*Low Reynolds Number Flow Past a Circular Cylinder**

SAUL KAPLUN

Communicated by P. A. LAGERSTROM

The ideas discussed in Ref. 1 are applied to the problem of finding a uniformly valid asymptotic expansion of the Navier-Stokes solution for incompressible stationary flow past a circular cylinder at low Reynolds numbers. Some additional remarks are made about the general two-dimensional problem. The following notation is used: L is a characteristic length of the object, in particular the circular cylinder is defined by

$$r^2 = x_1^2 + x_2^2 = \left(\frac{L}{2}\right)^2. \quad (1)$$

At infinity the velocity $\mathbf{q}$ is $U\mathbf{i}$ and the pressure p is p_∞ . The kinematic viscosity is ν . The following non-dimensional quantities are introduced

$$\text{Reynolds number} = \text{Re} = \frac{UL}{\nu}, \quad \mathbf{q}^* = \frac{\mathbf{q}}{U}, \quad (2a)$$

$$\tilde{x}_i = \frac{x_i U}{\nu}, \quad \tilde{r} = \frac{r U}{\nu} \quad (\text{Oseen variables}), \quad (2b)$$

$$x_i^* = \frac{x_i}{L} = \frac{\tilde{x}_i}{\text{Re}}, \quad r^* = \frac{r}{L} = \frac{\tilde{r}}{\text{Re}} \quad (\text{Stokes variables}), \quad (2c)$$

$$\nabla^* = \left(\frac{\partial}{\partial x_1^*}, \frac{\partial}{\partial x_2^*} \right), \quad \nabla \approx \left(\frac{\partial}{\partial \tilde{x}_1}, \frac{\partial}{\partial \tilde{x}_2} \right). \quad (2d)$$

Result. The velocity field has the following asymptotic representation uniformly valid for small Re :

$$\begin{aligned} \mathbf{q}^* &= \mathbf{S}_m + O(\epsilon^{m+1}), \\ \mathbf{S}_m &= \{\mathbf{i} + \epsilon[\mathbf{g}_1(\tilde{x}_i) + \mathbf{l}_1(x_i^*)]\} + \sum_{n=2}^m \epsilon^n \{\mathbf{k}_n(\tilde{x}_i) + a_n \epsilon[\mathbf{g}_1(\tilde{x}_i) + \mathbf{l}_1(x_i^*)]\} \end{aligned} \quad (3)$$

* Research supported by Office of Naval Research, N6 ONR-244, T. O. VIII, Project No. 041-156.

where

$$\mathbf{g}_1 = -2ie^{\frac{1}{2}\tilde{r}}K_0(\frac{1}{2}\tilde{r}) + 2\widetilde{\nabla}[e^{\frac{1}{2}\tilde{r}}K_0(\frac{1}{2}\tilde{r}) + \log \tilde{r}], \quad (4a)$$

$$\mathbf{g}_1 \sim i(\log \tilde{r} + \gamma - \log 4) - \frac{\tilde{x}}{\tilde{r}}\widetilde{\nabla}\tilde{r} + O(\tilde{r} \log \tilde{r}) \text{ (for small } \tilde{r}), \quad (4b)$$

$$\mathbf{l}_1 = -\frac{1}{8}\nabla^* \frac{x^*}{r^{*2}}, \quad (4c)$$

$$\epsilon = \left[\log \left(\frac{8}{\text{Re}} \right) + \frac{1}{2} - \gamma \right]^{-1}, \quad \gamma = \text{Euler's constant} \doteq 0.5772, \quad (5)$$

$$a_n \mathbf{i} = \mathbf{k}_n(0, 0). \quad (6)$$

The functions $\mathbf{k}_n(x_i)$ will be defined in the course of the derivation of (3).

The remainder term for the m^{th} partial sum $\mathbf{S}_m$ of (3), divided by ϵ^{m+1} , is uniformly bounded for the entire flow field as Re tends to zero. The 0^{th} partial sum (first curly bracket) remains valid to order ϵ . The entire series leaves a non-trivial remainder of the transcendental order Re . Note that $\text{Re} = o(\epsilon^n)$ for any n , *i.e.* Re is transcendentally small with respect to ϵ . This is evident from the relation $\text{Re} = \text{constant} \cdot e^{-1/\epsilon}$.

The construction of (1) is based on the principal expansions of $\mathbf{q}^*$ discussed in Ref. 1. The outer (or Oseen) expansion of $\mathbf{q}^*$ is

$$\mathbf{q}^* \sim \mathbf{i} + \sum_{n=1} \epsilon^n \mathbf{g}_n(x_i) \quad (7a)$$

where $\mathbf{g}_1$ is given by (4a) and

$$\mathbf{g}_2 = \mathbf{k}_2, \quad \mathbf{g}_n = \mathbf{k}_n + a_{n-1}\mathbf{g}_1 \quad \text{for } n > 2. \quad (7b)$$

The intermediate expansion is

$$\mathbf{q}^* \sim \sum_{n=0} \epsilon^n \mathbf{u}_n(x_i^*, \text{Re}) \quad (8a)$$

where the $\mathbf{u}_n$ are given by

$$\mathbf{u}_0 = \epsilon \mathbf{h}_1(x_i^*), \quad \mathbf{u}_1 = \mathbf{0}, \quad \mathbf{u}_n = a_n \mathbf{u}_0 \quad (n > 1) \quad (8b)$$

and

$$\mathbf{h}_1 = i(\log 2r^* + \frac{1}{2}) - \frac{x^*}{r^*} \nabla^* r^* - \frac{1}{8} \nabla^* \frac{x^*}{r^{*2}}. \quad (8c)$$

The domain in which the n^{th} partial sum of (8a) is uniformly valid to order ϵ^n is the convex set $\text{ord } \text{Re} \leq \text{ord } f < \text{ord } \epsilon^n$. The region of validity of the n^{th} sum of (7a), on the other hand, contains the set $\text{ord } \epsilon^n \leq \text{ord } f \leq \text{ord } 1$.

The inner (or Stokes) expansion of $\mathbf{u}_n$ is exactly

$$\mathbf{u}_n = \epsilon a_n \mathbf{h}_1(x_i^*). \quad (9)$$

The drag D may be computed from (9) and is given by

$$\frac{\text{Re } C_D}{8\pi} = \frac{D}{4\pi\mu U} = \epsilon(1 + \sum_{n=2} a_n \epsilon^n) \quad (10)$$

where the value of a_2 is found below as -0.87 , approximately.

Derivation of $\mathbf{u}_0$. The construction of (3) proceeds as follows. First, we note that $\lim_o \mathbf{q}^* = \mathbf{i}$. This limit is non-uniform near the body. Furthermore, $\lim_s \mathbf{q}^* = \mathbf{0}$; this value cannot be matched with the Oseen limit. We therefore seek a function $\mathbf{u}_0$, uniformly valid to order one for the intermediate domain, *i.e.* such that, given any $f(\text{Re})$ which tends to zero,

$$(\mathbf{q}^* - \mathbf{u}_0) \rightarrow \mathbf{0} \quad \text{uniformly for } \frac{1}{2}\text{Re} \leq \tilde{r} \leq f(\text{Re}). \quad (11a)$$

On the other hand, $\mathbf{i}$ is uniform in the sense

$$(\mathbf{q}^* - \mathbf{i}) \rightarrow \mathbf{0} \quad \text{uniformly for } A \leq \tilde{r} \quad (11b)$$

where A is any constant greater than zero. As discussed in Ref. 1, the domain of validity of $\mathbf{i}$ may therefore be extended so that it overlaps with the domain of validity of $\mathbf{u}_0$. Hence the matching condition is

$$\lim_f (\mathbf{i} - \mathbf{u}_0) = \mathbf{0}, \quad \text{for } \text{ord } f_0 < \text{ord } f < \text{ord } 1 \quad (12)$$

where $\lim_{f_0}$ is sufficiently slow so that $\text{ord } f_0$ is in the overlap domain.

If $\text{ord } \text{Re} \leq \text{ord } f < \text{ord } 1$ a formal application of $\lim_f$ to the Navier-Stokes equations gives the Stokes equations. Condition (11a) suggests therefore that $\mathbf{u}_0$ be a solution of the Stokes equations satisfying $\mathbf{u}_0 = \mathbf{0}$ at the body. We may furthermore tentatively restrict attention to solutions which grow as slowly as possible at infinity. Under these restrictions one finds that $\mathbf{u}_0$ is of the form $\mathbf{u}_0 = \epsilon(\text{Re})\mathbf{h}_1^\dagger$ where $\epsilon(\text{Re})$ is yet to be determined. The matching (12) then gives

$$\lim_{\text{Re} \rightarrow 0} (-\epsilon \log \text{Re}) = 1. \quad (13a)$$

Hence ϵ may be assumed to satisfy a relation

$$-\epsilon \log \text{Re} = 1 + b_1\epsilon + b_2\epsilon^2 + \dots \quad (13b)$$

where the b_n will be normalized later. Note that in this case the overlap domain is much smaller than that for the sphere. A necessary condition is that $\text{ord } \text{Re}^\alpha < \text{ord } f_0$ for any $\alpha > 0$. A sufficiently slow limit is obtained *e.g.* by taking $f = -1/\log \text{Re}$ in Eq. 12. It may be noted that for the sphere $\mathbf{u}_0$ is given by $\mathbf{h}_0$ while for the cylinder $\mathbf{h}_0 = \mathbf{0}$ and $\mathbf{u}_0 = \epsilon\mathbf{h}_1$ (or, more properly, $\mathbf{h}_0 + \epsilon\mathbf{h}_1$). The first term in the drag formula (10) is obtained from $\mathbf{u}_0$ (*cf.* Eq. 9).

Derivation of $\mathbf{g}_1$ and $\mathbf{u}_1$. For $n > 0$, if one seeks a uniform approximation to order ϵ^n , the transport term may be neglected in the domain $\text{ord } \text{Re} \leq \text{ord } f < \text{ord } \epsilon^n$. We therefore seek a sequence of functions $\mathbf{u}_0, \mathbf{u}_1, \dots$, where the $\mathbf{u}_n$ are

$\dagger$ or a transformation of $\epsilon(\text{Re}) \mathbf{h}_1$ obtained by rotation of the flow field about the origin.

solutions of the homogeneous Stokes equations, such that the n^{th} sum of the series $\sum \epsilon^n \mathbf{u}_n$ is uniformly valid to order ϵ^n in $\text{ord Re} \leq \text{ord } f < \text{ord } \epsilon^n$. Such a series is called an "intermediate expansion" and is given by (8). (Note, in general, by an *intermediate expansion* we mean a series uniformly valid in a prescribed domain, cf. e.g. Ref. 1, Eq. 21 *et seq.*, in contrast to a *limit-process expansion*, whose domain of validity cannot be prescribed *a priori*.)

In principle, one would need a second intermediate expansion for the domain $\text{ord } \epsilon^n \leq \text{ord } f < \text{ord } 1$. For simplicity, however, we shall assume that the latter expansion is identical with (7), i.e. that the n^{th} sum of (7a) is uniform to order ϵ^n in $\text{ord } \epsilon^n \leq \text{ord } f \leq 1$. This may easily be verified *a posteriori*. In applying the matching principle, therefore, it is to be noted that the overlap domain between the n^{th} sums of (7) and (8) lies within the domain of validity of (8), i.e. is restricted by the condition $\text{ord } f < \text{ord } \epsilon^n$.

In the next step we determine $\mathbf{g}_1$. For $\text{ord } f$ in the overlap domain $\mathbf{i} + \epsilon \mathbf{g}_1$ should match $\mathbf{u}_0 + \epsilon \mathbf{u}_1$ to order ϵ , i.e.

$$\lim_{\epsilon} \frac{(\mathbf{i} + \epsilon \mathbf{g}_1) - (\mathbf{u}_0 + \epsilon \mathbf{u}_1)}{\epsilon} = 0. \quad (14)$$

This relation involves two unknown functions $\mathbf{u}_1$ and $\mathbf{g}_1$. Intuitively it is obvious, however, that $\mathbf{g}_1$ is determined independently of $\mathbf{u}_1$ and must be given by (4a). Since the body radius, $\frac{1}{2}\text{Re}$, is transcendentally small from the point of view of the Oseen limit process, the effect of the body on the free stream $\mathbf{i}$ is that of a singular force concentrated at the origin, equal and opposite to the drag. The perturbation due to such a force is given by the fundamental solution of the Oseen equations (cf. Ref. 2) and corresponds to (4a). The same result follows from the standard matching principle (14). Assuming that $\mathbf{u}_1$ is uniformly bounded in the intermediate domain, $\lim_{\epsilon} \mathbf{u}_1$ must be bounded at the origin for $\text{ord } f$ in the overlap domain. Hence if $\epsilon \mathbf{u}_1$ is omitted in the numerator of (14) the right-hand side must be bounded at the origin. It follows that $\mathbf{g}_1$ is a solution of the Oseen equations which cancels the unbounded (i.e. logarithmic) term of $\lim_{\epsilon} [(\mathbf{u}_0 - \mathbf{i})/\epsilon]$ at the origin. This leads to (4a).

Since $\mathbf{u}_1$ satisfies the Stokes equations we assume it to be of the form

$$\mathbf{u}_1 = c_1 \mathbf{u}_0. \quad (15a)$$

The matching (14) then gives (cf. (13b))

$$c_1 = \gamma - \log 8 - \frac{1}{2} - b_1. \quad (15b)$$

It is convenient to choose b_1 so that c_1 and hence $\mathbf{u}_1$ is equal to zero. If in addition $b_n = 0$ for $n \geq 2$, one obtains the ϵ given by (5).

Composite Expansion. Lamb's Solution. So far, we have found the Oseen and intermediate expansions to order ϵ . A corresponding uniformly valid composite term is then found by adding to the Oseen expansion the part of the intermediate expansion which is not contained in the part $\mathbf{i} + \epsilon \mathbf{g}_1$ of the Oseen

expansion. This is the Stokes-type correction term $\epsilon \mathbf{l}_1$ (cf. (3) and (4c)) which is transcendentally small under the Oseen limit process.

The result obtained above, *i.e.* the first expression in curly brackets of Eq. 3, is simply Lamb's solution of the Oseen equations. This is due to the accident that, up to the order considered, the Oseen equations remain uniformly valid near the body. The systematic procedure used above is, however, valuable for more general problems, for example, higher-order approximations for the present problem or compressible flow. In the latter case the "Stokes equations" are non-linear, hence the linear "Oseen equations" are not valid near the body. Note also that in two dimensions the Oseen expansion is non-uniform near the body, no matter how many terms are considered. While $\epsilon \mathbf{l}_1$ is a solution of the Oseen equations it would not appear as part of any $\mathbf{g}_n$. For the sphere the Stokes correction term $\mathbf{l}_0$ could actually be obtained by an Oseen expansion. The reason is that $\tilde{r}$ for the sphere is $O(\epsilon)$ whereas it is transcendentally small for the cylinder.

Influence of Non-linear Terms: $\mathbf{u}_2$ and $\mathbf{g}_2$. The solution of the Oseen equations just constructed satisfies the boundary conditions to transcendental order $O(\text{Re})$. Hence higher-order corrections come from the non-linearity of the Navier-Stokes equations. In particular, for $n \geq 2$ the $\mathbf{g}_n$ satisfy the non-homogeneous Oseen equations

$$\left(\nabla^2 - \frac{\partial}{\partial \tilde{x}}\right)\mathbf{g}_n = \widetilde{\nabla} p_n + \mathbf{f}_n(\tilde{x}_i), \quad \text{div } \mathbf{g}_n = 0, \quad \mathbf{f}_n = \sum_{i=1}^{n-1} (\mathbf{g}_i \cdot \widetilde{\nabla}) \mathbf{g}_{n-i} \quad (16)$$

(while the $\mathbf{u}_n$ satisfy the homogeneous Stokes equations). Here it is assumed that $(p - p_\infty)/\rho U^2$ has an Oseen expansion $\sum_{n=0} \epsilon^n p_n$.

The particular solutions of (16) which vanish at infinity and are continuous at the origin will be denoted by $\mathbf{k}_n(\tilde{x}_i)$. The construction of the higher-order terms of (3) then proceeds as follows. If the term $\epsilon^2 \mathbf{k}_2$ is added to the expansion previously obtained, the boundary conditions are no longer satisfied. However, since the body is transcendentally small under the Oseen limit, only $\mathbf{k}_2(0, 0)$ matters. Thus, in finding $\mathbf{u}_2$, we may consider the body to be in an apparent uniform stream of velocity $a_2 \mathbf{i} = \mathbf{k}_2(0, 0)$. This problem has been solved above. Hence $\mathbf{u}_2 = a_2 \mathbf{u}_0$, the additional Oseen term is given by $\epsilon^2 (a_2 \epsilon \mathbf{g}_1)$ and the uniform composite term is given by the expression in curly brackets corresponding to $n = 2$ in the expansion (3). The solution thus obtained again satisfies the boundary and matching conditions to transcendental order. Hence higher-order terms may be found by the same procedure. The formal matching procedure leads to the same result. Taking $\mathbf{g}_2$ tentatively as $\mathbf{k}_2$ we have

$$\lim_{\epsilon} \frac{(\mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2) - (\mathbf{g}_0 + \epsilon \mathbf{g}_1 + \epsilon^2 \mathbf{k}_2)}{\epsilon^2} = 0 \quad (17)$$

in some overlap domain. For $\text{ord } f_0 < \text{ord } f < \text{ord } \epsilon^2$ (where $\text{ord } f_0$ is in the overlap domain) the terms $\mathbf{u}_0 + \epsilon \mathbf{u}_1$ and $\mathbf{g}_0 + \epsilon \mathbf{g}_1$ match by previous construction to order ϵ^2 and hence may be neglected. Furthermore $\lim_{\epsilon} \mathbf{k}_2 = \mathbf{k}_2(0, 0) = a_2 \mathbf{i}$.

Hence condition (17) is identical with (12) except for the factor a_2 and the fact that the overlap domain is more restricted. The term $\epsilon^2 \mathbf{u}_2$ then leads to an Oseen term $\epsilon^3 a_2 \mathbf{g}_1$, *etc.* Higher order terms may be constructed in the same manner.

The first non-linear correction term for the drag, $a_2 \epsilon^3$, may be computed as follows. From the fundamental solution of the Oseen equations, $\mathbf{k}_2$ is constructed to be

$$(k_2)_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t_{i,j}(x_k - \xi_k) f_j(\xi_k) d\xi_1 d\xi_2 \quad (18)$$

where

$$\mathbf{f}(x_i) = (\mathbf{g}_1 \cdot \widetilde{\nabla}) \mathbf{g}_1 = -\mathbf{g}_1 \times \widetilde{\text{curl}} \mathbf{g}_1 + \widetilde{\nabla} \frac{1}{2} g_1^2, \quad (19)$$

$$t_{i,j}(x_k) = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} -\psi & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial \tilde{x}} & \frac{\partial}{\partial \tilde{y}} \\ \frac{\partial}{\partial \tilde{y}} & -\frac{\partial}{\partial \tilde{x}} \end{pmatrix} (\psi + \phi), \quad (20a)$$

$$\psi = \frac{1}{2\pi} e^{\frac{1}{2}\tilde{x}} K_0(\frac{1}{2}\tilde{r}), \quad \phi = \frac{1}{2\pi} \log \tilde{r}. \quad (20b)$$

The function defined by the above integral is evidently continuous at the origin and vanishes at infinity. Furthermore the irrotational part of $\mathbf{f}$ gives rise only to a hydrostatic pressure; hence, for the purpose of computing the velocity field, the term $\widetilde{\nabla} \frac{1}{2} g_1^2$ in Eq. 19 may be omitted in Eq. 18. The value of $\mathbf{k}_2$ at the origin is then

$$\mathbf{k}_2(0, 0) = a_2 \mathbf{i} \quad (21a)$$

where

$$a_2 = -\frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} K_0(\frac{1}{2}\tilde{r}) K_1(\frac{1}{2}\tilde{r}) [1 + e^{\tilde{x}} - \tilde{r} e^{\frac{1}{2}\tilde{x}} K_1(\frac{1}{2}\tilde{r})] \sin^2 \theta d\theta d\tilde{r}. \quad (21b)$$

Integrating with respect to θ and then replacing $\frac{1}{2}\tilde{r}$ by ξ , one obtains

$$a_2 = -\int_0^{\infty} K_0(\xi) K_1(\xi) [\xi^{-1} I_1(2\xi) - 4K_1(\xi) I_1(\xi) + 1] d\xi = -0.87 \quad (22)$$

where I_1 , K_0 , and K_1 are modified Bessel functions of the first and second kind and $K_1 = -K'_0$.

Cylinder of Arbitrary Cross-section. Some remarks may now be made about an arbitrary finite body in two dimensions.

The leading term $\mathbf{u}_{0a}$ of the intermediate expansion may evidently be expected to be of the form

$$\mathbf{u}_{0a} = \epsilon_a (\text{Re}) \mathbf{h}_{1a}(x_i^*) \quad (23a)$$

where the x_i^* are based on a characteristic length L of the body and $\mathbf{h}_{1a}$ satisfies the Stokes equations. $\mathbf{h}_{1a}$ is defined by

$$\mathbf{h}_{1a} = \lim_s (\mathbf{q}^*/\epsilon) \quad (23b)$$

where $\mathbf{q}^*$ is the exact solution for the body and ϵ is given by Eq. 5.

The general solution of the Stokes equations is discussed in Ref. 3. It may be conveniently described in terms of complex variables $z = x_1^* + ix_2^*$, $\bar{z} = x_1^* - ix_2^*$, and $u^* - iv^*$ where u^* and v^* are non-dimensional Cartesian velocity components. If (u^*, v^*) is regular outside and on the circle $|z| = 1$, excepting possibly at the point at infinity, the pressure is continuous and $u^* - iv^* = O(\log r^*)$ for $|z| \rightarrow \infty$, then, save for a multiplicative constant, $u^* - iv^*$ is of the form

$$u^* - iv^* = H + M, \quad (|z| \geq 1) \quad (24)$$

$$H = \log r^* - \frac{\bar{z}}{2z} - \log \frac{1}{2} \left(\frac{L_a}{L} \right) + \frac{1}{8z^2} \left(\frac{L_a}{L} \right)^2$$

where L_a is a complex constant of dimension of length. Here M is a function of z and $\bar{z}$ satisfying the relation

$$M = O\left(\frac{1}{r^*}\right) \quad \text{for } r^* \rightarrow \infty. \quad (25)$$

Also

$$\int_0^{2\pi} M d\theta = 0 \quad \text{at } r^* = \text{const} \geq 1 \quad (26)$$

(cf. Ref. 3).

Without loss of generality, we may for the moment take L so that the body is inscribed within the circle $|z| = 1$. Hence, clearly, $\mathbf{h}_{1a}$ may be represented in the form (24). In particular, if the multiplicative constant is unity, as will be shown, the complex constant L_a corresponding to $\mathbf{h}_{1a}$ is related to an average velocity by

$$-\log \left(\frac{L_a}{L} \right) = u_{av}^* - iv_{av}^* = \int_0^{2\pi} (u^* - iv^*) d\theta \quad \text{at } r^* = 1. \quad (27)$$

We shall restrict attention to the case

$$v_{av}^* = 0. \quad (28)$$

This condition may be satisfied for any body by a suitable orientation to the free stream; in particular it is satisfied if the body is symmetric with respect to the x_1 -axis. If $v_{av} = 0$, L_a is real. We note that H is then the Stokes solution for a circular cylinder of diameter L_a (cf. Eq. 8c). Hence L_a may be called the *equivalent diameter* of the given body.

From (25) it follows that M is transcendentally small by any limit $\lim_r$ such that $\text{ord } f \geq \text{ord } \text{Re}^\alpha$ for some α where $\alpha < 1$; hence M plays no role in any of the matching processes described earlier. The matching problem is therefore

identical to that for a circle whose diameter is L_a .† In particular, if we use the equivalent diameter L_a as the characteristic length L of the body, *i.e.* put

$$\text{Re} = \frac{UL_a}{\nu}, \quad x_i^* = \frac{x_i}{L_a}, \quad r^* = \frac{r}{L_a}, \quad (29)$$

then the matching problem is solved and the expansions for the body are found simply by replacing $\mathbf{h}_1$, wherever it appears, by $\mathbf{h}_{1a}$. In particular,

$$\epsilon_a(\text{Re}) = \epsilon(\text{Re}), \quad (30a)$$

$$\mathbf{u}_{0a} = \epsilon \mathbf{h}_{1a} = \epsilon(\mathbf{h}_1 + \mathbf{m}_1) \quad (30b)$$

where $\epsilon(\text{Re})$ is defined by Eqs. 5 and 29 and $\mathbf{h}_1$ and $\mathbf{m}_1$ are the velocity fields corresponding to H and M respectively (Eq. 24). It also follows that the Oseen expansion is the same for the arbitrary body and for the circular cylinder. Since the drag may be computed from the Oseen expansion, it then follows that *the drag formula (10) is valid to all orders (ϵ^n) independently of the shape of the body*, provided only that Re is defined by Eq. 29. Corrections to the drag of order Re will, however, depend on the body shape. Finally, we observe that the intermediate and composite expansions for the arbitrary body are obtained from the corresponding expansions for the circular cylinder by replacing $\mathbf{h}_1$ and $\mathbf{l}_1$ by $\mathbf{h}_{1a}$ and $\mathbf{l}_{1a}$, respectively, where

$$\mathbf{l}_{1a} = \mathbf{l}_1 + \mathbf{m}_1. \quad (31)$$

Numerical Aspects. It is clear that (3) is only asymptotically equivalent to $\mathbf{q}^*$; in particular, as may be seen from the drag formula for an arbitrary body, some of the essential features of the full solution are missing and no minor modifications will repair the asymptotic character of (3). On the other hand, there is no indication that the series (3) does not converge, for Re sufficiently small. This makes it possible in principle to compute the transcendental corrections of orders Re , ϵRe , $\epsilon^2 \text{Re}$, *etc.* These corrections may be significant when Re is not very small, indeed, in the region where the corrections to Lamb's solution are most interesting from the numerical point of view. It should also be observed that the series (3) is unchanged if ϵ is replaced by a function ϵ^* of the form

$$\epsilon^* = [\tfrac{1}{2} - \gamma - \log \tfrac{1}{3} \text{Re} + O(\text{Re})]^{-1}. \quad (32)$$

A suitable choice of ϵ^* may increase the radius of convergence of the series (3) and (10) and also improve the numerical applicability of a partial sum of these series.

† The results here bear a certain resemblance to results in Ref. 4, p. 188 ff. The Laplace equation considered there is actually the relevant special case of the Stokes equation.

REFERENCES

1. S. KAPLUN & P. A. LAGERSTROM, Asymptotic Expansions of Navier-Stokes Solutions for Small Reynolds Numbers, paper (I.138) delivered at *IX International Congress of Applied Mechanics*, Brussels, Sept. 5–13, 1956.
2. C. W. OSEEN, *Hydrodynamik*, Akademische Verlagsgesellschaft, Leipzig, 1927.
3. M. KRAKOWSKI & A. CHARNES, Stokes Paradox and Biharmonic Flows, *Carnegie Inst. of Tech., Dept. of Math., Tech. Rept. No. 37*.
4. G. K. BATCHELOR, The Skin Friction on Infinite Cylinders Moving Parallel to their Length, *QJMM*, **7**, 172–192 (June, 1954).

Guggenheim Aeronautical Laboratory
California Institute of Technology
Pasadena, California