On the Propagation and Decay of Spherical Blast Waves

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1. Introduction. The following article¹ contains the results of the application of our recent work on *compatibility conditions*² to the problem of the propagation of spherical blast waves. As customary in treating a problem of this character it is assumed that the gas is devoid of viscosity and thermal conductivity; since the flow is spherically symmetric it is therefore governed by the equations

$$(1.1) \quad \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial r} + \rho v \frac{\partial v}{\partial r} = 0 \quad (\text{equation of motion}),$$

$$(1.2) \quad \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial v}{\partial r} + \frac{2\rho v}{r} = 0 \quad (\text{equation of continuity}),$$

$$(1.3) \quad \frac{\partial p}{\partial t} + v \frac{\partial p}{\partial r} + \gamma p \frac{\partial v}{\partial r} + \frac{2\gamma p v}{r} = 0 \quad (\text{entropy condition}),$$

where r denotes the distance from the center of symmetry O and p , ρ , v and t are, of course, the pressure, density, velocity and time with p , ρ and v functions of r and t . The first two equations (1.1) and (1.2) require no additional comment. Equation (1.3), in conjunction with (1.2), expresses the requirement that the quantity p/ρ^γ , where γ is the gas constant, remains unchanged following the motion of the fluid particles.

In this problem we are concerned with a spherical region, having its center at the point O , which is expanding into a gas at rest. Within this sphere the equations (1.1), (1.2) and (1.3) are satisfied. The boundary $r = R(t)$ of the sphere is a surface of discontinuity or *shock* where abrupt changes in the quantities p , ρ and v occur as determined by the well known shock conditions. Denoting by p_0 and ρ_0 the values of the pressure and density in the undisturbed gas and by $\bar{p}$, $\bar{\rho}$ and $\bar{v}$ the values of the pressure, density and velocity immediately behind the moving shock front we can express the shock conditions in the form³

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² *Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics*, Jour. Math. and Mech. **6** (1957) pp. 311-322.

³ See, T. Y. Thomas, *The fundamental hydrodynamical equations and shock conditions for gases*, Math. Mag. **22** (1949) pp. 169-189. Also, *The distribution of singular shock directions*, Jour. of Math. and Physics **28** (1949) p. 164.

$$(1.4) \quad [v] = \frac{2(G^2 - c_0^2)}{(\gamma + 1)G},$$

$$(1.5) \quad [p] = \frac{2\rho_0(G^2 - c_0^2)}{\gamma + 1},$$

$$(1.6) \quad [\rho] = \frac{2\rho_0(G^2 - c_0^2)}{2c_0^2 + (\gamma - 1)G^2},$$

where $c_0^2 = \gamma p_0/\rho_0$ is the square of the velocity of sound in the undisturbed gas, $G = dR/dt$ is the velocity of the wave in the radial or normal direction, and

$$[v] = \bar{v}; \quad [p] = \bar{p} - p_0; \quad [\rho] = \bar{\rho} - \rho_0.$$

The bracket [] operation will later be applied to derivatives of the quantities p , ρ and v in a similar manner.

It will be convenient, on occasion, to have the following equivalent forms of the relations (1.5) and (1.6), namely

$$(1.7) \quad \bar{p} = \frac{2\rho_0}{\gamma + 1} \left(G^2 - \frac{\gamma - 1}{2\gamma} c_0^2 \right),$$

$$(1.8) \quad \bar{\rho} = \frac{(\gamma + 1)\rho_0 G^2}{2c_0^2 + (\gamma - 1)G^2}.$$

The special form of the compatibility conditions of the first and second orders, appropriate to the problem under consideration, is given in §2. Now the equations (1.4), (1.7) and (1.8) express the quantities $\bar{v}$, $\bar{p}$ and $\bar{\rho}$ as simple rational functions of the wave velocity G . By application of the compatibility conditions of the first order it is shown in §3 that the first derivatives of the functions p , ρ and v , when evaluated behind the shock, are given by expressions linear in dG/dt with coefficients which are rational functions of G and R . Use of the compatibility conditions of the second order in §4 shows that the second derivatives of the quantities p , ρ and v behind the shock are uniquely determined by expressions which are linear in the second derivatives d^2G/dt^2 and quadratic in the first derivatives dG/dt and that the coefficients of these expressions are rational functions of the velocity G and the radius R . It is evident that this procedure can be continued to give a corresponding determination of the derivatives of p , ρ and v of any order behind the shock in terms of the radius R , the velocity G and its time derivatives.

Since the determination of the above derivatives of p , ρ and v exhausts all conditions at our disposal the velocity G and its various time derivatives will be unrestricted at any particular instant and hence G or R can be taken to be an arbitrary differentiable or analytic function of the time t . It is clear therefore that the solution of the problem of the determination of the velocity of propagation of a blast wave can not be attained unless we impose some requirement in addition to the equations (1.1), (1.2), (1.3) and the shock conditions (1.4),

(1.5) and (1.6). This requirement must express a characteristic property of the blast phenomenon.

After the velocity $G(t)$ has been found one can consider the matter of the determination of the flow within the sphere of radius $R(t)$ by formal computation. But we shall not be concerned with this latter problem.

We have attempted to solve the problem of the propagation of a blast wave by the introduction of an *energy hypothesis* in §5. We believe that this hypothesis is well suited to express, at least to a first approximation, a basic property of the energy behavior within a blast caused by an *atomic or thermonuclear explosion* which is our main interest in this investigation. For small values of the radius R , *i.e.* for very strong shocks, the propagation resulting from this hypothesis resembles that obtained by the so called similarity assumption used by G. I. TAYLOR.⁴ Moreover as $R \rightarrow \infty$ the shock strength approaches zero and the velocity of propagation of the shock tends to the value c_0 , *i.e.* the velocity of sound in still air. In §7 we have derived formulæ giving the pressure increment $[p]$ and the pressure ratio $\bar{p}/p_0$ in terms of the radius $R(t)$ of the blast. An equation expressing the relation between the radius $R(t)$ and the time t is given in §8.

Because of the above mentioned character of this solution for small and large values of the shock radius R we would anticipate good agreement between calculated and observed results throughout the entire course of the propagation of the blast wave. However calculations based on our formulæ have not been made since accurate data, necessary for the proper comparison of theory and experiment, is not readily available due to present security regulations.

2. Compatibility conditions. From equations (14) and (21) of the previous paper on compatibility conditions² we now obtain relations of the form

$$(2.1) \quad \left[\frac{\partial v}{\partial r} \right] = \lambda; \quad \left[\frac{\partial v}{\partial t} \right] = -\lambda G + \frac{\delta k}{\delta t},$$

$$(2.2) \quad \left[\frac{\partial p}{\partial r} \right] = \xi; \quad \left[\frac{\partial p}{\partial t} \right] = -\xi G + \frac{\delta m}{\delta t},$$

$$(2.3) \quad \left[\frac{\partial \rho}{\partial r} \right] = \zeta; \quad \left[\frac{\partial \rho}{\partial t} \right] = -\zeta G + \frac{\delta n}{\delta t},$$

where λ , ξ and ζ are functions defined over the spherical wave surface $r = R(t)$, also denoted by $\Sigma(t)$ for brevity, and the quantities k , m and n whose δ time derivatives occur in these equations are defined by

$$(2.4) \quad k = [v]; \quad m = [p]; \quad n = [\rho].$$

⁴ SIR GEOFFREY TAYLOR, F.R.S., *The formation of a blast wave by a very intense explosion I. Theoretical discussion*, Proc. Roy. Soc. London, Ser. A, 201 (1950) pp. 159–174. Also, *The formation of a blast wave by a very intense explosion II. The atomic explosion of 1954*, *ibid.*, pp. 175–186.

Equations (2.1), (2.2) and (2.3) are the compatibility conditions of the first order. Similarly the compatibility conditions of the second order are given by the equations (28), (40) and (41) of the previous paper. For the spherically symmetric problem under consideration we thus obtain

$$(2.5) \quad \left[\frac{\partial^2 v}{\partial r^2} \right] = \hat{\lambda}; \quad \left[\frac{\partial^2 v}{\partial r \partial t} \right] = -\hat{\lambda}G + \frac{\delta \lambda}{\delta t},$$

$$(2.6) \quad \left[\frac{\partial^2 v}{\partial t^2} \right] = \hat{\lambda}G^2 - 2G \frac{\delta \lambda}{\delta t} - \lambda \frac{\delta G}{\delta t} + \frac{\delta^2 k}{\delta t^2},$$

$$(2.7) \quad \left[\frac{\partial^2 p}{\partial r^2} \right] = \hat{\xi}; \quad \left[\frac{\partial^2 p}{\partial r \partial t} \right] = -\hat{\xi}G + \frac{\delta \xi}{\delta t},$$

$$(2.8) \quad \left[\frac{\partial^2 p}{\partial t^2} \right] = \hat{\xi}G^2 - 2G \frac{\delta \xi}{\delta t} - \xi \frac{\delta G}{\delta t} + \frac{\delta^2 m}{\delta t^2},$$

$$(2.9) \quad \left[\frac{\partial^2 \rho}{\partial r^2} \right] = \hat{\zeta}; \quad \left[\frac{\partial^2 \rho}{\partial r \partial t} \right] = -\hat{\zeta}G + \frac{\delta \zeta}{\delta t},$$

$$(2.10) \quad \left[\frac{\partial^2 \rho}{\partial t^2} \right] = \hat{\zeta}G^2 - 2G \frac{\delta \zeta}{\delta t} - \zeta \frac{\delta G}{\delta t} + \frac{\delta^2 n}{\delta t^2},$$

where the quantities $\hat{\lambda}$, $\hat{\xi}$ and $\hat{\zeta}$ are additional functions defined over $\Sigma(t)$. The δ time derivative of any quantity which is considered to be expressed on $\Sigma(t)$ as a function of the time t alone is identical with the ordinary time derivative of the quantity; this follows immediately from the definition of the δ time derivative in the previous paper. Thus the derivative $\delta G/\delta t$ in the above equations is equal to the ordinary time derivative of the velocity $G(t)$.

Since the pressure and density are constant and the velocity is zero for the undisturbed gas on the outer side of the wave surface $\Sigma(t)$ the bracket expressions in the above relations are identical with the corresponding derivatives on the rear of the shock. This circumstance will of course simplify the applications of these formulæ in the following sections.

3. First derivatives behind the shock surface. Evaluating the equations (1.1), (1.2) and (1.3) on the inner boundary of $\Sigma(t)$ and using the conditions (2.1), (2.2) and (2.3) we find

$$(3.1) \quad \xi = \bar{p}(G - \bar{v})\lambda - \bar{p} \frac{\delta k}{\delta t},$$

$$(3.2) \quad (G - \bar{v})\zeta = \bar{p}\lambda + \frac{\delta n}{\delta t} + \frac{2\bar{p}\bar{v}}{R},$$

$$(3.3) \quad \gamma\bar{p}\lambda = (G - \bar{v})\xi - \frac{\delta m}{\delta t} - \frac{2\gamma\bar{p}\bar{v}}{R}.$$

From (3.1) and (3.3) we can now eliminate ξ and hence obtain an expression for λ . Then ξ can be found from (3.1) and finally the quantity ζ can be deter-

mined from (3.2). The solution of the equations (3.1), (3.2) and (3.3) for λ , ξ and ζ can be simplified somewhat by application of the usual form of the first shock condition; this becomes

$$\bar{\rho}(G - \bar{v}) = \rho_0 G$$

for the propagation into a gas at rest which is under consideration. Thus we obtain

$$(\bar{\rho}^2 \bar{c}^2 - \rho_0^2 G^2) \lambda = -\rho_0 \bar{\rho} G \frac{\delta k}{\delta t} - \bar{\rho} \frac{\delta m}{\delta t} - \left(\frac{2\gamma}{R} \right) \bar{\rho} \bar{v},$$

$$(\bar{\rho}^2 \bar{c}^2 - \rho_0^2 G^2) \xi = -\bar{\rho}^3 \bar{c}^2 \frac{\delta k}{\delta t} - \rho_0 \bar{\rho} G \frac{\delta m}{\delta t} - \left(\frac{2\gamma \rho_0}{R} \right) G \bar{\rho} \bar{v},$$

$$(\bar{\rho}^2 \bar{c}^2 - \rho_0^2 G^2) \left(\frac{\rho_0}{\bar{\rho}} \right) G \zeta = -\rho_0 \bar{\rho}^2 G \frac{\delta k}{\delta t} - \bar{\rho}^2 \frac{\delta m}{\delta t} + (\bar{\rho}^2 \bar{c}^2 - \rho_0^2 G^2) \frac{\delta n}{\delta t} - \frac{2\rho_0^2 G}{R} \bar{\rho} \bar{v},$$

where $\bar{c}^2 = \gamma \bar{p} / \bar{\rho}$ is the square of the velocity of sound on the inner boundary of the shock surface $\Sigma(t)$.

Now by differentiation of (1.4), (1.5) and (1.6) the δ time derivatives of k , m and n are found to be given by

$$(3.4) \quad \frac{\delta k}{\delta t} = \frac{2}{\gamma + 1} \left(1 + \frac{c_0^2}{G^2} \right) \frac{dG}{dt}; \quad \frac{\delta m}{\delta t} = \frac{4\rho_0 G}{\gamma + 1} \frac{dG}{dt},$$

$$(3.5) \quad \frac{\delta n}{\delta t} = \frac{4(\gamma + 1)\rho_0 c_0^2 G}{(2c_0^2 + (\gamma - 1)G^2)^2} \frac{dG}{dt}.$$

Moreover, when use is made of (1.7) and (1.8), we readily find that

$$(3.6) \quad \bar{\rho}^2 \bar{c}^2 - c_0^2 G^2 = \rho_0 \bar{\rho} (G^2 - c_0^2).$$

Making the substitutions (3.4), (3.5) and (3.6) in the above equations for λ , ξ and ζ the resulting equations can be given the following form

$$(G^2 - c_0^2) \lambda = -\frac{2G}{\gamma + 1} \left(3 + \frac{c_0^2}{G^2} \right) \frac{dG}{dt} - \left(\frac{2\gamma}{\rho_0 R} \right) \bar{\rho} \bar{v},$$

$$(G^2 - c_0^2) \xi = -\frac{2\rho_0 G^2}{\gamma + 1} \left\{ 3 + \frac{c_0^2}{G^2} + \frac{\bar{\rho}}{\rho_0} \left(1 - \frac{c_0^4}{G^4} \right) \right\} \frac{dG}{dt} - \left(\frac{2\gamma G}{R} \right) \bar{\rho} \bar{v},$$

$$(G^2 - c_0^2) \zeta = -\frac{2\rho_0}{\gamma + 1} \left(\frac{\bar{\rho}}{\rho_0} \right)^2 \left\{ 3 + \left(2 \frac{\gamma + 2}{\gamma + 1} \frac{c_0^2}{G^2} - \frac{\gamma + 3}{\gamma + 1} \right) \frac{\bar{\rho}}{\rho_0} \frac{c_0^2}{G^2} \right\} \frac{dG}{dt} - \left(\frac{2G}{R} \right) \bar{\rho} \bar{v}.$$

But the values of λ , ξ and ζ given by these relations are equal to the values of the derivatives of v , p and ρ with respect to r immediately behind the shock front. Likewise from the above expressions for λ , ξ and ζ and the second relations in (2.1), (2.2) and (2.3) we obtain the formulæ for the time derivatives of v , p and ρ behind the shock. Eliminating $\bar{v}$, $\bar{p}$ and $\bar{\rho}$ by means of the shock conditions we see that the derivatives of v , p and ρ are given by linear expressions in dG/dt

with coefficients which (apart from the occurrence of the radius R in certain terms) are rational in the wave velocity G .

4. Determination of second and higher derivatives. If we differentiate the equations (1.1), (1.2) and (1.3) with respect to r , evaluate behind the shock surface $\Sigma(t)$, and then make use of the conditions (2.5), $\dots$, (2.10) we obtain relations of the form

$$\begin{aligned}\hat{\xi} &= \bar{p}(G - \bar{v})\hat{\lambda} + \hat{A}, \\ (G - \bar{v})\hat{\xi} &= \bar{p}\hat{\lambda} + \hat{B}, \\ \gamma\bar{p}\hat{\lambda} &= (G - \bar{v})\hat{\xi} + \hat{C},\end{aligned}$$

where $\hat{A}$, $\hat{B}$ and $\hat{C}$ represent quantities which can be expressed in terms of the velocity G , its first and second time derivatives, and the radius $R(t)$. We observe that the coefficients of $\hat{\lambda}$, $\hat{\xi}$ and $\hat{\xi}$ in these relations are identical with the corresponding coefficients in (3.1), (3.2) and (3.3). Hence solving, as above, we find

$$\begin{aligned}(G^2 - c_0^2)\hat{\lambda} &= \frac{G\hat{A}}{\bar{p}} + \frac{\hat{C}}{\rho_0}, \\ (G^2 - c_0^2)\hat{\xi} &= \left\{ \left(1 + \frac{\rho_0}{\bar{p}} \right) \frac{G^2}{c_0^2} - 1 \right\} c_0^2 \hat{A} + G\hat{C}, \\ (G^2 - c_0^2)\hat{\xi} &= \frac{\bar{p}}{\rho_0} \hat{A} + \frac{\bar{p}}{\rho_0} \left(1 - \frac{c_0^2}{G^2} \right) G\hat{B} + \left(\frac{\bar{p}}{\rho_0} \right)^2 \frac{\hat{C}}{G}.\end{aligned}$$

Using these relations and the conditions (2.5), $\dots$, (2.10) we can proceed to the determination of all second derivatives of v , p and ρ behind the shock front. Examination of the structure of the quantities $\hat{A}$, $\hat{B}$ and $\hat{C}$ shows that the expressions for the second derivatives are linear in d^2G/dt^2 and quadratic in dG/dt , and that the coefficients of these expressions are rational functions of the velocity G and the radius R .

Differentiation of the equations (1.1), (1.2) and (1.3) with respect to the time will lead to no new conditions on derivatives immediately behind the shock front. This could of course be shown by direct computation. However such computation can be avoided by the following consideration. Denote by J the expression in the left member of one of the equations (1.1), (1.2) or (1.3) and let $\bar{J}$ denote the evaluation of J immediately behind the shock. Then $\bar{J} = 0$ at all times from the conditions in §3. Hence

$$(4.1) \quad \frac{\delta \bar{J}}{\delta t} = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial r} G = 0,$$

where the derivatives $\partial J/\partial t$ and $\partial J/\partial r$ are evaluated just behind the shock. But the derivative $\partial J/\partial r = 0$ in (4.1) from the condition used above to determine second derivatives. Hence $\partial J/\partial t = 0$ from the identity (4.1) and hence

no new relations, immediately behind the shock front, can be obtained by differentiating (1.1), (1.2) and (1.3) with respect to the time.

The above procedure can evidently be continued with the result that the derivatives of p , ρ and v of any order will be determined behind the shock if the radius R , the velocity G and its successive time derivatives are known. It is therefore possible to choose $R(t)$ or $G(t)$ to be an arbitrary differentiable or analytic function of the time t since, after the determination of the above derivatives of p , ρ and v , there are no further conditions at our disposal. Hence, as mentioned in the introduction, the determination of the velocity of propagation of the wave will require an additional assumption which must evidently express some characteristic property of the blast phenomenon. This matter is considered in the following section.

5. Energy hypothesis. For the ideal gas under consideration the energy E per unit mass is given by the formula

$$(5.1) \quad E = \frac{1}{2}v^2 + \frac{p}{(\gamma - 1)\rho} + \text{const.},$$

where the constant in this equation depends on the selection of the zero point of energy and must evidently fail to appear in our final relations. On this account it is permissible to write the above equation without this constant term as is done by most writers.

Denote by Q the energy released by the explosion.⁵ Then the energy within the expanding sphere of radius $R(t)$ and volume $V(t)$ will thus consist of the energy Q plus the energy derived from the undisturbed gas which is continually being drawn into the sphere. Hence if E_0 is the energy per unit mass of the undisturbed gas we can write

$$(5.2) \quad \int_{V(t)} E \rho \, dV = Q + \int_{V(t)} E_0 \rho_0 \, dV = Q + \frac{4}{3} \pi E_0 \rho_0 R^3.$$

Let us assume that the actual mass added to the spherical region of radius $R(t)$ by the explosion is either zero or negligible. Then it follows that

$$(5.3) \quad \int_{V(t)} \rho \, dV = \int_{V(t)} \rho_0 \, dV = \frac{4}{3} \pi \rho_0 R^3.$$

Now if we introduce the energy E_0 as a factor in the equation (5.3) we have

$$(5.4) \quad \int_{V(t)} E_0 \rho \, dV = \int_0^{R(t)} 4\pi E_0 \rho r^2 \, dr = \frac{4}{3} \pi E_0 \rho_0 R^3,$$

where the right member gives the energy within the sphere which arises from the undisturbed gas. In this connection the above equation (5.4) can be inter-

⁵ More precisely Q is the part of the total energy released by the explosion which is available for mechanical work. Energy not considered in this theory, such as that producing radiation, must be disregarded in determining the value of Q .

puted as follows. *The energy, derived from the undisturbed gas, within a spherical shell of radius r and thickness dr is given by $4\pi E_0 \rho r^2 dr$ where ρ is the density of the gas in the shell.*

Now consider the spherical wave surface of radius $R(t)$ at time t and suppose that the radius increases by an amount ΔR in the time Δt . Let ΔQ be the energy, derived solely from the energy released by the explosion, which is in the spherical shell of thickness ΔR at the time $t + \Delta t$. Concerning this energy ΔQ we shall make the following

Energy hypothesis. The energy ΔQ is (a) proportional to the total energy Q released by the explosion, (b) proportional to the volume ΔV of the above spherical shell, and (c) inversely proportional to the volume $V(t)$ of the sphere of radius $R(t)$. The last part of this hypothesis expresses the condition that as the average concentration of the energy Q decreases, due to increasing volume $V(t)$, the contribution of this energy to the energy in the spherical shell will decrease in a corresponding manner.

On the basis of the energy hypothesis we have

$$(5.5) \quad \Delta Q = \frac{\alpha Q \Delta V}{V(t)} = \frac{\alpha Q \cdot 4\pi R^2 \Delta R}{4\pi R^3/3},$$

where the proportionality factor α will depend on the specific gas under consideration, *e.g.* the gas constant γ . Now the energy in the above spherical shell will consist of the energy ΔQ and the energy contributed by the undisturbed gas. But this latter energy is

$$(5.6) \quad 4\pi E_0 \bar{\rho} R^2 \Delta R$$

by the first italicized statement in this section. Hence, adding (5.6) to the expression for ΔQ given by (5.5), we see that the total energy in the spherical shell has the value

$$(5.7) \quad \frac{\alpha Q \cdot 4\pi R^2 \Delta R}{4\pi R^3/3} + 4\pi E_0 \bar{\rho} R^2 \Delta R.$$

The energy in the spherical shell can also be expressed by

$$(5.8) \quad 4\pi \bar{E} \bar{\rho} R^2 \Delta R,$$

since $\bar{E}$ represents the energy per unit mass and $\bar{\rho}$ is the density of the gas in the shell. Equating the expressions (5.7) and (5.8) we therefore have

$$(5.9) \quad [E] = \frac{3\alpha Q}{4\pi R^3 \bar{\rho}}.$$

Now the shock conditions imply that³

$$[E] = \frac{\bar{p}\bar{v}}{\rho_0 G},$$

and hence from (5.9) we have

$$(5.10) \quad \frac{\bar{p}\bar{v}}{\rho_0 G} = \frac{3\alpha Q}{4\pi R^3 \bar{p}}.$$

This relation leads to an equation giving an algebraic determination of the pressure increment $[p]$ or ratio $\bar{p}/p_0$ in terms of the radius $R(t)$ as shown in §7. It also leads, as shown in §8, to a differential equation of the first order for the determination of the radius $R(t)$ as a function of the time t .

Remark 1. The right member of (5.9) represents an energy per unit mass and this energy evidently results exclusively from the energy released by the explosion. Hence the condition given by the equation (5.9) can be expressed by saying that, immediately behind the shock, the total energy $\bar{E}$ per unit mass is equal to the energy E_0 per unit mass of the undisturbed gas plus the energy per unit mass due to the explosion.

Remark 2. If we replace the quantities E and E_0 in (5.2) by their values as given by (5.1) and there differentiate both members of the equation with respect to the time t we have

$$(5.11) \quad \frac{d}{dt} \int_0^{R(t)} \left(\frac{1}{2} \rho v^2 + \frac{p}{\gamma - 1} + e\rho \right) r^2 dr = \left(\frac{p_0}{\gamma - 1} + e\rho_0 \right) R^2 G,$$

where we have put $G = dR/dt$ in the right member and have used e to denote the constant in (5.1). Performing the indicated differentiation in the left member of (5.11) we now obtain

$$(5.12) \quad \int_0^{R(t)} \left(\frac{1}{2} \frac{\partial \rho v^2}{\partial t} + \frac{1}{\gamma - 1} \frac{\partial p}{\partial t} + e \frac{\partial \rho}{\partial t} \right) r^2 dr + \left(\frac{1}{2} \bar{\rho} \bar{v}^2 + \frac{\bar{p}}{\gamma - 1} + e\bar{\rho} \right) R^2 G.$$

When we substitute for the time derivatives and rearrange the terms in the resulting expression the integral in (5.12) becomes

$$(5.13) \quad \begin{cases} -\frac{1}{2} \int_0^R \frac{\partial}{\partial r} (\rho v^3) r^2 dr - \frac{\gamma}{\gamma - 1} \int_0^R \frac{\partial}{\partial r} (p v) r^2 dr - \int_0^R \rho v^3 r dr \\ -\frac{2\gamma}{\gamma - 1} \int_0^R p v r dr - e \int_0^R \frac{\partial}{\partial r} (\rho v) r^2 dr - 2e \int_0^R \rho v r dr. \end{cases}$$

Hence integrating by parts and combining with the second term in (5.12) we have

$$(5.14) \quad -\frac{1}{2} \bar{\rho} \bar{v}^3 R^2 - \frac{\gamma}{\gamma - 1} \bar{p} \bar{v} R^2 - e \bar{\rho} \bar{v} R^2 + \left(\frac{1}{2} \bar{\rho} \bar{v}^2 + \frac{\bar{p}}{\gamma - 1} + e\bar{\rho} \right) R^2 G.$$

The expression (5.14) must now be equated to the right member of (5.11). Doing this and dividing through by R^2 we obtain an equation which can be written in the following form

$$(5.15) \quad \frac{1}{2} \rho_0 G \bar{v}^2 + \frac{[p]}{\gamma - 1} G - \frac{\gamma}{\gamma - 1} \bar{p} \bar{v} = 0.$$

But when we eliminate the quantities $\bar{v}$, $[p]$ and $\bar{p}$ in (5.15) by means of the substitutions (1.4), (1.5) and (1.7) we find that the equation is satisfied identically. *We have thus given a direct proof of the fact that the two integrals in (5.2) must differ by a constant.* This constant of course is the energy Q released by the explosion in accordance with the equation (5.2).

6. Determination of the proportionality factor. Eliminate the quantities $\bar{v}$, $\bar{p}$ and $\bar{\rho}$ from the relation (5.10) by means of (1.4), (1.7) and (1.8). This gives

an equation which can be written in the form

$$(G^2 - c_0^2)^2 + \left\{ \frac{\gamma + 1}{2\gamma} c_0^2 - \frac{3(\gamma^2 - 1)\alpha Q}{16\pi\rho_0 R^3} \right\} (G^2 - c_0^2) = \frac{3(\gamma + 1)^2 \alpha Q c_0^2}{16\pi\rho_0 R^3}.$$

Solving this quadratic equation in $G^2 - c_0^2$ we find that

$$(6.1) \quad G^2 - c_0^2 = \frac{(\gamma + 1)c_0^2}{4\gamma} \left\{ 2(\gamma - 1) \left(\frac{a}{R^3} \right) - 1 \right. \\ \left. + \sqrt{4(\gamma - 1)^2 \left(\frac{a}{R^3} \right)^2 + 4(3\gamma + 1) \left(\frac{a}{R^3} \right) + 1} \right\},$$

where

$$(6.2) \quad a = \frac{3\alpha Q}{16\pi\rho_0}.$$

In order to deduce a value for the constant α let us assume the ideal situation for which the energy Q is released instantaneously by the explosion at the point O or center of the expanding shock wave. Let us also write the equation (5.9) in the form

$$\alpha = \frac{\bar{E} V \bar{\rho}}{Q} - \frac{E_0 V \bar{\rho}}{Q}, \quad V = \frac{4}{3} \pi R^3,$$

and let us consider this equation in the limit as V or R approaches zero, *i.e.*

$$(6.3) \quad \alpha = \lim_{R \rightarrow 0} \left(\frac{\bar{E} V \bar{\rho}}{Q} \right) - \lim_{R \rightarrow 0} \left(\frac{E_0 V \bar{\rho}}{Q} \right).$$

Now from the equation (6.1) it is seen immediately that $G \rightarrow \infty$ as $R \rightarrow 0$. Using this fact it follows from (1.8) that

$$(6.4) \quad \lim_{R \rightarrow 0} \frac{\bar{\rho}}{\rho_0} = \frac{\gamma + 1}{\gamma - 1}.$$

Hence the limiting value of $\bar{\rho}$ is finite and hence the second limit in (6.3) must vanish. Equation (6.3) can now be written

$$(6.5) \quad \alpha = \lim_{R \rightarrow 0} \left(\frac{\bar{E} V \rho_0}{Q} \right) \lim_{R \rightarrow 0} \left(\frac{\bar{\rho}}{\rho_0} \right) = \frac{\gamma + 1}{\gamma - 1} \lim_{R \rightarrow 0} \left(\frac{\bar{E} V \rho_0}{Q} \right).$$

when use is made of (6.4).

Let us assume furthermore that the nature of the explosion is such that the energy released can be regarded, with sufficient accuracy, as distributed uniformly at the instant at which the explosion takes place. But for the ideal case under consideration this means precisely that $\bar{E} V \rho_0 \rightarrow Q$ as $R \rightarrow 0$ since $V \rho_0$ is the mass in the volume V and the energy in V , other than that due to the explosion, will vanish in the limit as R approaches zero. Hence it follows from (6.5) that

$$(6.6) \quad \alpha = \frac{\gamma + 1}{\gamma - 1}.$$

7. Pressure equation. Using (6.6) the relation (6.2) becomes

$$(7.1) \quad a = \left(\frac{\gamma + 1}{\gamma - 1} \right) A, \quad A = \frac{3Q}{16\pi p_0}.$$

Hence equation (6.1) can be written

$$(7.2) \quad G^2 - c_0^2 = \frac{(\gamma + 1)c_0^2}{4\gamma} \left\{ 2(\gamma + 1) \left(\frac{A}{R^3} \right) - 1 + \sqrt{4(\gamma + 1)^2 \left(\frac{A}{R^3} \right)^2 + 4 \frac{(3\gamma + 1)(\gamma + 1)}{\gamma - 1} \left(\frac{A}{R^3} \right) + 1} \right\}.$$

The equation (7.2) can be used to determine the pressure increment $[p]$ in terms of the radius $R(t)$ of the spherical blast wave. In fact if we eliminate the quantity $G^2 - c_0^2$ from (7.2) by means of (1.5) we obtain

$$(7.3) \quad [p] = p_0 \left\{ -\frac{1}{2} + (\gamma + 1) \left(\frac{A}{R^3} \right) + \sqrt{(\gamma + 1)^2 \left(\frac{A}{R^3} \right)^2 + \frac{(3\gamma + 1)(\gamma + 1)}{\gamma - 1} \left(\frac{A}{R^3} \right) + \frac{1}{4}} \right\}.$$

Dividing both members of (7.3) by p_0 we are led to an equivalent equation for the determination of the pressure ratio $\bar{p}/p_0$, namely

$$(7.4) \quad \frac{\bar{p}}{p_0} = \frac{1}{2} + (\gamma + 1) \left(\frac{A}{R^3} \right) + \sqrt{(\gamma + 1)^2 \left(\frac{A}{R^3} \right)^2 + \frac{(3\gamma + 1)(\gamma + 1)}{\gamma - 1} \left(\frac{A}{R^3} \right) + \frac{1}{4}}.$$

Remark 1. It follows immediately from (7.3) and (7.4) that $[p] \rightarrow 0$ and $\bar{p}/p_0 \rightarrow 1$ as $R \rightarrow \infty$. In other words the shock strength approaches zero, or the shock decays, as the radius of the spherical blast becomes indefinitely large.

Remark 2. Equation (7.2) can be used to give an algebraic determination of the shock velocity G in terms of the radius $R(t)$ of the spherical blast. Moreover $G \rightarrow c_0$ as $R \rightarrow \infty$, i.e. the shock velocity approaches the velocity of sound in the undisturbed gas as the radius of the blast increases indefinitely.

8. Determination of the shock radius. The relation (7.2) can be transformed readily into a differential equation for the determination of the radius $R(t)$ as a function of the time t . In the first instance this differential equation is given by

$$(8.1) \quad \frac{dR}{dt} = \frac{c_0}{2\sqrt{\gamma}} \cdot \sqrt{(3\gamma - 1) + 2(\gamma + 1) \left(\frac{A}{R^3} \right) + (\gamma + 1) \sqrt{4(\gamma + 1)^2 \left(\frac{A}{R^3} \right)^2 + 4 \frac{(3\gamma + 1)(\gamma + 1)}{\gamma - 1} \left(\frac{A}{R^3} \right) + 1}},$$

as follows from (7.2) by putting $G = dR/dt$ and solving for the derivative of R .

Let us now write

$$(8.2) \quad B = \sqrt[3]{A} = \sqrt[3]{\frac{3Q}{16\pi p_0}},$$

and let us furthermore make the change of variable from R to $x = B/R$ in (8.1). Then the solution of (8.1) such that $R = 0$ when $t = 0$ is given by the equation

$$(8.3) \quad t = \frac{2\sqrt{\gamma}B}{c_0} W\left(\frac{B}{R}, \gamma\right),$$

where the function $W(x, \gamma)$ is defined by the integral

$$(8.4) \quad W(x, \gamma) = \int_x^\infty \frac{dx}{\phi(x, \gamma)}$$

in which

$$(8.5) \quad \phi(x, \gamma) = x^2 \sqrt{(3\gamma - 1) + 2(\gamma + 1)^2 x^3 + (\gamma + 1) \sqrt{4(\gamma + 1)^2 x^6 + 4 \frac{(3\gamma + 1)(\gamma + 1)}{\gamma - 1} x^3 + 1}}.$$

If one knows the energy Q released by the explosion and also the atmospheric conditions prevailing at the time of the explosion, *e.g.* the constants p_0 and ρ_0 , then the equation (8.3) can be used to calculate the time required for the blast to move any given distance R . For this calculation it will be convenient to have a table of values of the function W computed for the appropriate value of the gas constant γ which, in the case of air, is usually taken to be 1.4.

Remark 1. For large values of R the quantity $x = B/R$ is small and the above function $\phi(x, \gamma)$ is given by $2\sqrt{\gamma}x^2$ approximately. Limiting ourselves to values of R for which this approximation is justified the solution of (8.1) such that $R = R_1$ for $t = t_1$ is determined by the equation

$$\int_{t_1}^t dt = \frac{-2\sqrt{\gamma}B}{c_0} \int_{B/R_1}^{B/R} \frac{dx}{2\sqrt{\gamma}x^2},$$

from which we find

$$(8.6) \quad R = c_0(t - t_1) + R_1,$$

for $t \geq t_1$. But (8.6) states that the spherical wave is expanding with the velocity c_0 . Hence we conclude that for R sufficiently large the velocity of propagation of a blast wave approximates the velocity of sound in the undisturbed gas (Cf. Remark 2 in §7).

Remark 2. If R is small the variable x is large and we see from (8.5) that

$$\phi(x, \gamma) = 2(\gamma + 1)x^{7/2},$$

approximately. Hence

$$W(x, \gamma) = \int_x^\infty \frac{dx}{2(\gamma + 1)x^{7/2}} = \frac{x^{-5/2}}{5(\gamma + 1)},$$

and hence from (8.3) we obtain

$$(8.7) \quad t = \frac{2\sqrt{\gamma}BR^{5/2}}{5(\gamma+1)c_0B^{5/2}}.$$

The equation (8.7) is seen to be equivalent to the relation

$$(8.8) \quad Q = \frac{64\pi}{75(\gamma+1)^2} \rho_0 R^5 t^{-2}.$$

It follows from (8.8) that the quantity $R^5 t^{-2}$ must be constant for sufficiently small values of the shock radius R . This result is obviously independent of the value of α given by (6.6) although the value of the energy Q will depend on the value of the constant α . If the above value of α is associated with energies Q not in accord with experiment the special assumptions of §6, used in the determination of α , should be re-examined.

Remark 3. For small values of R we can take

$$(8.9) \quad \frac{\bar{p}}{\rho_0} = \frac{\gamma+1}{\gamma-1},$$

approximately, in accordance with the equation (6.4). Hence the right member of (5.9), which we have seen to give the energy per unit mass due to the explosion immediately behind the shock, becomes

$$(8.10) \quad \frac{3\alpha Q}{4\pi R^3 \bar{p}} = \frac{3\alpha Q}{4\pi R^3 \rho_0} \frac{\rho_0}{\bar{p}} = \frac{3Q}{4\pi R^3 \rho_0}$$

when use is made of (6.6) and (8.9). But the right member of (8.10) is equal to the mean value of the energy per unit mass, released by the explosion, in the sphere of radius $R(t)$ since $4\pi R^3 \rho_0/3$ is the mass of the gas in this sphere. *Hence, considering only energy due to the explosion, the energy per unit mass behind the shock for small values of the radius R is approximately equal to the mean value of the energy per unit mass in the sphere of radius R .*

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