] because of the restriction that $0 < kb < \pi$. That is, only one mode can propagate in the parallel plate regions.* This serves to change the lower index of the product from 2 to 1 since now all factors in the last expression are complex. Indeed, the first product is free of zeros in the lower half plane $\text{Im } w < \sqrt{(\pi^2/b^2) - k^2}$, while the second is free of zeros in the upper half plane $\text{Im } w > -\sqrt{(\pi^2/b^2) - k^2}$. Hence we write $K(w) = \bar{K}_-(w)/\bar{K}_+(w)$ where $\bar{K}_-$ and $1/\bar{K}_+(w)$ are those factors of $K(w)$ which are regular in the half planes $\text{Im } w < 0$ and $\text{Im } w > -\epsilon$ respectively. (The bar henceforth denotes the fact that there are some slight changes from the case we had in [1]). The factoring of

$$M(w) = 2[\cos \sigma b - \cos(k\rho - wa)]$$

is the same, and we simply use the same notation as in our previous paper, that is, $M(w) = M_-(w)/M_+(w)$ where $M_-(w)$ is free of zeros for $\text{Im } w < 0$, while $1/M_+(w)$ is free of zeros in the half plane $\text{Im } w > -\epsilon$, ($\epsilon < 0$). On the other hand,

$$L(w) = \frac{\bar{L}_-(w)}{\bar{L}_+(w)}$$

where now

$$\bar{L}_-(w) = \prod \left[\sqrt{1 - \frac{k^2 b^2}{n^2 \pi^2}} + \frac{iwb}{n\pi} \right] e^{-iwb/n\pi}$$

and $\bar{L}_-(w) = \bar{L}_+(-w)$. This then enables us to write

$$\bar{K}_-(w) = \bar{L}_-(w) \frac{\exp \{\bar{\chi}(w)\}}{M_-(w)}$$

and

$$\frac{1}{\bar{K}_+(w)} = \bar{L}_+(w) \frac{\exp \{-\bar{\chi}(w)\}}{M_+(w)}$$

where $\bar{\chi}(w)$ is the $\chi(w)$ from [1] modified as a result of the change of a factor in $\bar{K}_-(w)$. In this case $\bar{K}_-(w) = O(w^{-\frac{1}{2}})$, $|w| \rightarrow \infty$, $\text{Im } w < 0$ if there are no diffracted waves, and in general $\bar{K}_-(w) = O(w^{-p-\frac{1}{2}})$ in the same half plane if there are p diffracted waves present. Correspondingly, $\bar{K}_+(w) = O(w^{-\frac{1}{2}})$, $|w| \rightarrow \infty$, $\text{Im } w > -\epsilon$ ($\epsilon > 0$) in the case that there are no diffracted waves, and in general $\bar{K}_+(w)$ is asymptotic to $w^{-p-\frac{1}{2}}$ for $|w| \rightarrow \infty$, $\text{Im } w > -\epsilon$ if there are p diffracted waves present. The argument we used in [1] now gives us

$$\begin{aligned} \bar{K}_-(w)\Phi(w) &= \frac{\bar{A}\bar{K}_+(\sigma_1)}{i(w - \sigma_1)} + \frac{\bar{B}\bar{K}_+(\sigma_2)}{i(w - \sigma_2)} + \sum \frac{\bar{F}_n\bar{K}_+(\gamma)}{i(w - \gamma)} \\ &\quad + \frac{c_1\bar{K}_+(k)}{i(w - k)} + \frac{c_2\bar{K}_+(-k)}{i(w + k)}. \end{aligned} \tag{3.3}$$

* If we had considered this wave-length range in [1], there would be no propagation in the parallel plate regions since κ would be imaginary.

One point to be noted is that $\Phi(w)$ violates the Riemann-Lebesgue lemma in that, according to (3.3), $\Phi(w) = O(w^{\frac{1}{2}})$, $|w| \rightarrow \infty$, $\text{Im } w < 0$, if there are no diffracted waves and $O(w^{p+\frac{1}{2}})$ if there are p diffracted waves. Hence, in order to satisfy the Riemann-Lebesgue lemma, we have the following linear relation between the coefficients $\bar{A}$, $\bar{B}$, $\dots$ for the case $p = 0$:

$$\bar{A}\bar{K}_+(\sigma_1) + \bar{B}\bar{K}_+(\sigma_2) + c_1\bar{K}_+(k) + c_2\bar{K}_+(-k) = 0$$

and

$$\bar{A}\sigma_1\bar{K}(\sigma_1) + \bar{B}\sigma_2\bar{K}(\sigma_2) + c_1k\bar{K}_+(k) - c_2k\bar{K}_+(-k) = 0.$$

In general, for p diffracted waves, we have

$$\begin{aligned} \bar{A}\bar{K}_+(\sigma_1)\sigma_1^j + \bar{B}\bar{K}_+(\sigma_2)\sigma_2^j + c_1k^j\bar{K}_+(k) + c_2(-k)^j\bar{K}_+(-k) \\ + \sum F_n\gamma^j\bar{K}_+(\gamma) = 0, \quad j = 0, \dots, p+1. \end{aligned}$$

We should explain the need for two linear relations between the coefficients A , B , $\dots$ for the case of specular reflection and, in general, $p+2$ for the case of p diffracted waves. In [1], the discontinuity in the magnetic field involved the x components of the magnetic field. As such, as we have seen in [2], this discontinuity was $O(x^{-\frac{1}{2}})$, $x \rightarrow 0^+$ and therefore its Fourier transform was $O(w^{-\frac{1}{2}})$, $|w| \rightarrow \infty$, $\text{Im } w < 0$. In the case at hand, we are involved (in the electromagnetic case) with a situation for which $[\phi]$ is the discontinuity of the z component of the magnetic field on the plates and $[\phi]$ is $O(x^{\frac{1}{2}})$, $x \rightarrow 0^+$. Its Fourier transform is now $O(w^{-\frac{1}{2}})$, $|w| \rightarrow \infty$, $\text{Im } w < 0$, and hence we require the two linear relations to satisfy the Riemann-Lebesgue lemma for the case of specular reflection and the $p+2$ relations when p diffracted waves are present.

4. The determination of the far fields. Thus far we have found the Fourier transform of the discontinuity of the magnetic field (or the acoustic potential) on any sheet of the parallel plate medium. We are now confronted with the task of computing the far fields so that we can determine the reflection and transmission coefficients as we did in [1]. We also have to account for the two arbitrary constants c_1 and c_2 . To this end, we note that the integral in the representation (2.3) may be written as a contour integral by the Fourier transform theorem as

$$\begin{aligned} \phi(x, y) = & A \exp \{ik(x \cos \theta + y \sin \theta)\} \\ & + B \exp \{ik[x \cos (2\alpha - \theta) + y \cos (2\alpha - \theta)]\} \\ (4.1) \quad & + \sum F_n \exp \{ix\gamma + iy\delta\} \\ & + \frac{1}{2\pi} \int_P \frac{\exp \{iwx\} [\exp \{ik\rho - iwa\} \cos(\sigma y) - \cos\{\sigma(b-y)\}] \Phi(w) dw}{2 [\cos(k\rho - wa) - \cos(\sigma b)]} \end{aligned}$$

where P is a path in the strip $-\epsilon < \text{Im } w < 0$ and ϵ is the smaller non-vanishing Δ_n or Δ'_n . Let us note that since $\Phi(w)$ is analytic in the lower half plane $\text{Im } w < 0$ and is $O(w^{-\frac{1}{2}})$, $|w| \rightarrow \infty$ in this region, the path P can be closed in

the lower half plane if $x < 0$. Since the integrand in (4.1) has only poles in this half plane and these poles all possess negative imaginary parts, we shall see that the integral will only contribute terms which vanish for $x \rightarrow -\infty$. That is, in the (ξ, η) coordinate system we have a sum of terms of the type

$$\exp \left\{ \frac{\xi}{c} [(2n\pi + k\rho)^2 - c^2 k^2]^{\frac{1}{2}} + \frac{i\eta}{c} [2n\pi + k\rho] \right\}$$

for $(2n\pi + k\rho)^2 - c^2 k^2 > 0$, and this vanishes as $\xi \rightarrow -\infty$, that is, in free space.

$$\Phi(w) = O(w^{-\frac{3}{2}}), \quad |w| \rightarrow \infty, \quad \text{Im } w < 0$$

and therefore the coefficients of the series derived from the residue calculation are $O(n^{-\frac{3}{2}})$, $n \rightarrow \infty$. It follows that the series converges. Hence, for $\xi \rightarrow -\infty$, $\phi(x, y)$ is asymptotic to a finite series of bounded terms containing the incident field, the specularly reflected field and the propagating diffracted fields.

In order to study the case $x > a$, we observe that $K_-(w)$ can be replaced by

$$\bar{K}_-(w) = \frac{\bar{K}_+(w) \sin(\sigma b)}{2\sigma [\cos(k\rho - wa) - \cos(\sigma b)]}.$$

Since we now have to close the path P in the upper half plane, this relation will relieve us of the burden of calculating the residues of the term $1/\bar{K}_-(w)$ and shift this part of the problem on to the function $\sigma/\sin(\sigma b)$ and the simple real poles at $w = \pm k$, $w = \cos \theta$, $w = \cos(2\alpha - \theta)$ and $w = \gamma_n$. The integrand in (4.1) may now be written

$$\frac{\exp\{iwx\} [\exp\{ik\rho - iwa\} \cos(\sigma y) - \cos\{\sigma(b - y)\}]}{\bar{K}_+(w) \sin \sigma b} \\ \times \left[\frac{\bar{A}\bar{K}_+(\sigma_1)}{i(w - \sigma_1)} + \frac{\bar{B}\bar{K}_+(\sigma_2)}{i(w - \sigma_2)} + \sum \frac{\bar{F}_n\bar{K}_+(\gamma)}{i(w - \gamma)} + \frac{c_1\bar{K}_+(k)}{i(w - k)} + \frac{c_2\bar{K}_+(-k)}{i(w + k)} \right].$$

Since $\bar{K}_+(w)$ is analytic in the upper half plane, contributions will arise from the poles $w = k \cos \theta = \sigma_1$, $k \cos(2\alpha - \theta) = \sigma_2$, γ_n and $\pm k$ as well as the zeros of $\sin(\sigma b)/\sigma$. In the first place, a direct calculation will show us that the contributions from the poles σ_1 , σ_2 and γ_n will serve to annihilate the terms outside of the integrand. From the poles $w = \pm k$, we get the contribution

$$\frac{1}{b} \{c_1 [\exp\{ik\rho - ika\} - 1] \exp\{ikx\} + c_2 [\exp\{ik\rho + ika\} - 1] \\ \times \exp\{-ikx\}\}$$

while from the zeros of $\sin(b\sigma)$ we obtain terms

$$O\left(\exp\left\{-x\sqrt{\frac{n^2\pi^2}{b^2} - k^2}\right\}\cos\left[\frac{n\pi y}{b}\right]\right), \quad x \rightarrow \infty, \quad n \geq 1,$$

that is, the attenuated parallel plate modes. Hence the amplitude of the parallel-plate mode travelling to the right is

$$\frac{c_1}{b} [\exp \{ik\rho - ika\} - 1] = C_1$$

while the one travelling to the left is

$$\frac{c_2}{b} [\exp \{ik\rho + ika\} - 1] = C_2.$$

These last two relations plus the $p + 2$ linear relations we found between the coefficients $A, B, \dots$ will provide us with the various reflection and transmission coefficients.

5. The reflection and transmission coefficients. In order to define the various reflection and transmission coefficients of the propagating waves, we write an energy conservation relation as in [1]. The one change which now occurs is that $\phi(x, y)$ for $x \rightarrow \infty, 0 \leq y \leq b$ has the form

$$\phi(x, y) = C_1 e^{ikx} + C_2 e^{-ikx}.$$

If we now apply the method of Section 6, [1] to the case at hand, we find that $c[k|A|^2 \sin(\alpha - \theta) - k|B|^2 \sin(\alpha - \theta) - \sum w|F_n|^2] = bk[|C_1|^2 - |C_2|^2]$. In particular, if $C_2 = 0$, that is, $c_2 = 0$, we have the ratio $|B/A|$ is the magnitude of the reflection coefficient of the specularly reflected wave. Further,

$$\left| \frac{w_n^{\frac{1}{2}} F_n}{A k^{\frac{1}{2}} \sin^{\frac{1}{2}}(\alpha - \theta)} \right|,$$

where

$$cw_n = [c^2 k^2 - (2n\pi + k\rho)^2]^{\frac{1}{2}} > 0,$$

is the magnitude of the reflection coefficient for the propagating diffracted waves, while

$$\left| \frac{C_1 \sin^{\frac{1}{2}} \alpha}{A \sin^{\frac{1}{2}}(\alpha - \theta)} \right|$$

is the magnitude of the transmission coefficient in any duct. Similarly if $A = 0$, that is, the ducts have been excited, the ratio $|C_1/C_2|$ is the reflection coefficient in any duct, while

$$\left| \frac{B \sin^{\frac{1}{2}}(\alpha - \theta) \csc^{\frac{1}{2}} \alpha}{C_2} \right|$$

and

$$\left| \frac{w_n^{\frac{1}{2}} F_n \csc^{\frac{1}{2}} \alpha}{C_2 k^{\frac{1}{2}}} \right|$$

are the transmission coefficients of the plane waves in free space.

Clearly, then, these various coefficients may be determined once we know the ratios $|B/A|$, $|F_n/A|$ and $|C_1/A|$ for $C_2 = 0$ and $|C_1/C_2|$, $|B/C_2|$ and $|F_n/C_2|$

for $A = 0$. We are in a position to derive this information since, as noted in Section 3, we had linear relations of the form

$$\begin{aligned} \bar{A}\sigma_1^n\bar{K}_+(\sigma_1) + \bar{B}\sigma_2^n\bar{K}_+(\sigma_2) + \sum_i \bar{F}_i\gamma_i^n\bar{K}_+(\gamma_i) \\ + c_1k^n\bar{K}_+(k) + c_2(-k)^n\bar{K}_+(-k) = 0 \end{aligned}$$

for $n = 0, \dots, p+1$. Here p is the number of propagating diffracted waves. This system of linear equations involves $p+4$ unknowns with $p+2$ equations. For the two major cases of interest, either $\bar{A} = 0$ or $c_2 = 0$. In these cases we have $p+3$ unknowns with $p+2$ equations and we can solve for the desired ratios, since the determinant of the system is of the van der Monde type and does not vanish. We shall give details for the case $c_2 = 0$. We have

$$\begin{aligned} \frac{\bar{B}}{\bar{A}} &= -\frac{(k-\sigma_1)}{(k-\sigma_2)} \frac{\bar{K}_+(\sigma_1)}{\bar{K}_+(\sigma_2)} \prod_i \frac{\gamma_i - \sigma_1}{\gamma_i - \sigma_2}, \quad i \neq 0, \quad \gamma_0 = \sigma_2, \\ \frac{\bar{F}_n}{\bar{A}} &= -\frac{(k-\sigma_1)}{(k-\gamma_n)} \frac{\bar{K}_+(\sigma_1)}{\bar{K}_+(\gamma_n)} \prod_i \frac{\gamma_i - \sigma_1}{\gamma_i - \gamma_n}, \quad i \neq n, \end{aligned}$$

and

$$\frac{c_1}{\bar{A}} = -\frac{\bar{K}_+(\sigma_1)}{\bar{K}_+(k)} \prod_i \frac{\gamma_i - \sigma_1}{\gamma_i - k}.$$

Hence the magnitude of the reflection coefficient of the specularly reflected wave is

$$|R_0|^2 = \tan^2(\tfrac{1}{2}\theta) \cot^2(\alpha - \tfrac{1}{2}\theta) \prod_i \frac{\gamma_i - \sigma_1}{\gamma_i - \sigma_2} \frac{\bar{\gamma}_i - \sigma_2}{\bar{\gamma}_i - \sigma_1}, \quad i \neq 0.$$

Here

$$c\gamma_i = -(\sin \alpha)[c^2k^2 - (2i\pi + k\rho)^2]^{\frac{1}{2}} + (\cos \alpha)[2i\pi + k\rho]$$

and

$$c\bar{\gamma}_i = (\sin \alpha)[c^2k^2 - (2i\pi + k\rho)^2]^{\frac{1}{2}} + (\cos \alpha)[2i\pi + k\rho].$$

The index i in the product is taken over all positive and negative integers for which the γ 's are real. Further, the $|R_n|^2$, the squares of the magnitudes of the reflection coefficients of the diffracted waves, are

$$\begin{aligned} |R_n|^2 &= \frac{4kw_n(k+\gamma_n)\sin^2\alpha\sin(\alpha-\theta)}{(k-\gamma_n)(\gamma_n-\sigma_2)(\bar{\gamma}_n-\sigma_1)} \tan^2(\tfrac{1}{2}\theta) \\ &\quad \cdot \prod_i \frac{\gamma_i - \sigma_1}{\gamma_i - \gamma_n} \frac{\bar{\gamma}_i - \gamma_n}{\bar{\gamma}_i - \sigma_1}, \quad i \neq 0, n. \end{aligned}$$

$|T|^2$, the square of the magnitude of the transmission coefficient in any duct, is

$$|T|^2 = \frac{4\sin\alpha\sin(\alpha-\theta)}{[1-\cos(2\alpha-\theta)][1+\cos\theta]} \prod_i \frac{\gamma_i - \sigma_1}{\gamma_i - k} \frac{\bar{\gamma}_i - k}{\bar{\gamma}_i - \sigma_1}.$$

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