Note on the Preceding Two Papers

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References [1] and [2] were read at the IXth International Congress of Applied Mechanics, Brussels, 1956. The present texts (except for subsequent correction of a few minor errors) were submitted for publication in the Proceedings of the Congress. Due to the uncertainty of the publication date it was, however, decided to try to obtain a more speedy publication. In this note some comments are made about the connection between the results presented in [1] and [2] and some related work on asymptotic expansions.

The problem of asymptotic expansions of solutions of the Navier-Stokes equations and related equations has been studied at GALCIT over a period of years. The extension of earlier methods to the case of low Reynolds numbers rests on two ideas, both of which are due to S. KAPLUN: (1) The discovery of suitable inner and outer limits; (2) An extension of the technique of matching between various expansions.

The classical principles for the matching of boundary layer flow, outer flow and flow due to displacement thickness are contained in the following proposition: If first the inner limit and then the outer limit are applied to a function describing a physical quantity, such as the velocity, the result is the same as that obtained by reversing the order of the limits (*cf.* [3], p. 122). This proposition appears to be correct in a large number of cases, in particular if the non-uniformities are characterized by exponential decay. It is highly plausible, for example, for expansions for large Reynolds numbers of the type described in [4], p. 862 ff. and p. 870 ff. It is related to the assumption of a uniformly valid composite expansion of the form described in [4] pp. 863, 871 and in [5] pp. 128, 130. For some similar methods used in a problem in elasticity compare [6]. This restricted matching principle fails, however, for expansions for small Reynolds numbers. In the study of this case, a generalized matching principle was introduced, which in essence reduces to the assertion that the inner expansion of the outer expansion should, to appropriate orders, agree with the outer expansion of the inner expansion. Actually, this technique has been known to various investigators of asymptotic expansions. It was mentioned to the authors of [1] and [2] by C. C. LIN and it is described in [7] p. B-184 ff. However, KAPLUN arrived at this result independently, on the basis of a fundamental analysis of the reasons for matching. The important new concepts used in this analysis, such as extension principle, intermediate expansion, overlap domain, *etc.*, are described and illus-

trated in [1] and [2]. While these ideas lead to a solution of the expansion problem of Navier-Stokes solutions for low Reynolds numbers, the main importance of this work lies in the analysis of a basic problem in asymptotic expansions. It is believed that these ideas contain the essential principles for the study of a very large class of singular perturbation problems.

The two ideas stated at the beginning of this note were announced in [4] where various applications were given. They have subsequently been used in [8] in the determination of various terms in the expansions for flow past a sphere and a circular cylinder at low Reynolds numbers. These results overlap somewhat in [1] and [2]. An interesting new result given in [8] is the discovery of the fact that the function $\epsilon_2(\text{Re})$ (see [1]) is $\text{Re}^2 \log \text{Re}$ rather than Re^2 . The drag correction to this order was computed in [8]. In connection with [8] it should be noted that the authors refer to [4] for certain basic ideas but fail to mention that in [4] the two ideas mentioned above are clearly attributed to KAPLUN.

REFERENCES

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