*Asymptotic Expansions of Navier-Stokes Solutions for Small Reynolds Numbers**

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Introduction. The present paper and Ref. 1 are essentially based on Ref. 2. Some of the basic ideas of Ref. 2 have been described and applied in Ref. 3; pp. 872–877 in particular may serve as an introduction to the present paper and to Ref. 1. A concrete application of the ideas described below is given in Ref. 1.

We consider viscous incompressible stationary flow past an arbitrary finite body in two or three dimensions. The following notation is used

$\mathbf{q}$ = velocity, ρ = density = constant, p = pressure, ν = kinematic viscosity,
 L = characteristic length of body, (x_i) = Cartesian coordinates, $x = x_1$, (1)
 $r^2 = \sum_{i=1}^n x_i^2$ where n is the dimension of the space.

The governing equations are the Navier-Stokes equations

$$\frac{D\mathbf{q}}{Dt} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \mathbf{q}, \quad (2a)$$

$$\nabla \cdot \mathbf{q} = 0, \quad (2b)$$

with the boundary conditions

$$\mathbf{q} = \mathbf{0} \quad \text{at body}, \quad (3a)$$

$$\mathbf{q} = U\mathbf{i}, \quad p = p_\infty \quad \text{at infinity}, \quad (3b)$$

where $\mathbf{i}$ is the unit vector in the x -direction. Non-dimensional quantities are defined as follows:

$$\text{Re} = \frac{UL}{\nu} \quad (\text{Reynolds number}), \quad (4a)$$

$$\mathbf{q}^* = \frac{\mathbf{q}}{U}, \quad (4b)$$

$$\tilde{x}_i = \frac{x_i U}{\nu}, \quad \tilde{r} = \frac{r U}{\nu} \quad (\text{Oseen variables}), \quad (4c)$$

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$$x_i^* = \frac{x_i}{L} = \frac{\tilde{x}_i}{\text{Re}}, \quad r^* = \frac{r}{L} = \frac{\tilde{r}}{\text{Re}} \quad (\text{Stokes variables}), \quad (4d)$$

$$\tilde{\nabla} = \left(\frac{\partial}{\partial \tilde{x}_i} \right), \quad \nabla^* = \left(\frac{\partial}{\partial x_i^*} \right). \quad (4e)$$

This paper deals with the problem of finding asymptotic expansions

$$\mathbf{q}^* \sim \sum_{i=0} \mathbf{q}_i(\tilde{x}_i, \text{Re}) \quad (5)$$

which are uniformly valid for the entire flow field for small values of Re . Here $\mathbf{q}^*$ is the exact solution of Eqs. 2 and 3. The degree of approximation is measured with a suitable sequence of functions of Re , $\epsilon_i(\text{Re})$, having the property

$$\epsilon_0 = 1, \quad \lim_{\text{Re} \downarrow 0} \frac{\epsilon_{i+1}}{\epsilon_i} = 0. \quad (6)$$

We require the n^{th} partial sum to be valid to order ϵ_n , *i.e.*

$$\lim_{\text{Re} \downarrow 0} \frac{\mathbf{q}^* - \sum_{i=0}^n \mathbf{q}_i}{\epsilon_n} = \mathbf{0} \quad \text{uniformly in space.} \quad (7)$$

A corresponding expansion of the pressure should also be found. This expansion is easily constructed from (5) and will not be discussed here.

A recursive method for constructing a special form of (5), called the composite expansion, is given below. In this method the main problem is to construct two associated principal expansions called the outer (Oseen) expansion and the intermediate expansion. The relation of the leading terms of the various expansions to the Stokes and Oseen approximations (Ref. 4) is discussed. Flow past a sphere is used as an illustrative example. Flow past a circular cylinder is discussed in Ref. 1. The investigation of the problem has led to a development of certain methods in the theory of singular perturbations. One of the main purposes of Ref. 2 is to investigate certain aspects of the theory of singular perturbations, using the present problem as an illustration.

Outer and Inner Limits and Expansions. Consider a flow quantity F (such as $\mathbf{q}^*$) depending on $\tilde{x}_i$ and Re (or, equivalently, on x_i^* and Re). The Oseen limit of F is defined by

$$\lim_o F = \lim F \text{ as } \text{Re} \downarrow 0, \tilde{x}_i \text{ fixed}, \tilde{x}_i \neq 0. \quad (8)$$

Similarly, the Stokes limit is defined by

$$\lim_s F = \lim F \text{ as } \text{Re} \downarrow 0, x_i^* \text{ fixed}, (x_i^*) \text{ on or outside the body.} \quad (9)$$

By a repeated application of the Oseen limit with a suitably chosen sequence (ϵ_i) (*cf.* Eq. 6) one obtains the Oseen expansion of $\mathbf{q}^*$ (or other flow quantities)

$$\mathbf{q}^* \sim \sum_{i=0} \epsilon_i(\text{Re}) \mathbf{g}_i(\tilde{x}_i) \quad (10a)$$

where

$$\mathbf{g}_0 = \lim_o \mathbf{q}^*, \quad (10b)$$

$$\mathbf{g}_{n+1} = \lim_o \frac{\mathbf{q}^* - \sum_{j=0}^n \epsilon_j \mathbf{g}_j}{\epsilon_{n+1}}. \quad (10c)$$

Similarly, a repeated application of the Stokes limit gives a Stokes expansion

$$\mathbf{q}^* \sim \sum_{j=0} \epsilon_j (\text{Re}) \mathbf{h}_j(x_i^*). \quad (11)$$

By insertion of these expansions into Eq. 2 (written in Oseen and Stokes variables respectively) one obtains the equations for the $\mathbf{g}_i$ and the $\mathbf{h}_i$. For a large class of bodies, including all finite bodies, (*cf.* Ref. 3, pp. 872-877) $\mathbf{g}_0 = \mathbf{i}$ and $\mathbf{g}_1$ then satisfies the Oseen equations. The Oseen limit corresponds to the outer limit (Euler limit) in the theory of high Reynolds number flow (Ref. 3). The Oseen expansion should then satisfy the outer boundary condition (Eq. 3b). In general this expansion cannot be expected to be uniformly valid near the body. This fact makes the problem a singular perturbation problem. A second limit is needed to handle the "boundary layer" near the body. This is the Stokes limit which corresponds to the inner limit (Prandtl limit) in high Reynolds number flow. The Stokes expansion should then satisfy the inner boundary condition (Eq. 3a); in general it cannot be expected to be uniformly valid at infinity.

Disregarding for a moment the problem of determining the sequence (ϵ_i) , we see that the equations for the $\mathbf{g}_i$ and the $\mathbf{h}_i$ are given, whereas the boundary conditions are incomplete. One needs to find inner boundary conditions for the $\mathbf{g}_i$ and outer boundary conditions for the $\mathbf{h}_i$; such conditions follow from principles of matching of the outer and inner expansions. A classical example of matching occurs in boundary layer theory where the tangential velocity component of the boundary layer takes on a value at infinity which agrees with the external flow at the body; a similar condition exists for flow due to displacement thickness (*cf.* Ref. 5). In the present case the matching is more complicated. For two-dimensional flow $\mathbf{g}_0 = \mathbf{i}$ and $\mathbf{h}_0 = \mathbf{0}$; hence *the first terms of the inner and outer expansion cannot be matched*. The problem of matching various expansions will therefore have to be reconsidered; Ref. 2 contains a detailed investigation of this problem. It is found that the matching principle used in boundary layer theory may be generalized so as to apply to the present case (this is applied in Ref. 3, p. 873 ff.). A brief review of the reasoning developed in Ref. 2 is given immediately below.

Intermediate Limits. In order to find the connection between the outer and inner limits (Eqs. 8 and 9) we consider other limits which in a certain sense are intermediate. As a preliminary step we study a partially ordered set of equivalence classes of functions. Consider functions $f(\text{Re})$ which are positive and continuous on some interval $0 < \text{Re} < A$ and which tend to a definite limit as Re tends to

zero. We say that $f(\text{Re})$ and $g(\text{Re})$ belong to the same equivalence class (denoted by $\text{ord } f$ or $\text{ord } g$) if

$$0 < \lim_{\text{Re} \downarrow 0} \frac{f}{g} < \infty \quad \text{as } \text{Re} \downarrow 0. \quad (12)$$

An ordering of the equivalence classes is defined by

$$\text{ord } f < \text{ord } g, \quad \text{whenever} \quad \lim_{\text{Re} \downarrow 0} \frac{f}{g} = 0 \quad \text{as } \text{Re} \downarrow 0. \quad (13)$$

This ordering is only partial; there are functions f and g which are incompatible in the sense that neither $\lim (f/g)$ nor $\lim (g/f)$ exist. A set S of equivalence classes is called convex if, for every $\text{ord } f$, $\text{ord } g$ in S

$$\text{ord } f < \text{ord } h < \text{ord } g \quad \text{implies} \quad \text{ord } h \text{ is in } S. \quad (14)$$

A convex set S is called open if for every $\text{ord } f$ in S there exist functions f_1 and f_2 such that $\text{ord } f_1, \text{ord } f_2$ are in S and $\text{ord } f_1 < \text{ord } f, \text{ord } f_2 > \text{ord } f$. A convex set S is called a closed interval if there exists an $\text{ord } f_1$ in S such that, for every $\text{ord } f$ in S , $\text{ord } f_1 \leq \text{ord } f$ and if a similar statement is true with "smaller than" replaced by "greater than".

We now define variables (*cf.* Eqs. 4c, d)

$$x_i^{(\prime)} = \frac{\tilde{x}_i}{f}, \quad r^{(\prime)} = \frac{\tilde{r}}{f} \quad (15)$$

and the corresponding limits (*cf.* Eqs. 8 and 9)

$$\lim_{\text{Re} \downarrow 0} F(\tilde{x}_i, \text{Re}) = \lim_{\text{Re} \downarrow 0} F \text{ as } \text{Re} \downarrow 0, x_i^{(\prime)} \text{ fixed}, x_i^{(\prime)} \neq 0. \quad (16)$$

The Oseen and Stokes limits correspond to $f(\text{Re}) \equiv 1$ and $f(\text{Re}) \equiv \text{Re}$ respectively. The intermediate limits are the $\lim_{\text{Re} \downarrow 0}$ for which $\text{ord } \text{Re} < \text{ord } f < \text{ord } 1$. In the $\tilde{x}_i$ -space, a point $x_i^{(\prime)} = \text{constant}$ moves towards the origin for $\text{ord } f < \text{ord } 1$. The point $x_i^{(\prime)} = \text{constant}$ moves faster than the point $x_i^{(g)} = \text{constant}$ if $\text{ord } f < \text{ord } g$; we then say that $\lim_{\text{Re} \downarrow 0}$ is faster than $\lim_{\text{Re} \downarrow 0}$. If $f = \text{Re}$ then the point $(x_i^{(\prime)}) = (x_i^*) = \text{constant}$ moves towards the origin at the same rate as the body surface; this is the fastest limit considered.

Let F_1 be an approximation to a function F depending on $\tilde{x}_i$ and Re . We say that F_1 constitutes a *uniform approximation to order ϵ* (is uniformly valid to order ϵ) in a convex set S of equivalence classes if

$$\lim_{\text{Re} \downarrow 0} \frac{F - F_1}{\epsilon(\text{Re})} = 0 \quad \text{uniformly for } f_1(\text{Re}) \leq \tilde{r} \leq f_2(\text{Re}) \quad (17)$$

whenever $\text{ord } f_1$ and $\text{ord } f_2$ are in the set S . If f is a function such that $\lim_{\text{Re} \downarrow 0} \mathbf{q}^*$ exists, it may be assumed that $\lim_{\text{Re} \downarrow 0} \mathbf{q}^*$ is a uniform approximation to $\mathbf{q}^*$ to order unity in the set consisting of $\text{ord } f$ alone, *i.e.*

$$\lim_{\text{Re} \downarrow 0} |\mathbf{q}^* - \lim_{\text{Re} \downarrow 0} \mathbf{q}^*| = 0 \quad \text{uniformly for } Af(\text{Re}) \leq \tilde{r} \leq Bf(\text{Re}) \quad (18)$$

where A and B are any positive constants. This is implied by the definition of $\lim_r \mathbf{q}^*$ (save for cases which are of no interest in the present context). In the particular case in which $f(\text{Re}) = \text{Re}$, the lower limit in (18) may be replaced by the body surface (*i.e.* A is a function of the angular coordinates); for $f(\text{Re}) = 1$ the upper limit may be replaced by infinity. On the other hand there is no reason to expect that $\lim_r \mathbf{q}^*$ is a uniform approximation for an arbitrary set S . We can now see that the reason why $\lim_o \mathbf{q}^*$ and $\lim_s \mathbf{q}^*$ cannot always be matched is that there is no *a priori* reason for their regions of validity to overlap.

Matching. Intermediate Expansions. The theory of matching developed in Ref. 2 follows from an overlap principle which rests on two basic ideas. The first one is the introduction of a Stokes *solution* (not necessarily a Stokes limit!) which is uniformly valid to order unity in the convex set $\text{ord Re} \leq \text{ord } f < \text{ord } 1$. The second idea is the observation (extension theorem) that the Oseen limit must be valid to order unity not only in the set consisting of $\text{ord } 1$ but in some extended set containing orders smaller than $\text{ord } 1$.

Regarding the first idea we observe the following. The number of limit processes is very large and there is a large number of possible values of $\lim_r \mathbf{q}^*$ (it will follow from the discussion in Ref. 1 that for two-dimensional flow $\lim_r \mathbf{q}^* = 1 - \alpha$ for $f(\text{Re}) \equiv \text{Re}^\alpha$). However, the number of different systems of equations obtained by the various limit processes is very small. More precisely, if we introduce the variables $x_i^{(f)}$ into the Navier-Stokes equations and then let Re tend to zero keeping $x_i^{(f)}$ fixed the Stokes equations are obtained whenever $\text{ord Re} \leq \text{ord } f < \text{ord } 1$. This suggests finding a Stokes solution $\mathbf{u}_0$ which is uniformly valid in the convex set just described. We shall call $\mathbf{u}_0$ an *intermediate solution* valid to order unity if

$$\lim_{\text{Re} \downarrow 0} |\mathbf{q}^* - \mathbf{u}_0| = 0 \quad \text{uniformly for } A \cdot \text{Re} \leq \tilde{r} \leq f(\text{Re}) \quad (19)$$

for any $f(\text{Re})$ such that $\text{ord } f < \text{ord } 1$. (Here the surface of the body is defined by $\tilde{r} = A \cdot \text{Re}$, A being a function of the angular coordinates only.)

The extension theorem may be justified as follows. Consider the function $w = |\mathbf{q}^* - \mathbf{i}|$. It is defined between $\tilde{r} = A \cdot \text{Re}$ (body surface) and $\tilde{r} = \text{infinity}$. For any $\tilde{r}$ fixed ($\tilde{r} \neq 0$) w tends to zero as Re tends to zero; this convergence is uniform in any domain $\tilde{r} \geq B$, $B = \text{constant} > 0$. It is therefore possible to find a sequence (A_n) decreasing monotonically to zero such that $w \leq 1/n$ for $0 < \text{Re} \leq A_n$ and $\tilde{r} \geq 1/n$. Let $f(\text{Re})$ be a monotonic continuous function such that $f(A_n) = 1/n$ and g a function such that $\text{ord } f < \text{ord } g < \text{ord } 1$. From this it may be concluded that $\lim_r w = 0$ and that this limit is uniform for the set $\text{ord } f < \text{ord } g \leq \text{ord } 1$. (A more complete proof is given in Ref. 2.) Thus we see that if $\mathbf{i}$ is an approximation to $\mathbf{q}^*$ uniformly valid to order unity in the convex set S consisting of $\text{ord } 1$ alone the domain of validity may be extended to a larger convex set S_1 containing S . This is a fundamental theorem of extension. It may easily be generalized: *If an approximation is valid to order ϵ in a closed interval S , its domain of validity may be extended to an open convex set S_1 containing S .*

We now have a possibility of matching. *The domain where $\mathbf{u}_0$ is valid to order unity (Eq. 19) overlaps with the extended domain of validity of the approximation $\mathbf{i}$ (overlap principle).* The part common to both domains will be called the overlap domain. Thus $\text{ord } f$ is in the overlap domain if $\text{ord } f < \text{ord } 1$ and at the same time $\lim_r$ is a sufficiently slow limit so that $\lim_r \mathbf{q}^* = 1$. Hence we obtain the fundamental matching principle

$$\lim_r (\mathbf{i} - \mathbf{u}_0) = \mathbf{0} \quad \text{in some overlap domain.} \quad (20)$$

Now $\mathbf{u}_0$ was defined as a Stokes solution satisfying the inner boundary conditions. The matching condition (20) furnishes the missing boundary condition and determines $\mathbf{u}_0$ to order unity.

Extending the above ideas to higher order approximations we replace the Stokes expansion by an intermediate expansion

$$\mathbf{q}^* \sim \sum_{i=0} \epsilon_i (\text{Re}) \mathbf{u}_i. \quad (21)$$

We require the n^{th} partial sum to be valid to order $\epsilon_n(\text{Re})$ in some convex domain $\text{ord } \text{Re} \leq \text{ord } f < \text{ord } f_n$ where the limit corresponding to $f_n(\text{Re})$ is sufficiently slow so that the domain just described overlaps with the corresponding domain of validity of the Oseen expansion. Thus, for $n = 1$,

$$\lim_r \frac{(\mathbf{g}_0 + \epsilon_1 \mathbf{g}_1) - (\mathbf{u}_0 + \epsilon_1 \mathbf{u}_1)}{\epsilon_1} = \mathbf{0} \quad (22)$$

uniformly in some convex overlap domain.

The matching principle discussed above is of great generality. Applied to expansions for high Re it yields the classical matching rules of boundary layer theory. It will be illustrated below for the case of three-dimensional flow at low Re and in Ref. 1 for the corresponding two-dimensional case. In both these cases it will be seen that the intermediate expansions are simply related to the Stokes expansions, although in a different manner in the two cases.

Application to Flow Past a Sphere. As a specific example we consider flow past a sphere with center at the origin. The diameter of the sphere will be taken as the characteristic length L .

Since the sphere is a finite three-dimensional body there exists a solution of the Stokes equations satisfying the boundary conditions at the body as well as at infinity. The velocity field is given by a function $\mathbf{A}$

$$\mathbf{A} = \mathbf{i} - \frac{3}{4}\mathbf{A}_1 + \frac{1}{32}\mathbf{A}_2 \quad (23)$$

where

$$\begin{aligned} \mathbf{A}_1 &= \frac{\mathbf{i}}{r^*} - \nabla^* \left(\frac{\tau^*}{r^*} \right), & \tau^* &= \frac{1}{2}(x^* - r^*), \\ \mathbf{A}_2 &= \nabla^* \frac{\partial}{\partial x^*} \left(\frac{1}{r^*} \right), & \nabla^* &= \left(\frac{\partial}{\partial x_i^*} \right). \end{aligned}$$

This solution is the first term of the intermediate expansion and also of the Stokes expansion, *i.e.*

$$\mathbf{u}_0 = \mathbf{h}_0 = \mathbf{A}. \quad (24)$$

Due to the simple structure of $\mathbf{u}_0$ the matching condition (20) reduces to the boundary condition $\mathbf{u}_0 = \mathbf{i}$ at $r^* = \infty$. The fact that $\mathbf{u}_0$ satisfies the "correct" boundary condition at infinity, *i.e.* the same condition as the solution of the full Navier-Stokes equations, should be regarded as an exceptional coincidence (*cf.* Ref. 1, the discussion of $\mathbf{u}_1$ below, and the corresponding problem in boundary layer theory).

To find $\mathbf{g}_1$, $\mathbf{u}_1$ and ϵ_1 we proceed heuristically and verify the results *a posteriori*. If we assume that $\mathbf{u}_1$ is bounded at the origin in the overlap domain we see that (22) implies that $\mathbf{g}_1$ must be a solution of the Oseen equations (as well as an Oseen limit) which cancels the part of $(\mathbf{u}_0 - \mathbf{i})/\epsilon_1$ which is unbounded, *i.e.* the term $-\frac{3}{4}A_1/\epsilon_1$. This gives

$$\mathbf{g}_1 = \frac{3}{4} \left(-\mathbf{i} \frac{e^{\tilde{r}}}{\tilde{r}} + \nabla \frac{e^{\tilde{r}} - 1}{\tilde{r}} \right), \quad \tilde{r} = \frac{1}{2}(x - \tilde{r}) \quad (25)$$

and also

$$\epsilon_1 = \text{Re}. \quad (26)$$

We note that for small values of $\tilde{r}$

$$\mathbf{g}_1 = \frac{3}{4}\mathbf{B}_1 + \frac{3}{4}\mathbf{B}_2 + O(\tilde{r}) \quad (27)$$

where

$$\begin{aligned} \mathbf{B}_1 &= -\frac{\mathbf{i}}{\tilde{r}} + \nabla \frac{\tilde{r}}{\tilde{r}} = -\frac{\mathbf{A}_1}{\text{Re}}, \\ \mathbf{B}_2 &= -\frac{\tilde{r}}{\tilde{r}} \mathbf{i} + \nabla \frac{\tilde{r}^2}{2\tilde{r}}. \end{aligned}$$

Comparing (23) and (27) we find

$$\lim_f \frac{(\mathbf{i} + \text{Re } \mathbf{g}_1) - \mathbf{u}_0}{\text{Re}} = \frac{3}{4}\mathbf{B}_2 \quad (28)$$

when f is in the overlap domain

$$\text{ord Re}^{\frac{3}{4}} < \text{ord } f < \text{ord } 1. \quad (29)$$

To find $\mathbf{u}_1$ we first consider the term $\mathbf{h}_1$ of the Stokes expansion. It satisfies the inhomogeneous Stokes equations

$$(\mathbf{h}_0 \cdot \nabla^*) \mathbf{h}_0 + \nabla^* \frac{p_1 L}{\mu U} = \nabla^{*2} \mathbf{h}_1, \quad (30a)$$

$$\nabla^* \cdot \mathbf{h}_1 = 0 \quad (30b)$$

and the condition

$$\mathbf{h}_1 = \mathbf{0} \quad \text{at the sphere.} \quad (31a)$$

Now (22) and (28) imply that

$$\lim_{r \rightarrow \infty} \mathbf{u}_1 = \frac{3}{4} \mathbf{B}_2 \quad (31b)$$

for ord f in the overlap domain (29). It has been verified by actual calculation (see Eq. 33 below) that (30) has a solution $\mathbf{h}_1$ which vanishes at the sphere and which for large values of r^* behaves like $\frac{3}{4} \mathbf{B}_2 + O(1/r^*)$. Identifying $\mathbf{u}_1$ with this $\mathbf{h}_1$ we then see that (31b) and hence (22) are satisfied. Note that the "correct" boundary conditions for $\mathbf{h}_1$ at infinity (which would be $\mathbf{h}_1 = \mathbf{0}$) can no longer be satisfied. As pointed out, this should be regarded as the normal state of affairs.

The ideas used above may in principle be used for computing higher-order terms. The term $\mathbf{g}_2$ satisfies a non-homogeneous Oseen equation; its behavior near the origin is determined by a matching condition.

Composite Expansion. From the intermediate and outer expansions one may easily construct a composite expansion (5) which is uniformly valid for the entire flow field. As the first term, uniformly valid to order unity, one may choose $\mathbf{h}_0$. To simplify the successive terms it is more convenient to choose

$$\mathbf{q}_0(x_i, \text{Re}) = \mathbf{i} + \text{Re } \mathbf{g}_1(x_i) + \mathbf{l}_0(x_i^*). \quad (32a)$$

The term $\mathbf{l}_0$ is the part of $\mathbf{u}_0$ which is not contained in $\text{Re } \mathbf{g}_1$, *i.e.*

$$\frac{1}{32} \mathbf{A}_2 = \frac{1}{32} \nabla^* \frac{\partial}{\partial x^*} \left(\frac{1}{r^*} \right),$$

cf. (23) and (27). This term is of order unity near the body ($r^* \leq \text{constant}$). Rewriting it in Oseen variables one sees that it is of order Re^3 away from the body ($\tilde{r} \geq \text{constant}$). It would thus eventually be accounted for in an Oseen expansion. The term $\mathbf{l}_0$, and hence $\mathbf{q}_0$, is a solution of the Oseen equations. This is due to the fact that the Oseen equations happen to be valid near the body, as shown by the fact that the Stokes limit applied to the Oseen equations gives the Stokes equation. This is a special coincidence; the corresponding statement is no longer true in compressible flow. As a second term in the composite expansion one may use

$$\mathbf{q}_1(x_i, \text{Re}) = \text{Re } [\text{Re } \mathbf{g}_2(x_i) + \mathbf{l}_1(x_i^*)] \quad (32b)$$

where $\mathbf{l}_1$ is a suitable part of $\mathbf{h}_1$.

A further example of a composite expansion is given in Ref. 1.

Drag. The drag may be computed from the $\mathbf{u}_i$. Without computing $\mathbf{u}_1$ it can be seen by a symmetry argument that for bodies symmetrical about a plane perpendicular to the flow direction, the part of $\mathbf{u}_1$ which comes from the non-linear terms (the forcing terms $D\mathbf{h}_0/Dt^*$) gives no net contribution to the drag. This

fact (*cf.* discussion of l_0 above) implies that the part of q_1 which determines the drag may be obtained by solving the homogeneous Oseen equations only. If drag is computed from the Oseen equations only and the result is expanded in powers of Re the first two terms thus agree with the corresponding expansion of drag on the non-linear equations. This is stated in Ref. 6, p. 492. However, to obtain the local distribution of skin friction and pressure at the sphere, the full term u_1 must be used.

Explicit Expression for h_1 for the Sphere. By a straightforward calculation one finds that the solution of Eqs. 30a, b with conditions (31a, b) is

$$h_1 = h_{1p} + h_{1c} + \frac{3}{16}h_0,$$

where

$$\begin{aligned} h_{1p} = & \frac{3}{4} \left[-i \frac{x^*}{2r^*} + \frac{1}{8} \nabla^* \left(\frac{x^{*2}}{r^*} + r^* \right) \right] \\ & + \frac{27}{128} \left[i \frac{x^*}{r^{*2}} - \frac{1}{3} \nabla^* \left(\log r^* + \frac{x^{*2}}{r^{*2}} \right) \right] + \frac{3}{1024} \frac{\partial}{\partial x^*} \left(x^* \nabla^* \frac{1}{r^{*2}} \right), \\ h_{1c} = & \frac{1}{32} \left[3 \left(\frac{1}{4r^{*2}} - \frac{x^{*2}}{r^{*4}} \right) \nabla^* r^* + \frac{3}{4} \frac{x^*}{r^{*3}} i + \frac{1}{8} \left(r^{*2} - \frac{1}{4} \right) \nabla^* \frac{\partial}{\partial x^*} \left(\frac{x^*}{r^{*3}} \right) \right] \end{aligned} \quad (33a)$$

and h_0 is given by Eq. 24. The corresponding pressure is

$$\begin{aligned} \frac{p_1 L}{\mu U} = & \frac{1}{128r^{*2}} \left[9 \left(\frac{2x^{*2}}{r^{*2}} - 1 \right) - \frac{7}{2r^*} \left(\frac{3x^{*2}}{r^{*2}} - 1 \right) + \frac{3}{4r^{*2}} \left(\frac{6x^{*2}}{r^{*2}} - 1 \right) \right. \\ & \left. - \frac{1}{32r^{*4}} \left(\frac{3x^{*2}}{r^{*2}} + 1 \right) \right] - \frac{9}{128} \frac{x^*}{r^{*3}}. \end{aligned} \quad (33b)$$

Here the last term on the right corresponds to $\frac{3}{16}h_0$ in (33a).

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