Variation of Flow Quantities Along Streamlines and Their Principal Normals and Binormals in Three-Dimensional Gas Flows

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Introduction. In this paper* we discuss three-dimensional steady rotational gas flows of non-viscous and non-heat-conducting fluids. The main purpose is to investigate various dynamical and kinematical relations connecting the flow quantities with the geometrical parameters of the streamline trajectories. The expressions for the tangent, principal normal and binormal vectors and the curvature and torsion of the streamlines are given in terms of the velocity components, pressure and density. The variation of pressure along the streamlines and their principal normals and binormals is found. It is observed that the pressure along the binormals remains constant. Furthermore the relations connecting the vorticity components with the gradient of entropy are investigated. Finally the vorticity components are found in terms of the curvature of the streamlines and the velocity gradients along the streamlines and their principal normals and binormals. In plane gas flows these results reduce to well known results of CROCCO and THOMAS.

1. Equations of motion. We consider the equations of motion governing the flow of a stationary ideal gas with viscosity and thermal conductivity zero, namely [1]

$$(1) \quad p_{,i} + \rho u_i u_{i,i} = 0 \quad (\text{equations of motion}),$$

$$(2) \quad \rho_{,i} u_i + \rho u_{i,i} = 0 \quad (\text{equation of continuity}),$$

in which p , ρ and u_i denote the pressure, density and velocity components respectively. We assume the motion referred to a system of rectangular coordinates x^i ; then comma in the above equations and in the following equations represents

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partial differentiation. It is to be understood in the above and in the following discussion that an index which occurs twice in a term is to be summed over the admissible values of the index. Since there is no distinction between covariant and contravariant indices within a rectangular system we may write any index as a superscript or a subscript without modification of the term in which the index occurs.

In addition there is the equation which states that the entropy remains constant along streamlines. This condition when combined with the equations (1) results in the relation

$$(3) \quad \frac{\gamma p}{\rho(\gamma - 1)} + \frac{1}{2} u_i u_i = H$$

where γ is the ratio of two specific heats c_p and c_v (assumed constant) and H is constant along streamlines but can vary from streamline to streamline (weak Bernoulli equation). Conversely the condition that S , the entropy, be constant along streamlines can be recovered from (1) and (3). Hence the general equations determining the motion of the gas under consideration are given completely by (1), (2) and (3). In the following discussion we shall assume that H is an absolute constant, *i.e.* its value is independent of the streamline (strong Bernoulli condition). It can be assumed moreover that the values of the constants γ and H are known from the conditions of the problem. Then the number of equations in the set (1), (2) and (3) is exactly equal to the number of functions u_i , p and ρ to be determined.

By means of (3) the density ρ and its derivatives $\rho_{,i}$ can be eliminated from (1) and (2). The resulting equations can be written in the form

$$(4) \quad c^2 p_{,i} + \gamma p u_{i,j} u_{i,j} = 0,$$

$$(5) \quad p_{,i} u_i + \gamma p u_{i,j} = 0,$$

where $c^2 = \gamma p / \rho$ is the adiabatic speed of sound in the gas. From (3) the quantity c^2 is given explicitly in terms of the velocity components u_i by

$$(6) \quad c^2 = \frac{\gamma - 1}{2} (2H - u_i u_i).$$

Conversely from (4), (5) and (6) the equations (1), (2) and (3) can be recovered.

If we multiply (5) by c^2 , equations (4) by u_i , and subtract the corresponding members of the equations we obtain

$$(7) \quad u_{i,j} u_i u_j = c^2 u_{k,k}.$$

This relation expresses the condition that the entropy is constant along streamlines. This assumption also implies that we have the equation of state [1]

$$(8) \quad p = N \rho^\gamma,$$

$$(9) \quad N = e^{S/Jc_v},$$

in which J is the mechanical equivalent of heat. S and therefore N remain constant along the streamlines.

2. Variation of pressure along and perpendicular to the streamlines. The differential equations for a streamline can be expressed by the proportionality

$$(10) \quad dx_1 : dx_2 : dx_3 = u_1 : u_2 : u_3 .$$

Let λ^i , μ^i and ν^i denote respectively the components of the unit tangent, principal normal and binormal vectors to the streamline at a point. We then have

$$(11) \quad \frac{dx^i}{ds} = \lambda^i = \frac{u_i}{q} ,$$

where $q^2 = u_i u_i$ and s denotes arc length measured along the streamline in the direction of the flow. Hence

$$(12) \quad \frac{d^2 x^i}{ds^2} = \frac{u_{i,j} \lambda_j}{q} - \frac{u_i}{q^2} \frac{dq}{ds} = \frac{u_{i,j} \lambda_j}{q} - \frac{u_i u_j u_{l,j} \lambda_l}{q^3} .$$

But from one of the Frenet formulas we have [3]

$$(13) \quad \frac{d^2 x^i}{ds^2} = \frac{d\lambda^i}{ds} = K \mu^i ,$$

where K is the curvature of the streamline. Thus the relation (12) becomes

$$(14) \quad K \mu^i = \frac{u_{i,j} u_j}{q^2} - \frac{u_i u_j u_{l,j} \lambda_l}{q^3} .$$

Using (1) the equations (14) reduce to

$$(15) \quad \mu^i = -\frac{1}{K} \left(\frac{p_{,i}}{\rho q^2} + \frac{u_i u_j u_{l,j} \lambda_l}{q^3} \right) .$$

If we multiply both sides of (15) by λ_i , μ_i and ν_i in turn and use equation (7) we obtain

$$(16) \quad \frac{1}{\rho q} p_{,i} \lambda_i = -\frac{c^2}{q^2} u_{k,k} = -\frac{u_{k,k}}{M^2} ,$$

$$(17) \quad p_{,i} \mu_i = -K \rho q^2 ,$$

$$(18) \quad p_{,i} \nu_i = 0 ,$$

where $M = q/c$ is the Mach number. From the last result we observe that *the pressure remains constant along the binormals to the streamlines*.

Futhermore from the relations (11) and (15) we obtain the components of the binormal vector, namely

$$(19) \quad \nu_i = e_{ijk} \lambda_j \mu_k = -\frac{1}{\rho K q^3} e_{ijk} u_j p_{,k} ,$$

where the quantities e_{ijk} are the components of the permutation tensor with the property that

$$(20) \quad \begin{cases} (i) & e_{ijk} = 0 \quad \text{if any two of the indices } i, j \text{ and } k \text{ are equal,} \\ (ii) & e_{123} = e_{231} = e_{312} = +1, \\ (iii) & e_{132} = e_{321} = e_{213} = -1. \end{cases}$$

differentiating both sides of (19), using the Frenet formula

$$\frac{dv^i}{ds} = -\tau \mu^i$$

and the equations (1), (2) and (7) we find the relation connecting the torsion τ of the streamlines with the parameters of the flow, *viz.*

$$(21) \quad \tau \mu^i = -\frac{e_{ijk}}{\rho K^2 q^6} [u_i p_{,k} \{3Kc^2 u_{l,i} + q^2 K_{,l} u_l - K q^2 u_{l,i}\} - K q^2 u_{,i} u_l p_{,kl}].$$

3. Vorticity components. Equations (3) and (8) can be written in the form

$$(3) \quad \frac{2\gamma}{\gamma - 1} \frac{p}{\rho} = (2H - q^2),$$

$$(8) \quad p = N \rho^\gamma.$$

Solution of these equations for p and ρ yields

$$(22) \quad p = \left(\frac{\gamma - 1}{2\gamma} \right)^{\gamma/(\gamma-1)} \frac{(2H - q^2)^{\gamma/(\gamma-1)}}{N^{1/(\gamma-1)}},$$

$$(23) \quad \rho = \left(\frac{\gamma - 1}{2\gamma} \right)^{1/(\gamma-1)} \frac{(2H - q^2)^{1/(\gamma-1)}}{N^{1/(\gamma-1)}}.$$

Differentiating both sides of the relation (22) with respect to x^i and simplifying we get

$$(24) \quad p_{,i} = -\rho u_i u_{i,i} - \frac{p}{\gamma - 1} \frac{N_{,i}}{N},$$

which with the help of (9) becomes

$$(25) \quad p_{,i} = -\rho u_i u_{i,i} - \frac{p}{Jc_v(\gamma - 1)} S_{,i}.$$

Also from the equations of motion we have

$$(1) \quad p_{,i} = -\rho u_i u_{i,i}.$$

Subtracting the corresponding sides of the above two relations we find

$$(26) \quad u_i(u_{i,i} - u_{i,i}) = \frac{p}{Jc_v(\gamma - 1)\rho} S_{,i}.$$

Now the vorticity components w_k are defined as

$$(27) \quad w_k = e_{klm} u_{m,l} .$$

Multiplying both sides of this equation by e^{ijk} and summing on the index k we obtain

$$(28) \quad u_{i,j} - u_{j,i} = -e_{ijk} w_k .$$

With the help of this relation the equation (26) becomes

$$(29) \quad e_{ijk} w_k u_j = -\frac{p}{Jc_v(\gamma - 1)\rho} S_{,i} .$$

By using the result (11) the above equations give

$$(30) \quad e_{ijk} w_k \lambda_j = -\frac{p}{Jc_v(\gamma - 1)\rho q} S_{,i} .$$

If we multiply both sides of (30) by λ_i , μ_i and ν_i in turn and use the skew symmetry of e_{ijk} and the identities [4]

$$(31) \quad \begin{aligned} e_{ijk} \mu_j \nu_k &= \lambda_i ; & e_{ijk} \lambda_j \mu_k &= -\mu_i , \\ e_{ijk} \lambda_j \mu_k &= \nu_i ; & e_{ijk} \lambda_i \mu_j \nu_k &= 1 , \end{aligned}$$

which state that λ , μ and ν form a triad of positively oriented mutually orthogonal unit vectors, we obtain

$$(32) \quad \begin{aligned} \frac{dS}{ds} &= 0 ; & w_i \mu_i &= -\frac{p}{Jc_v(\gamma - 1)} \frac{dS}{dn_2} , \\ w_i \nu_i &= -\frac{p}{Jc_v(\gamma - 1)} \frac{dS}{dn_1} , \end{aligned}$$

where d/ds , d/dn_1 and d/dn_2 denote differentiation along the streamline, the principal normal and the binormal respectively. The first of the above results is merely another version of the result (7).

Furthermore the substitution of the relation (11) into (27) yields

$$(33) \quad w_i = q e_{ijk} \lambda_{k,i} + e_{ijk} \frac{\lambda_k}{q} u_l u_{l,i} .$$

Multiplying both sides of this equation by λ_i , μ_i and ν_i in turn, using the skew symmetry of e_{ijk} and the identities (31) we get

$$(34) \quad w_i \lambda_i = q e_{ijk} \lambda_{k,i} \lambda_i ,$$

$$(35) \quad w_i \mu_i = q e_{ijk} \lambda_{k,i} \mu_i + \frac{u_l u_{l,i} \nu_i}{q} ,$$

$$(36) \quad w_i \nu_i = q e_{ijk} \lambda_{k,i} \nu_i - \frac{u_l u_{l,i} \mu_i}{q} .$$

Now the gradient of λ_k in the x^i -coordinate system can be written as the vector sum of its gradients in the system of axes determined by λ , μ and ν as

$$(37) \quad \lambda_{k,i} = \frac{d\lambda_k}{ds} \lambda_i + \frac{d\lambda_k}{dn_1} \mu_i + \frac{d\lambda_k}{dn_2} \nu_i .$$

In order to evaluate the quantities occurring on the right hand side of the above equation consider the congruence determined by (i) the principal normals and (ii) the binormals to the streamlines in such a way that the former have μ as the tangent vector, λ as the principal normal vector and ν as the binormal vector, while the latter have ν as the tangent vector, λ as the principal normal vector and μ as the binormal vector. Let K' and σ' denote the curvature and torsion of the curves in the congruence determined by the principal normals and K'' and σ'' the curvature and torsion of the curves in the congruence determined by the binormals to the streamlines. The Frenet formulas then give

$$(38) \quad \begin{aligned} \frac{d\lambda_k}{ds} &= K\mu_k , \\ \frac{d\lambda_k}{dn_1} &= -K'\mu_k + \sigma'\nu_k , \\ \frac{d\lambda_k}{dn_2} &= \sigma''\mu_k - K''\nu_k . \end{aligned}$$

From (37) and (38) we have

$$(39) \quad \lambda_{k,i} = K\mu_k\lambda_i + (-K'\mu_k + \sigma'\nu_k)\mu_i + (\sigma''\mu_k - K''\nu_k)\nu_i .$$

If we multiply both sides of (39) by $e_{ijk}\lambda_i$, $e_{ijk}\mu_i$ and $e_{ijk}\nu_i$ in turn and make use of the identities (31) and the skew symmetry of e_{ijk} , we get

$$(40) \quad e_{ijk}\lambda_{k,i}\lambda_i = \sigma' - \sigma''; \quad e_{ijk}\lambda_{k,i}\mu_i = 0, \quad e_{ijk}\lambda_{k,i}\nu_i = K.$$

Substituting the above results in the relations (34) to (36) we have

$$(41) \quad w_i\lambda_i = q(\sigma' - \sigma''); \quad w_i\mu_i = \frac{dq}{dn_2}, \quad w_i\nu_i = qK - \frac{dq}{dn_1}.$$

If we solve the above equations we arrive at the explicit determination of the vorticity components as

$$(42) \quad w_i = q(\sigma' - \sigma'')\lambda_i + \frac{dq}{dn_2} \mu_i + q\left(K - \frac{dq}{dn_1}\right)\nu_i .$$

Futhermore, if we combine the results (32) and (41) we find

$$(43) \quad w_i\mu_i = \frac{dq}{dn_2} = -\frac{p}{R\rho q} \frac{dS}{dn_2} ,$$

$$(44) \quad w_i \nu_i = qK - \frac{dq}{dn_1} = -\frac{p}{R\rho q} \frac{dS}{dn_1},$$

where $R = Jc_v(\gamma - 1)$ is an absolute constant of the flow.

In two-dimensional gas flows we have only one component of vorticity, viz. $w = w_i \nu_i$ where ν_i are the components of the unit normal to the plane of the flow. In this case the result (44) reduces to the well known result of Crocco [5], i.e.

$$(45) \quad w = qK - \frac{dq}{dn_1} = -\frac{p}{R\rho q} \frac{dS}{dn_1},$$

where the quantity $(dS/dn_1)/R\rho q$ is constant along the streamlines. This is not the case for three-dimensional gas flows. In the axially symmetric flows also there is only one component of vorticity at every point in the space and it is given by equation (44), but the ν_i in this case reduce to the components of the unit normal to the meridian plane through the point. This has been pointed out by VAZSONYI [6]. However CROCCO and VAZSONYI find this result by introducing the stream function. Thereby the results in these two special cases take respectively the forms

$$(46) \quad w = \frac{p}{R} \frac{dS}{d\psi},$$

$$(47) \quad w = \frac{p\varpi}{R} \frac{dS}{d\psi},$$

where ψ is the stream function and ϖ denotes the distance from the axis of symmetry.

Another result may be noticed from the equation (39) by contracting the indices j and k , namely

$$(48) \quad \lambda_{k,k} = -(K' + K'').$$

Differentiating (11) and using (7) and (48) we get

$$(49) \quad \frac{u_{k,k}}{q} \left(\frac{M^2 - 1}{M^2} \right) = -(K' + K'').$$

Finally the substitution of the relation (49) into the equation (16) yields

$$(50) \quad \frac{1}{\rho q^2} \frac{dp}{ds} = (K' + K'') \frac{1}{M^2 - 1},$$

which in plane flows reduces to the result found by THOMAS [2].

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