Some Remarks on Classical Lie Algebras

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If $\mathfrak{F}$ is a field of characteristic zero, it is known that there is a Lie algebra $\mathfrak{g}$ over $\mathfrak{F}$ having the properties (i)—(v) of $[C]^*$ relative to an abelian Cartan subalgebra $\mathfrak{S}$, with a fundamental system of roots (necessarily linearly independent) spanning $\mathfrak{S}^*$ of each of the types $A_r(r \geq 1)$, $B_r(r \geq 3)$, $C_r(r \geq 2)$, $D_r(r \geq 4)$, E_6 , E_7 , E_8 , F_4 , G_2 , and that no two of these algebras are isomorphic [1], [11]. Each of these algebras is simple.

The purpose of this note is to study somewhat more carefully than has heretofore been done the corresponding situation when $\mathfrak{F}$ is of prime characteristic p . In [6] and [8] one can find a full discussion of the existence and non-isomorphism of simple algebras satisfying the conditions (i)—(v) of $[C]$ with fundamental systems of types $A_r(r \geq 1, p \nmid r + 1)$; $B_r(r \geq 3)$; $C_r(r \geq 2)$; $D_r(r \geq 4)$. The remaining types are somewhat inadequately discussed in [8], particularly the types E , F , G . We make use of CHEVALLEY'S work [2] and the general theory at characteristic zero to give a more nearly complete treatment of these types. The question of the existence of isomorphisms among these algebras is also settled. By the results of [6] it is only necessary to eliminate the possibility of isomorphisms between the "exceptional" types E , F , G , or between these and the types of [6]. Dimension considerations reduce this problem to the question of isomorphisms between our algebra of type E_6 and either B_6 or C_6 . Although this could be attacked directly, it seems preferable to obtain a solution as a corollary of a theorem on conjugacy of Cartan subalgebras, a result of some interest in its own right.

1. The type $A_r(p \mid r + 1)$. The set $\mathfrak{M}_{r+1}$ of all $(r + 1) \times (r + 1)$ matrices over $\mathfrak{F}$, $p \mid r + 1$, forms a Lie algebra $\mathfrak{A}$ under $[ab] = ab - ba$. $[\mathfrak{A}]$ is of dimension $(r + 1)^2 - 1 = r^2 + 2r$ and has a one-dimensional center $\mathfrak{C}$. $\mathfrak{g} = [\mathfrak{A}]/\mathfrak{C}$ is a simple Lie algebra of dimension $r^2 + 2r - 1$. $\mathfrak{C}$ consists of scalar multiples of the identity matrix, and if one denotes by $\bar{a}$ the coset of $a \in [\mathfrak{A}]$ by $\mathfrak{C}$, $\mathfrak{g}$ has as basis the cosets of the matrix units $\bar{e}_{ij}$, $i \neq j$, and $\bar{h}_i = \bar{e}_{i+1, i+1} - \bar{e}_{ii}$, $1 \leq i \leq r - 1$. The $\bar{h}_i$ span an $(r - 1)$ -dimensional abelian Cartan subalgebra $\mathfrak{S}'$ relative to which the $\bar{e}_{ij}$ are root-vectors. The roots to which $\bar{e}_{12}$, $\bar{e}_{23}$, $\dots$,

$\bar{e}_{r-1,r}$ belong are linearly independent and span $\mathfrak{S}^*$. Together with the root to which $\bar{e}_{r,r+1}$ belongs, they form a fundamental system of roots in our sense, and (i)—(v) are satisfied. The fundamental system so obtained is of type A_r [8].

2. The types E, F, G . For the remaining algebras with $p > 5$, we make use of the theory over algebraically closed fields of characteristic zero. There are simple Lie algebras of characteristic zero corresponding to these types. These algebras have non-degenerate Killing forms (x, y) , as do all semi-simple Lie algebras of characteristic zero. CHEVALLEY [2] has also shown that if $h_\alpha \in \mathfrak{S}$ is defined by $(h_\alpha, h) = \alpha(h)$ for all h in the Cartan subalgebra $\mathfrak{S}$, each of these algebras $\mathfrak{L}$ has a basis consisting of the set of $h_i = \alpha_i(h_{\alpha_i})^{-1}2h_{\alpha_i}$ for α_i running through a fundamental system (necessarily simple), and of a complete set of root-vectors e_α satisfying $[e_{-\alpha}e_\alpha] = \alpha(h_\alpha)^{-1}2h_\alpha$, such that the structure constants relative to this basis are rational integers. For such algebras it is also known that $[e_{-\alpha}e_\alpha] = (e_{-\alpha}, e_\alpha)h_\alpha$. We further assume that it is known for these algebras that if $\alpha_1, \dots, \alpha_r$ is a fundamental system of roots, then the complete set of roots relative to $\mathfrak{S}$ is that listed in Theorem 14.1, 13.1, or 12.1 of [8] if $\alpha_1, \dots, \alpha_r$ is of type E, F , or G , respectively.

Utilizing these results, we observe that we can construct a Lie algebra over the prime field contained in a given field $\mathfrak{F}$ of characteristic $p > 0$ by reducing the structure constants of $\mathfrak{L}$ modulo p , and then we can obtain an algebra over $\mathfrak{F}$ by extension of the base field. The resulting algebra will have non-degenerate Killing form provided p does not divide the determinant of the Killing form of $\mathfrak{L}$ relative to CHEVALLEY's basis. We now show for exactly which values of p this is the case.

The determinant of the Killing form of $\mathfrak{L}$ is, up to sign, the product of the determinant of the form on $\mathfrak{S}$ by all $(e_{-\alpha}, e_\alpha)$, where α runs through all non-zero roots. Now $(h_i, h_j) = \sum_\beta \beta(h_i) \beta(h_j)$, the sum running over all roots β ; since these are completely enumerated as linear combinations of the α_k , $1 \leq k \leq r$, and since all $\alpha_k(h_i)$ are known by knowing the type of $\alpha_1, \dots, \alpha_r$, the computation here is straightforward. We have seen above that

$$\frac{2h_\alpha}{\alpha(h_\alpha)} = [e_{-\alpha}e_\alpha] = (e_{-\alpha}, e_\alpha)h_\alpha$$

for all non-zero roots α . Hence $(e_{-\alpha}, e_\alpha) = \alpha(h_\alpha)^{-1}2$. If $\alpha = \sum m_i(\alpha)\alpha_i$, then $h_\alpha = \sum m_i(\alpha)h_{\alpha_i}$ where $h_{\alpha_i} = \frac{1}{2} \alpha_i(h_{\alpha_i})h_i$. Thus $\alpha_i(h_i) = 2 = (h_{\alpha_i}, h_i) = \frac{1}{2} \alpha_i(h_{\alpha_i}) (h_i, h_i)$, or $\frac{1}{2} \alpha_i(h_{\alpha_i}) = 2/(h_i, h_i)$, and

$$h_\alpha = 2 \sum_{i=1}^r \frac{m_i(\alpha)h_i}{(h_i, h_i)}.$$

Thus

$$(e_{-\alpha}, e_\alpha) = \frac{2}{\alpha(h_\alpha)} = \left(\sum_{i=1}^r \frac{m_i(\alpha)\alpha(h_i)}{(h_i, h_i)} \right)^{-1},$$

and these quantities are again computed without difficulty from the results of the calculation of the (h_i, h_j) and from our knowledge of the complete set of roots.

In fact, if the type of the fundamental system is prescribed, it is enough to calculate *one* (h_i, h_i) , say (h_1, h_1) , to determine them all. For if (h_1, h_1) is known, we have

$$\begin{aligned}(h_1, h_i) &= \frac{2}{\alpha_1(h_{\alpha_1})} (h_{\alpha_1}, h_i) = \frac{4}{\alpha_1(h_{\alpha_1})\alpha_i(h_{\alpha_i})} (h_{\alpha_1}, h_{\alpha_i}) \\ &= \frac{2}{\alpha_1(h_{\alpha_1})} A_{1i} = \frac{1}{2}(h_1, h_1) A_{1i},\end{aligned}$$

where the A_{ij} are as in [C]. Since the fundamental system is indecomposable, we may assume $A_{12} \neq 0$. Then the appropriate adaptation of the above calculation yields $(h_1, h_2) = \frac{1}{2}(h_2, h_2)A_{21}$, and (h_2, h_2) is determined. The procedure for determining all (h_i, h_j) from (h_1, h_1) is now evident. We also see that once all (h_i, h_i) are calculated, the determinant of the Killing form on $\mathfrak{S}$ will be simply $|A_{ij}| \cdot \prod_{i=1}^r \frac{1}{2}(h_i, h_i)$. This is the form in which the results are expressed in what follows.

G_2 : The determinant on $\mathfrak{S}$ is $8 \cdot 24 \cdot |G_2| = 2^6 \cdot 3$. For six of the twelve roots α , $(e_{-\alpha}, e_\alpha) = 8$; for the rest, $(e_{-\alpha}, e_\alpha) = 24$. Thus the determinant K of the Killing form is $K = 2^{42} \cdot 3^7$, and the algebra modulo p has non-degenerate Killing form if $p > 3$.

F_4 : The determinant on $\mathfrak{S}$ is $18 \cdot 18 \cdot 36 \cdot 36 \cdot |F_4| = 2^6 \cdot 3^8$. For 24 of the 48 roots, $(e_{-\alpha}, e_\alpha) = 18$; for the rest, $(e_{-\alpha}, e_\alpha) = 36$. Thus the determinant K is given by $K = 2^{78} \cdot 3^{104}$, and the algebra modulo p has non-degenerate Killing form if $p > 3$.

E_6 : The determinant on $\mathfrak{S}$ is $24^6 \cdot |E_6| = 2^{18} \cdot 3^7$. For all 72 roots, $(e_{-\alpha}, e_\alpha) = 24$. Thus $K = 2^{234} \cdot 3^{79}$, and the algebra modulo p has non-degenerate Killing form if $p > 3$.

E_7 : The determinant on $\mathfrak{S}$ is $36^7 \cdot |E_7| = 2^{15} \cdot 3^{14}$. For all 126 roots, $(e_{-\alpha}, e_\alpha) = 36$. Thus $K = -2^{267} \cdot 3^{266}$, and the algebra modulo p has non-degenerate Killing form if $p > 3$.

E_8 : The determinant on $\mathfrak{S}$ is $60^8 \cdot |E_8| = 2^{16} \cdot 3^8 \cdot 5^8$. For all 240 roots, $(e_{-\alpha}, e_\alpha) = 60$. Thus $K = 2^{496} \cdot 3^{248} \cdot 5^{248}$, and the algebra modulo p has non-degenerate Killing form if $p > 5$.

The Cartan subalgebra $\mathfrak{S}$ goes into a Cartan subalgebra over our field $\mathfrak{F}$ of characteristic $p > 5$, and the fundamental system of type E , F , or G goes into one of the same type (in these cases, none of CHEVALLEY's structure constants has prime divisors other than 2, 3, 5). The resulting algebra must be simple, since the fundamental system $\alpha_1, \dots, \alpha_r$ is indecomposable. Thus there does exist a simple algebra corresponding to each admissible type of indecomposable fundamental system.

3. Conjugacy of Cartan subalgebras. Our solution of this problem is essentially an adaptation of the treatment for characteristic zero in the mimeographed notes of HARISH-CHANDRA (Chicago, 1951). The first two lemmas belong properly to algebraic geometry, but they do not seem to be readily available in the literature in the form needed here. Therefore we shall include proofs.

Lemma 1. *If $P_1, \dots, P_n$ are n polynomials in n algebraically independent indeterminates $s_1, \dots, s_n$ over a perfect field $\mathfrak{F}$ and if the Jacobian*

$$\left| \frac{\partial P_i}{\partial s_j} \right| \neq 0,$$

then:

- (1) $P_1, \dots, P_n$ are algebraically independent over $\mathfrak{F}$.
- (2) $s_1, \dots, s_n$ are separable and algebraic over $\mathfrak{K} = \mathfrak{F}(P_1, \dots, P_n)$.
- (3) $\mathfrak{F}(s_1, \dots, s_n) = \mathfrak{K}(\theta)$, where θ is algebraic over $\mathfrak{K}$.

Proof. (1): Suppose $Q(P_1, \dots, P_n) = 0$, where Q is a non-zero polynomial with coefficients in $\mathfrak{F}$. We may assume Q to be of minimal degree with this property. But then we have

$$0 = \frac{\partial Q(P_1, \dots, P_n)}{\partial s_j} = \sum_{i=1}^n Q_i(P_1, \dots, P_n) \frac{\partial P_i}{\partial s_j},$$

$1 \leq j \leq n$, where Q_i denotes the partial derivative of Q with respect to its i -th variable. If the Jacobian $J \neq 0$, this implies that all $Q_i(P_1, \dots, P_n) = 0$; since Q_i has lower degree than Q , we have $Q_i = 0$, $1 \leq i \leq n$. But this means that Q is a polynomial in the p -th powers of its variables:

$$\begin{aligned} Q(x_1, \dots, x_n) &= \sum \alpha_{i_1 \dots i_n} x_1^{p i_1} \dots x_n^{p i_n} \\ &= \left(\sum \beta_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \right)^p = (G(x_1, \dots, x_n))^p \end{aligned}$$

where

$$\beta_{i_1 \dots i_n} \in \mathfrak{F}, \quad (\beta_{i_1 \dots i_n})^p = \alpha_{i_1 \dots i_n}.$$

Thus $G(P_1, \dots, P_n) = 0$, $G \neq 0$. But G has lower degree than Q unless Q is constant. Therefore Q is constant, in which case $Q(P_1, \dots, P_n) = 0$ implies $Q = 0$. This completes the proof of (1) and has as an immediate consequence that $\mathfrak{F}(s_1, \dots, s_n)$ and its subfield $\mathfrak{K} = \mathfrak{F}(P_1, \dots, P_n)$ have the same transcendence degree over $\mathfrak{F}$. Hence $\mathfrak{F}(s_1, \dots, s_n)$ is algebraic over $\mathfrak{K}$; in particular, all s_i are algebraic over $\mathfrak{K}$. Thus (2) will be proved as soon as we show the s_i separable over $\mathfrak{K}$ and (3) is then a consequence of a well known theorem in the theory of fields (*e.g.*, see VAN DER WAERDEN, *Moderne Algebra*, Ch. V).

(2): We may assume $p > 0$ and that some s_i , say s_1 , satisfies an irreducible equation over $\mathfrak{K}$ of the form

$$(i) \quad Q_m(P)s_1^{mp} + Q_{m-1}(P)s_1^{(m-1)p} + \dots + Q_1(P)s_1^p + Q_0(P) = 0,$$

where the Q_i are polynomials in n variables with coefficients in $\mathfrak{F}$ and where Q_m is of minimal degree among all leading coefficients in such equations. Differentiating with respect to s_k , $1 \leq k \leq n$, gives

$$\frac{\partial}{\partial s_k} (Q_m(P)) s_1^{m_p} + \cdots + \frac{\partial}{\partial s_k} (Q_1(P)) s_1^p + \frac{\partial}{\partial s_k} (Q_0(P)) = 0.$$

But

$$\frac{\partial}{\partial s_k} (Q_i(P)) = \sum_{i=1}^n \frac{\partial Q_i}{\partial P_i} \frac{\partial P_i}{\partial s_k}.$$

Thus

$$0 = \sum_{j=0}^m \sum_{i=1}^n \frac{\partial Q_i}{\partial P_i} \frac{\partial P_i}{\partial s_k} s_1^{j_p} = \sum_{i=1}^n \left(\sum_{j=0}^m \frac{\partial Q_i}{\partial P_i} s_1^{j_p} \right) \frac{\partial P_i}{\partial s_k},$$

$1 \leq k \leq n$. Since $J \neq 0$,

$$\sum_{j=0}^m \frac{\partial Q_i}{\partial P_i} s_1^{j_p} = 0, \quad 1 \leq i \leq n.$$

It follows from the minimality property of (i) that

$$\frac{\partial Q_i}{\partial P_i} = 0, \quad 0 \leq j \leq m, \quad 1 \leq i \leq n.$$

Thus $Q_i(P) \in \mathfrak{F}[P_1^p, \dots, P_n^p]$, and since $\mathfrak{F}$ is perfect $Q_i(P) = (G_i(P))^p$, $0 \leq j \leq m$, where $G_i(P) \in \mathfrak{F}[P_1, \dots, P_n]$. Hence s_1 satisfies the equation

$$G_m(P) s_1^m + G_{m-1}(P) s_1^{m-1} + \cdots + G_1(P) s_1 + G_0(P) = 0,$$

in contradiction to the choice of (i). This completes the proof.

Lemma 2. *Let $\mathfrak{F}$ be algebraically closed; let $s_1, \dots, s_n$ be algebraically independent indeterminates over $\mathfrak{F}$. Let $P_1, \dots, P_n \in \mathfrak{F}[s_1, \dots, s_n]$, $|\partial P_i / \partial s_i| \neq 0$. Then there exists a polynomial $D(x_1, \dots, x_n)$ in n variables over $\mathfrak{F}$ such that if $\xi_1, \dots, \xi_n \in \mathfrak{F}$ and $D(\xi_1, \dots, \xi_n) \neq 0$, then there exist $\zeta_1, \dots, \zeta_n \in \mathfrak{F}$ such that $\xi_i = P_i(\zeta_1, \dots, \zeta_n)$, $1 \leq i \leq n$.*

Proof. By Lemma 1, $\mathfrak{E} = \mathfrak{F}(s_1, \dots, s_n)$ is a simple algebraic extension $\mathfrak{R}(\theta)$, where $\mathfrak{R} = \mathfrak{F}(P_1, \dots, P_n)$. Hence we can write $s_i = U_i(P, \theta)$, where the $U_i(P, \theta)$ are polynomials in θ with coefficients in $\mathfrak{R}$; i.e., the coefficients are rational functions of $P_1, \dots, P_n$ with coefficients in $\mathfrak{F}$. Let the monic irreducible equation over $\mathfrak{R}$ satisfied by θ be

$$\lambda^m + R_{m-1}(P) \lambda^{m-1} + \cdots + R_1(P) \lambda + R_0(P) = 0,$$

where the R_i are rational functions over $\mathfrak{F}$.

Now there is a polynomial D in n variables such that $D(P_1, \dots, P_n) \cdot R_j(P_1, \dots, P_n) \in \mathfrak{F}[P_1, \dots, P_n]$, $0 \leq j \leq m-1$, and $D(P_1, \dots, P_n) \cdot U_i(P_1, \dots, P_n, \theta) \in \mathfrak{F}[P_1, \dots, P_n, \theta]$, $1 \leq i \leq n$. Now let $\xi_1, \dots, \xi_n \in \mathfrak{F}$,

$D(\xi_1, \dots, \xi_n) \neq 0$. Then all $R_i(\xi_1, \dots, \xi_n)$ and all $U_i(\xi_1, \dots, \xi_n, \theta)$ are defined. Since

$$\lambda^m + R_{m-1}(\xi)\lambda^{m-1} + \dots + R_1(\xi)\lambda + R_0(\xi) = 0$$

is a polynomial equation with coefficients in the algebraically closed field $\mathfrak{F}$, it has a root $\theta_0 \in \mathfrak{F}$.

Now let Q be a polynomial in $n + 1$ variables over $\mathfrak{F}$ such that

$$Q(P_1, \dots, P_n, \theta) = 0.$$

Then

$$Q(P_1, \dots, P_n, \lambda) = S(P_1, \dots, P_n, \lambda) (\lambda^m + R_{m-1}(P)\lambda^{m-1} + \dots + R_0(P)),$$

where $S(P_1, \dots, P_n, \lambda) \in \mathfrak{R}[\lambda]$. Whenever $\xi_1, \dots, \xi_n \in \mathfrak{F}$ are such that all $R_i(\xi)$ are defined we have

$$Q(\xi_1, \dots, \xi_n, \lambda) = S(\xi_1, \dots, \xi_n, \lambda) (\lambda^m + R_{m-1}(\xi)\lambda^{m-1} + \dots + R_0(\xi)),$$

$S(\xi_1, \dots, \xi_n, \lambda) \in \mathfrak{F}[\lambda]$. In particular,

$$\begin{aligned} 0 &= S(\xi_1, \dots, \xi_n, \theta_0) (\theta_0^m + R_{m-1}(\xi)\theta_0^{m-1} + \dots + R_0(\xi)) \\ &= Q(\xi_1, \dots, \xi_n, \theta_0). \end{aligned}$$

That is, whenever $D(\xi) \neq 0$, $Q(P_1, \dots, P_n, \theta) = 0$ implies $Q(\xi_1, \dots, \xi_n, \theta_0) = 0$, or $(\xi_1, \dots, \xi_n, \theta_0)$ is a specialization of $(P_1, \dots, P_n, \theta)$.

Next consider the quantities $P_i - P_i(U(P, \theta))$, $1 \leq i \leq n$. Since $s_i = U_i(P, \theta)$, we have

$$P_i = P_i(s_1, \dots, s_n) = P_i(U_1(P, \theta), \dots, U_n(P, \theta)),$$

or

$$P_i - P_i(U(P, \theta)) = 0.$$

Multiplying by a sufficiently high power of $D(P_1, \dots, P_n)$ gives a polynomial relation $D(P)^h (P_i - P_i(U(P, \theta))) = 0$, $1 \leq i \leq n$. If $D(\xi) \neq 0$, all $U_i(\xi, \theta_0)$ are defined, and we have

$$D(\xi)^h (\xi_i - P_i(U(\xi, \theta_0))) = 0, \quad \text{or} \quad \xi_i = P_i(U(\xi, \theta_0)).$$

Thus we may take $\zeta_i = U_i(\xi, \theta_0)$ to obtain the result.

Now let $\mathfrak{L}$ be a Lie algebra satisfying (i)—(v) of [C] with respect to an abelian Cartan subalgebra $\mathfrak{S}$ of dimension r over an algebraically closed field of characteristic $p \neq 2, 3, 5$. Let $f_1, \dots, f_n$ be any basis for $\mathfrak{L}$ over $\mathfrak{F}$. Let $h_1, \dots, h_r$ be a basis for $\mathfrak{S}$. Let $s_1, \dots, s_n$ be n algebraically independent indeterminates over $\mathfrak{F}$, let $\mathfrak{E} = \mathfrak{F}(s_1, \dots, s_n)$, and let $\mathfrak{L}_{\mathfrak{E}}$ be the Lie algebra over $\mathfrak{E}$ obtained by extending the base field $\mathfrak{F}$ of $\mathfrak{L}$ to $\mathfrak{E}$. If $\alpha_{r+1}, \dots, \alpha_n$ are the non-zero roots of $\mathfrak{L}$ with respect to $\mathfrak{S}$, and if $0 \neq e_{\alpha_i} \in \mathfrak{L}_{\alpha_i}$, $r+1 \leq i \leq n$, then these e_{α_i} , together

with the h_i , $1 \leq i \leq r$, form a basis for $\mathfrak{L}_{\mathfrak{E}}$, as do $f_1, \dots, f_n$. Since $\text{ad}(e_{\alpha_i})$ is a derivation of $\mathfrak{L}$ with $(\text{ad}(e_{\alpha_i}))^4 = 0$, the same holds for $\text{ad}(s_i e_{\alpha_i})$ acting in $\mathfrak{L}_{\mathfrak{E}}$. Since $p \neq 2, 3$, $\exp(\text{ad}(s_i e_{\alpha_i}))$ is defined by its power series and is an automorphism of $\mathfrak{L}_{\mathfrak{E}}$, $r+1 \leq i \leq n$. Set

$$h = \sum_{i=1}^r s_i h_i, \quad \sigma(s) = \prod_{i=r+1}^n \exp(\text{ad}(s_i e_{\alpha_i})).$$

Then $h\sigma(s) = \sum_{i=1}^n P_i(s) f_i$, where $P_i(s) \in \mathfrak{F}[s_1, \dots, s_n]$.

Lemma 3. *The Jacobian $|\partial P_i / \partial s_i| \neq 0$.*

Proof. Denote the Jacobian by J . If the basis $f_1, \dots, f_n$ is replaced by a new basis $g_1, \dots, g_n$ for $\mathfrak{L}$ over $\mathfrak{F}$, we have $f_i = \sum \tau_{ij} g_j$, $\tau_{ij} \in \mathfrak{F}$, where the matrix (τ) is non-singular. Thus

$$h\sigma(s) = \sum_{i=1}^n P_i(s) \sum_{j=1}^n \tau_{ij} g_j = \sum_{i=1}^n \sum_{j=1}^n P_i(s) \tau_{ij} g_j = \sum_{j=1}^n Q_j(s) g_j,$$

where

$$\frac{\partial Q_j}{\partial s_k} = \sum_{i=1}^n \frac{\partial P_i}{\partial s_k} \tau_{ij},$$

or

$$\left(\frac{\partial Q}{\partial s} \right) = (\tau)' \left(\frac{\partial P}{\partial s} \right), \quad \text{and} \quad K = \left| \frac{\partial Q}{\partial s} \right| = |(\tau)| \cdot J.$$

Since $|(\tau)| \neq 0$, $J = 0$ if and only if $K = 0$.

In particular, we can take the basis g_i to be $g_i = h_i$, $1 \leq j \leq r$; $g_i = e_{\alpha_i}$, $r+1 \leq j \leq n$. Now we evaluate the Jacobian $K(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$, where $\lambda_1, \dots, \lambda_r$ are elements of $\mathfrak{F}$ such that $\alpha(\lambda_1 h_1 + \dots + \lambda_r h_r) \neq 0$ for all roots α relative to $\mathfrak{S}$. That such elements exist follows from the fact that

$$\prod_{j=r+1}^n \alpha_j(s_1 h_1 + \dots + s_r h_r) = \prod_{j=r+1}^n (s_1 \alpha_j(h_1) + \dots + s_r \alpha_j(h_r)),$$

as a product of non-zero polynomials, is a non-zero polynomial in $s_1, \dots, s_r$ over an infinite field. First note that

$$\sum_{i \neq j} \lambda_i h_i + s_j h_j \sigma(0, \dots, 0) = \sum_{i \neq j} \lambda_i h_i + s_j h_j.$$

Thus the coefficient $Q_i(\lambda_1, \dots, s_j, \dots, \lambda_r, 0, \dots, 0)$ of g_i is λ_i if $i \leq r$, $i \neq j$; 0 if $i > r$; and s_j if $i = j$. Hence

$$\frac{\partial Q_i}{\partial s_j}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) = \delta_{ij}, \quad 1 \leq j \leq r.$$

For $j > r$,

$$\begin{aligned}\sum \lambda_i h_i \sigma(0, \dots, s_j, \dots, 0) &= (\sum \lambda_i h_i) \exp(\operatorname{ad}(s_j e_{\alpha_j})) \\ &= h_0 - s_j \alpha_j(h_0) e_{\alpha_j},\end{aligned}$$

where

$$h_0 = \sum \lambda_i h_i.$$

Hence the coefficient Q_i of g_i is $-s_j \alpha_j(h_0)$, while $Q_i = 0$ for all other $i > r$ and $Q_i = \lambda_i$ for $i \leq r$. Therefore

$$\frac{\partial Q_i}{\partial s_j}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) = -\alpha_j(h_0) \delta_{ij}, \quad j > r,$$

and

$$K(\lambda_1, \dots, \lambda_r, 0, \dots, 0) = \prod_{i=r+1}^n (-\alpha_i(h_0)) \neq 0.$$

Thus $K \neq 0$, and the lemma is proved.

Lemma 4. Let $\mathfrak{L}$, (f_i) , (h_i) , (e_{α_i}) be as above. Then there is a polynomial D in n variables over $\mathfrak{F}$ such that whenever $D(\xi_1, \dots, \xi_n) \neq 0$ ($\xi_i \in \mathfrak{F}$) then

$$\sum_{i=1}^n \xi_i f_i = \left(\sum_{i=1}^r \zeta_i h_i \right) \sigma(\zeta)$$

for some $\zeta_1, \dots, \zeta_n \in \mathfrak{F}$.

This lemma is an immediate consequence of Lemmas 2 and 3.

A *regular element* of $\mathfrak{L}$ is an element $b \in \mathfrak{L}$ such the characteristic root 0 of $\operatorname{ad}(b)$ has minimal multiplicity.

Lemma 5. Denote by $\mathfrak{A}$ the set of elements of $\mathfrak{L}$ of the form $h\sigma(\zeta)$, where $h \in \mathfrak{S}$ and $\zeta_{r+1}, \dots, \zeta_n \in \mathfrak{F}$. Then there is a polynomial Q in n variables over $\mathfrak{F}$ such that if $Q(\xi_1, \dots, \xi_n) \neq 0$ then $\sum \xi_i f_i$ is a regular element and lies in $\mathfrak{A}$.

Proof. If $x_1, \dots, x_n$ are independent indeterminates over $\mathfrak{F}$, the characteristic polynomial of $\operatorname{ad}(\sum x_i f_i)$ in $\mathfrak{L}_{\mathfrak{F}(x_1, \dots, x_n)}$ (the *Killing polynomial*) is of the form

$$\lambda^n + Q_{n-1}(x_1, \dots, x_n) \lambda^{n-1} + \dots + Q_0(x_1, \dots, x_n) \lambda^k,$$

where Q_k is a non-zero polynomial. Thus if we set $Q = DQ_k$, where D is as in Lemma 4, $Q(\xi_1, \dots, \xi_n) \neq 0$ implies that $Q_k(\xi_1, \dots, \xi_n) \neq 0$, hence that $\sum \xi_i f_i$ is regular, and it also implies $D(\xi_1, \dots, \xi_n) \neq 0$, hence that $\sum \xi_i f_i \in \mathfrak{A}$.

Let $\mathfrak{G}$ be the group of automorphisms of $\mathfrak{L}$ generated by the automorphisms $\sigma(\zeta)$, $\zeta \in \mathfrak{F}$, formed for all abelian Cartan subalgebras $\mathfrak{S}$ of $\mathfrak{L}$ relative to which the properties (iii)—(v) of [C] are satisfied. We shall call $\mathfrak{G}$ the group of *invariant automorphisms* of $\mathfrak{L}$.

Theorem. *Let $\mathfrak{S}_1, \mathfrak{S}_2$ be any two abelian Cartan subalgebras of $\mathfrak{L}$ relative to which (iii)—(v) are satisfied. Then there is an invariant automorphism σ such that $\mathfrak{S}_2 = \mathfrak{S}_1\sigma$.*

Proof. We have seen that the automorphisms of the form $\sigma(\zeta)$ can be defined relative to each Cartan subalgebra of the type under consideration. Let $\mathfrak{A}_1, \mathfrak{A}_2$, respectively, be the set of transforms of elements of $\mathfrak{S}_1$ under mappings $\sigma(\zeta)$ relative to $\mathfrak{S}_1$ and the set of transforms of $\mathfrak{S}_2$ under mappings $\sigma(\xi)$ relative to $\mathfrak{S}_2$. We can apply Lemma 5 to find polynomials Q_1, Q_2 such that $Q_i(\xi) \neq 0$ implies $\sum \xi_i f_i$ is in $\mathfrak{A}_i$ and is regular, $j = 1, 2$. Then if $Q = Q_1 Q_2$, $Q(\xi) \neq 0$ implies $\sum \xi_i f_i$ is in $\mathfrak{A}_1 \cap \mathfrak{A}_2$ and is regular.

Thus there are elements $h_1 \in \mathfrak{S}_1, h_2 \in \mathfrak{S}_2$, and $\sigma_1, \sigma_2 \in \mathfrak{G}$ such that $\sum \xi_i f_i = h_1 \sigma_1 = h_2 \sigma_2$ is regular. Since σ_1 and σ_2 are automorphisms, h_1 and h_2 are regular. Setting $\sigma = \sigma_1 \sigma_2^{-1}$, we have $h_2 = h_1 \sigma \in \mathfrak{S}_1 \sigma$. Since there are elements of $\mathfrak{S}_2$ at which no non-zero roots with respect to $\mathfrak{S}_2$ vanish, and since h_2 is regular, the subspace of $\mathfrak{L}$ belonging to the characteristic root 0 of $\text{ad}(h_2)$ is precisely $\mathfrak{S}_2$. A similar assertion holds for h_1 and $\mathfrak{S}_1$, therefore for $h_1 \sigma = h_2$ and $\mathfrak{S}_1 \sigma$. Hence $\mathfrak{S}_2 = \mathfrak{S}_1 \sigma$.

Using the results of the first five sections of [8], namely, that all the conditions required for the above theorem are satisfied by every Cartan subalgebra, we have the following corollaries.

Corollary 1. *If $\mathfrak{L}$ is a restricted Lie algebra with restricted representation with non-degenerate trace form over an algebraically closed field of characteristic $p > 5$, then any two Cartan subalgebras of $\mathfrak{L}$ are conjugate under an invariant automorphism.*

Corollary 2. *If $\mathfrak{L}$ is a Lie algebra with non-degenerate Killing form over an algebraically closed field of characteristic $\neq 2, 3, 5$, then any two Cartan subalgebras of $\mathfrak{L}$ are conjugate under an invariant automorphism.*

With the work of W. H. MILLS on characteristic 5 appearing elsewhere in THIS JOURNAL, it appears that these results can be extended to that case. A recently discovered example of RICHARD BLOCK proves that they cannot be expected to hold even for all simple Lie algebras of characteristic p over algebraically closed fields. In contrast to this situation we have the result of CHEVALLEY that all Cartan subalgebras are conjugate in every Lie algebra over an algebraically closed field of characteristic zero.

4. Isomorphism questions. Since abelian Cartan subalgebras satisfying (i)—(v) of [C] over an algebraically closed field are now known to have invariant dimension, one can simply check dimensions of Cartan subalgebras against dimensions of the algebras to note that there can be no isomorphisms among the algebras given as examples in [8], except possibly between algebras with fundamental systems of types B_r and C_r ($r \geq 3$) and between the algebra with

fundamental system of type E_6 and one with fundamental system of type B_6 or C_6 (see table below).

<i>Fundamental system</i>	<i>dim. $\mathfrak{L}$</i>	<i>dim. $\mathfrak{L}$</i>
A_r ($r \geq 1, p \nmid (r+1)$)	r	$r^2 + 2r$
A_r ($p \mid (r+1)$)	$r-1$	$r^2 + 2r - 1$
B_r ($r \geq 3$)	r	$2r^2 + r$
C_r ($r \geq 2$)	r	$2r^2 + r$
D_r ($r \geq 4$)	r	$2r^2 - r$
E_r ($r = 6, 7, 8$)	r	78, 133, 248
F_r	4	52
G_2	2	14

In [6] JACOBSON has shown that there is no isomorphism between the algebras of type B_r and those of type C_r . Finally, we note that if an algebra $\mathfrak{L}$ of type E_6 is isomorphic to one of type B_6 (or C_6), there is an automorphism of $\mathfrak{L}$ mapping a Cartan subalgebra with fundamental system of roots of type E_6 onto one with fundamental system of type $B_6(C_6)$ and giving a 1-1 mapping between the sets of roots with the property of carrying sums of roots into the sums of their images. But, if one checks the complete set of roots for a fundamental system of type E_6 as given in [8], he finds that for no two roots α, β , $\alpha \neq -\beta$, are $\alpha + \beta$ and $\alpha + 2\beta$ both roots. For both B_6 and C_6 there are such roots; in the notation of [C], we can take $\alpha = \alpha_5, \beta = \alpha_6$ in B_6 ; $\alpha = \alpha_6, \beta = \alpha_5$ in C_6 . Thus the absence of isomorphisms between algebras of different types is completely established. (This same argument was used in [8], but the result of §3 is needed to make it valid.) If the base field $\mathfrak{F}$ is not algebraically closed, we could extend any isomorphism between algebras $\mathfrak{L}, \mathfrak{L}'$ of different types to an isomorphism between $\mathfrak{L}_{\overline{\mathfrak{F}}}$ and $\mathfrak{L}'_{\overline{\mathfrak{F}}}$, where $\overline{\mathfrak{F}}$ is the algebraic closure of $\mathfrak{F}$. Since $\mathfrak{L}_{\overline{\mathfrak{F}}}$ and $\mathfrak{L}'_{\overline{\mathfrak{F}}}$ have the same respective types as $\mathfrak{L}$ and $\mathfrak{L}'$, there can be no isomorphism between $\mathfrak{L}$ and $\mathfrak{L}'$.

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*The letter [C] refers to the paper "On Lie algebras of classical type" by W. H. MILLS & G. B. SELIGMAN appearing elsewhere in THIS JOURNAL. The numbers in brackets refer to the bibliography at the end of [C].