The Growth and Decay of Sonic Discontinuities in Ideal Gases

T. Y. THOMAS

1. Introduction. We shall be concerned in this paper¹ with an ideal gas, devoid of viscosity and thermal conductivity, whose behavior is governed by the following system of non-linear differential equations

$$(1) \quad \rho \frac{\partial v_i}{\partial t} + \rho_{,i} + \rho v_k v_{i,k} = 0,$$

$$(2) \quad \frac{\partial \rho}{\partial t} + \rho_{,k} v_k + \rho v_{k,k} = 0,$$

$$(3) \quad \frac{\partial \rho}{\partial t} - \rho v_k \frac{\partial v_k}{\partial t} - \rho v_i v_k v_{i,k} + \gamma p v_{k,k} = 0,$$

where t denotes the time, p the pressure, ρ the density, v_i the components of velocity of the gas and γ is the gas constant. The above equations are referred to a system of rectangular coordinates x and the comma indicates partial differentiation with respect to these coordinates in the usual manner. Relations (1) are the equations of motion and the relation (2) is the equation of continuity. The entropy condition² is expressed by the equation (3). In the special case for which the motion is stationary this latter equation gives the well known condition that the entropy is constant along stream lines although it may vary from stream line to stream line.³

A moving surface $\Sigma(t)$ is said to be singular of order 1 relative to the pressure, density and velocity if these quantities are continuous across the surface but at least one of the first derivatives of p , ρ , or v_i with respect to the coordinates

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² See, T. Y. THOMAS, *The fundamental hydrodynamical equations and shock conditions for gases*, Math. Mag. 22 (1949) pp. 169-189.

³ In the stationary case the relation between the pressure and density of the gas is given by $p = N\rho^\gamma$ where N is a function which is constant along stream lines. A specialization of this case arises when we assume that N is constant throughout the gas, or, more generally, if we assume an equation of state of the form $p = f(\rho)$ relating the pressure and the density. It may be noted in this connection that the method of this paper is applicable, with trivial modifications, when the equation (3) is replaced by the above equation of state provided the function $f(\rho)$ is continuous and twice differentiable.

x_i is discontinuous. Such a surface will be called a *sonic wave of order 1* or simply a *sonic wave* in the present gas dynamical problem and the associated discontinuity in the derivatives will be referred to as a *sonic discontinuity*.⁴

Denoting the discontinuity or "jump" in a quantity by means of a bracket [] we thus have

$$(4) \quad [p] = [\rho] = [v_i] = 0,$$

over $\Sigma(t)$. Also, under the proper assumptions of regularity of the surface $\Sigma(t)$ and the existence of limiting values of the functions and their derivatives as one approaches this surface from each side, it is well known that

$$(5) \quad [v_{i,i}] = \lambda_i v_i ; \quad \left[\frac{\partial v_i}{\partial t} \right] = -G \lambda_i ,$$

$$(6) \quad [p_{,i}] = \xi \nu_i ; \quad \left[\frac{\partial p}{\partial t} \right] = -G \xi ,$$

$$(7) \quad [\rho_{,i}] = \zeta \nu_i ; \quad \left[\frac{\partial \rho}{\partial t} \right] = -G \zeta ,$$

where the quantities ξ , ζ and λ_i are suitable functions defined over the surface $\Sigma(t)$ and G is the *coordinate velocity* of the surface. The two quantities ξ and ζ are scalars on Σ and the quantities λ_i can be replaced by a single scalar λ since it is shown in §2 that $\lambda_i = \lambda \nu_i$ where the ν_i are the components of the unit normal vector ν to $\Sigma(t)$.

In §2 we have derived certain relations between the scalars λ , ξ and ζ which are necessary for the subsequent discussion and in this connection we have obtained the well known expression for the velocity of propagation of the sonic wave. The discussion following §2 is based on the assumption that the wave is propagated into a gas which is at rest and uniform, *i.e.* the pressure and density are constant. It is first shown for this case in §3 that the successive positions of the sonic wave $\Sigma(t)$ form a family of parallel surfaces. The position of the sonic wave Σ at time $t > t_0$ is therefore determined by a simple geometrical construction when the position of the wave is known at time t_0 .

From the relations (5), (6) and (7) it is seen that the magnitude of the sonic discontinuity is determined by the above scalars λ , ξ and ζ . Now as the sonic wave progresses the values of these scalars on $\Sigma(t)$ will either approach zero, in which case the sonic discontinuity will be said to *decay* or to be *damped out*, or they will become infinite. In the latter case the sonic wave will no longer be tenable and must terminate in a shock in the sense that condition (4) will fail to be satisfied. This is discussed in §8 for the special case of plane and spherical waves. In general, the question of growth or decay of sonic discontinuities is determined by the algebraic sign of one of the scalars λ , ξ or λ and by the signs

⁴ The term *sonic discontinuity* is used in the above sense by R. COURANT & K. O. FRIEDRICHS, *Supersonic Flow and Shock Waves*, Interscience Publishers, N. Y. (1948) p. 116.

of the Gaussian and mean curvatures of the surface Σ at a particular instant t_0 . We have treated the general situation in §9 and have there tabulated the various possible cases.

Finally in §10 we have introduced a non-dimensional quantity of the form Qdt which gives the relative increase in the pressure which occurs, at a point P , within the interval of time dt following the advent of the sonic wave at this point. While it would appear that the quantity Qdt expresses a significant property of the sonic discontinuity we have merely called it the *pressure increment* for lack of a more suitable designation. It is of course possible to define density and velocity increments in a corresponding manner. The growth or decay of these increments as the wave progresses in the gas is determined directly from the appropriate equations of this paper.

2. Velocity of propagation of the sonic wave. From the equations (1), (2), (3) and the compatibility conditions (5), (6), (7) we find

$$(8) \quad \rho(v_n - G)\lambda_i + \xi\nu_i = 0,$$

$$(9) \quad (v_n - G)\zeta + \rho\lambda_i\nu_i = 0,$$

$$(10) \quad \rho(G - v_n)\lambda_i\nu_i - G\xi + \gamma p\lambda_i\nu_i = 0,$$

where v_n denotes the component of velocity in the direction of the unit normal ν to the surface $\Sigma(t)$. Now multiply (8) by ν_i and again by ν_i and in each case sum on the repeated index i to obtain

$$(11) \quad \begin{cases} \rho(v_n - G)\lambda_i\nu_i + \xi v_n = 0, \\ \rho(v_n - G)\lambda_i\nu_i + \xi = 0. \end{cases}$$

Add (10) and the first relation (11). This gives

$$(12) \quad (v_n - G)\xi + \gamma p\lambda_i\nu_i = 0.$$

Then, multiplying the second equation (11) by γp and (12) by $\rho(v_n - G)$ and subtracting the resulting equations, we have

$$(13) \quad \{(G - v_n)^2 - c^2\}\xi = 0, \quad c^2 = \gamma p/\rho.$$

We assume that the *speed* s of the sonic wave, defined by $s = G - v_n$, is different from zero. Then, if $\xi = 0$ on $\Sigma(t)$, it follows from (8) that $\lambda_i = 0$. Hence $\zeta = 0$ from (9). But this means that the sonic wave is not a singular surface of order 1 as postulated. Hence $\xi \neq 0$ and we have from (13) that

$$(14) \quad (G - v_n)^2 = \gamma p/\rho.$$

Let us now observe that the equations (8) and (9) can be written

$$(15) \quad \lambda_i = \lambda\nu_i, \quad \lambda = \frac{\xi}{\rho(G - v_n)},$$

$$(16) \quad \zeta = \frac{\rho\lambda}{G - v_n} = \frac{\xi}{(G - v_n)^2}.$$

Thus the quantities λ_i can effectively be replaced by the single scalar λ .

It will henceforth be assumed that the sonic wave is propagated into a gas at rest and within which the pressure p and the density ρ are constant. For this case v_i vanishes on $\Sigma(t)$ and hence the speed of propagation of the wave is a constant given by $G^2 = \gamma p/\rho$ from (14). Use of this relation will be made without especial mention in the following discussion. Also the above relations (15) and (16) between the scalars λ , ξ and ζ become

$$(17) \quad \xi = \rho G \lambda; \quad \zeta = \rho \lambda / G.$$

3. Propagation into a uniform gas at rest. The discussion in this and the following sections will depend essentially on the results obtained in our previous communication in THIS JOURNAL under the title *Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics*.⁵ One of the relations derived in this article is

$$(18) \quad \frac{\delta \nu_i}{\delta t} = -g^{\alpha\beta} G_{,\alpha} x_{i,\beta},$$

in which the left members are the δ time derivatives of the components ν_i of the unit normal vector ν . The quantities $g^{\alpha\beta}$ are the contravariant components of the fundamental metric tensor of the sonic surface $\Sigma(t)$ which we consider to be defined by the differentiable functions $x_i(u_1, u_2, t)$ of the two curvilinear coordinates u_1, u_2 and the time t .

In equations (18) and subsequent equations Latin indices have the range 1, 2, 3 and Greek indices the range 1, 2 corresponding to the three space coordinates x and the two curvilinear or surface coordinates u . The summation convention applies subject to these ranges of indices and the comma in this paper denotes, unless otherwise specified, partial differentiation with respect to the coordinates x or u as indicated by the Latin or Greek letters following the comma.

Under the assumption made at the end of §2 we have $G = \text{const.}$ and it follows from (18) that $\delta \nu_i / \delta t = 0$. Hence the normal trajectories to the family of surfaces $\Sigma(t)$ in the three dimensional space of the coordinates x_i must be *straight lines*. But under this condition it can be shown that the surfaces $\Sigma(t)$ form a family of parallel surfaces.⁶ Hence the sonic wave surface $\Sigma(t)$ for $t > t_0$ can be constructed from the surface $\Sigma(t_0)$ by laying off equal intervals of length along the normals in the direction of the propagation.

⁵ Jour. of Math. and Mech. **6** (1957) pp. 311-322.

⁶ See, E. P. LANE, *Metric Differential Geometry of Curves and Surfaces*, Univ. of Chicago Press, Chicago (1940) pp. 208-209.

4. Conditions of compatibility of the second order. Since the quantities p , ρ and v_i are continuous across the sonic wave the compatibility conditions (29), (42) and (43) of the above mentioned article⁵ are applicable. When we furthermore make use of the fact that $G = \text{const.}$ we find that the conditions of compatibility of the second order for the velocity components v_i are given by equations of the form

$$(19) \quad [v_{i,jk}] = \bar{\lambda}_i \nu_j \nu_k + g^{\alpha\beta} \lambda_{i,\alpha} (\nu_j x_{k,\beta} + \nu_k x_{j,\beta}) - \lambda_i g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{j,\beta} x_{k,\tau},$$

$$(20) \quad \left[\frac{\partial^2 v_i}{\partial x_j \partial t} \right] = \left(-G \bar{\lambda}_i + \frac{\delta \lambda_i}{\delta t} \right) \nu_j - G g^{\alpha\beta} \lambda_{i,\alpha} x_{j,\beta},$$

$$(21) \quad \left[\frac{\partial^2 v_i}{\partial t^2} \right] = G^2 \bar{\lambda}_i - 2G \frac{\delta \lambda_i}{\delta t},$$

in which the quantities $\bar{\lambda}_i$ are new functions defined on the surface $\Sigma(t)$ and the $b_{\alpha\sigma}$ are the components of the second fundamental form of the surface. We have called the relations (19) the *geometrical conditions of compatibility of the second order* and the relations (20) and (21) the *kinematical conditions of compatibility of the second order* for the quantities v_i . Corresponding conditions of compatibility for the functions p and ρ are given by

$$(22) \quad [p_{,ij}] = \bar{\xi} \nu_i \nu_j + g^{\alpha\beta} \xi_{,\alpha} (\nu_i x_{j,\beta} + \nu_j x_{i,\beta}) - \xi g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{i,\beta} x_{j,\tau},$$

$$(23) \quad \left[\frac{\partial^2 p}{\partial x_i \partial t} \right] = \left(-G \bar{\xi} + \frac{\delta \xi}{\delta t} \right) \nu_i - G g^{\alpha\beta} \xi_{,\alpha} x_{i,\beta},$$

$$(24) \quad \left[\frac{\partial^2 p}{\partial t^2} \right] = G^2 \bar{\xi} - 2G \frac{\delta \xi}{\delta t},$$

and,

$$(25) \quad [\rho_{,ij}] = \bar{\zeta} \nu_i \nu_j + g^{\alpha\beta} \zeta_{,\alpha} (\nu_i x_{j,\beta} + \nu_j x_{i,\beta}) - \zeta g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{i,\beta} x_{j,\tau},$$

$$(26) \quad \left[\frac{\partial^2 \rho}{\partial x_i \partial t} \right] = \left(-G \bar{\zeta} + \frac{\delta \zeta}{\delta t} \right) \nu_i - G g^{\alpha\beta} \zeta_{,\alpha} x_{i,\beta},$$

$$(27) \quad \left[\frac{\partial^2 \rho}{\partial t^2} \right] = G^2 \bar{\zeta} - 2G \frac{\delta \zeta}{\delta t},$$

where the $\bar{\xi}$ and $\bar{\zeta}$ are two additional functions defined on the surface $\Sigma(t)$.

We shall find that it will be helpful in the later work to have certain special relations which can be derived from the above compatibility conditions. These relations will be obtained in the following section.

5. Special relations. From the relation (15) we have

$$(28) \quad \lambda_k x_{k,\beta} = \lambda \nu_k x_{k,\beta} = 0,$$

since the $x_{k,\beta}$ are the components of vectors tangent to the surface $\Sigma(t)$. Likewise we find that

$$(29) \quad \begin{aligned} \lambda_{k,\alpha} x_{k,\beta} &= \lambda_{,\alpha} \nu_k x_{k,\beta} + \lambda \nu_{k,\alpha} x_{k,\beta} = \lambda \nu_{k,\alpha} x_{k,\beta} \\ &= -\lambda \nu_k x_{k,\alpha\beta} = -\lambda \nu_k \nu_k b_{\alpha\beta} = -\lambda b_{\alpha\beta}, \end{aligned}$$

where the $x_{k,\alpha\beta}$ are the components of the second covariant derivative based on the metric of the surface $\Sigma(t)$. Contracting the indices i and j in (19) and using (28) and (29) we now obtain

$$(30) \quad [v_{i,ik}] = (\bar{\lambda}_i \nu_i) \nu_k + g^{\alpha\beta} \lambda_{i,\alpha} \nu_i x_{k,\beta} - 2\lambda \Omega \nu_k,$$

where Ω is the mean curvature of the surface $\Sigma(t)$. But

$$(31) \quad \lambda_{i,\alpha} \nu_i = (\lambda_i \nu_i)_{,\alpha} - \lambda_i \nu_{i,\alpha} = \lambda_{,\alpha} - \lambda \nu_{i\nu_{i,\alpha}},$$

when we make use of the first relation (15). Now, since ν is a unit vector, the last term in (31) vanishes and hence (30) becomes

$$(32) \quad [v_{i,ik}] = (\bar{\lambda}_i \nu_i - 2\lambda \Omega) \nu_k + g^{\alpha\beta} \lambda_{i,\alpha} x_{k,\beta}.$$

If we multiply (32) by ν_k and sum on the repeated index k we obtain

$$(33) \quad [v_{i,ik}] \nu_k = \bar{\lambda}_i \nu_i - 2\lambda \Omega.$$

Other relations involving multiplication by the components ν_k are obtained from (20), (22), (23) and (26). Thus we have

$$(34) \quad \left[\frac{\partial^2 v_i}{\partial x_j \partial t} \right] \nu_i = -G \bar{\lambda}_i + \frac{\delta \lambda_i}{\delta t},$$

$$(35) \quad [p_{,ij}] \nu_i = \bar{\xi} \nu_i + g^{\alpha\beta} \xi_{,\alpha} x_{i,\beta},$$

$$(36) \quad \left[\frac{\partial^2 p}{\partial x_i \partial t} \right] \nu_i = -G \bar{\xi} + \frac{\delta \xi}{\delta t},$$

$$(37) \quad \left[\frac{\partial^2 \rho}{\partial x_i \partial t} \right] \nu_i = -G \bar{\zeta} + \frac{\delta \zeta}{\delta t}.$$

Let us now denote by P_1 and P_2 the values of a quantity P on sides 1 and 2 respectively of the surface $\Sigma(t)$. For the discontinuity in the product PQ we shall then have

$$[PQ] = P_2 Q_2 - P_1 Q_1 = (P_2 - P_1) Q_2 + P_1 Q_2 - P_1 Q_1 = Q_2 [P] + P_2 [Q] - [P][Q].$$

If the gas is at rest on the side 2 of $\Sigma(t)$ and if the pressure and density are constant on this side of the surface, as assumed at the end of §2, then

$$(38) \quad [PQ] = -[P][Q],$$

provided the quantities P and Q involve derivatives of the pressure p or density ρ as a factor or have, as a factor, the velocity components v_i or their derivatives.

Hence from (38) and the conditions (5), (6) and (7) we can deduce the following relations

$$\begin{aligned}
 [\rho, v_{i,i}] &= -\zeta v_i \lambda_i v_i = -\zeta \lambda v_i, \\
 [\rho, v_{i,i}] &= -\zeta v_i \lambda_i v_i = -\zeta \lambda v_i, \\
 [p, v_{i,i}] &= -\xi v_i \lambda_i v_i = -\xi \lambda v_i, \\
 [v_{k,i} v_{i,k}] &= -\lambda_k v_i \lambda_i v_k = -\lambda^2 v_i v_i, \\
 \left[\rho, v_i \frac{\partial v_i}{\partial t} \right] &= G \zeta v_i \lambda_i = G \zeta \lambda v_i, \\
 \left[v_{i,i} \frac{\partial v_i}{\partial t} \right] &= G \lambda v_i \lambda_i = G \lambda^2 v_i,
 \end{aligned}$$

in which use has been made of the relation (15). From the above relations we obtain immediately the following relations which will have direct application in the next section, namely

$$\begin{aligned}
 [\rho, v_{i,i}] v_i &= [\rho, v_{i,i}] v_i = -\zeta \lambda; & [p, v_{i,i}] v_i &= -\xi \lambda, \\
 [v_{k,i} v_{i,k}] v_i v_i &= -\lambda^2; & \left[\rho, v_i \frac{\partial v_i}{\partial t} \right] v_i v_i &= G \zeta \lambda; & \left[v_{i,i} \frac{\partial v_i}{\partial t} \right] v_i &= G \lambda^2.
 \end{aligned}$$

6. Application of the compatibility conditions. Differentiating the equations (1), (2) and (3) with respect to the coordinates x and making use of the fact that $v_i = 0$ on $\Sigma(t)$, we now find that

$$(39) \quad \left[\rho, v_i \frac{\partial v_i}{\partial t} \right] + \rho \left[\frac{\partial^2 v_i}{\partial x_i \partial t} \right] + [p, v_i] + \rho [v_{k,i} v_{i,k}] = 0,$$

$$(40) \quad \left[\frac{\partial^2 \rho}{\partial x_i \partial t} \right] + [\rho, v_{i,i}] + [\rho, v_{i,i}] + \rho [v_{i,i}] = 0,$$

$$(41) \quad \left[\frac{\partial^2 p}{\partial x_i \partial t} \right] - \rho \left[v_{i,i} \frac{\partial v_i}{\partial t} \right] + \gamma [p, v_{i,i}] + \gamma p [v_{i,i}] = 0.$$

Next multiply (39) by v_i and sum on the repeated index j . Then, when we make use of (34), (35) and the proper relations at the end of §5, we obtain

$$(42) \quad \rho \frac{\delta \lambda_i}{\delta t} + \bar{\xi} v_i - \rho G \bar{\lambda}_i + (G \zeta - \rho \lambda) \lambda v_i + g^{\alpha\beta} \xi_{,\alpha} x_{i,\beta} = 0.$$

But the parenthesis expression in (42) is equal to zero from the second equation (17) and, furthermore, we have

$$\frac{\delta \lambda_i}{\delta t} = \frac{\delta}{\delta t} (\lambda v_i) = \frac{\delta \lambda}{\delta t} v_i,$$

since $\delta\nu_i/\delta t$ vanishes as shown in §3. Hence (42) becomes

$$(43) \quad \left(\rho \frac{\delta\lambda}{\delta t} + \bar{\xi} \right) \nu_i - \rho G \bar{\lambda}_i + g^{\alpha\beta} \xi_{,\alpha} x_{i,\beta} = 0.$$

When we multiply (43) by ν_i we now obtain

$$(44) \quad \rho \frac{\delta\lambda}{\delta t} + \bar{\xi} - \rho G \bar{\lambda}_{,i} \nu_i = 0.$$

In a similar manner if we multiply (40) and (41) by ν_j and sum on the repeated index j we are led to the following relations

$$(45) \quad \frac{\delta\zeta}{\delta t} = 2\zeta\lambda + 2\rho\lambda\Omega + G\bar{\zeta} - \rho\bar{\lambda}_{,i}\nu_i,$$

$$(46) \quad \frac{\delta\xi}{\delta t} = (\gamma + 1)\xi\lambda + 2\gamma\rho\lambda\Omega + G\bar{\xi} - \gamma\rho\bar{\lambda}_{,i}\nu_i.$$

7. The fundamental differential equations. Taking the δ time derivative of the relations (17) we have

$$(47) \quad \frac{\delta\xi}{\delta t} = \rho G \frac{\delta\lambda}{\delta t}; \quad \frac{\delta\zeta}{\delta t} = \frac{\rho}{G} \frac{\delta\lambda}{\delta t}.$$

Let us now eliminate the time derivatives in (47) by means of the equations (44), (45) and (46). We thus obtain

$$(48) \quad (\gamma + 1)\xi\lambda + 2\gamma\rho\lambda\Omega + G\bar{\xi} - \gamma\rho\bar{\lambda}_{,i}\nu_i = -G(\bar{\xi} - \rho G \bar{\lambda}_{,i}\nu_i),$$

$$(49) \quad 2\zeta\lambda + 2\rho\lambda\Omega + G\bar{\zeta} - \rho\bar{\lambda}_{,i}\nu_i = -\frac{1}{G}(\bar{\xi} - \rho G \bar{\lambda}_{,i}\nu_i).$$

But (48) can be written in the form

$$(50) \quad G\bar{\xi} - \gamma\rho\bar{\lambda}_{,i}\nu_i = -\left(\frac{\gamma + 1}{2}\right)\xi\lambda - \gamma\rho\lambda\Omega,$$

or

$$(51) \quad \bar{\xi} - \rho G \bar{\lambda}_{,i}\nu_i = -\left(\frac{\gamma + 1}{2G}\right)\xi\lambda - \rho G \lambda\Omega.$$

We can now use the form (51) of this relation to eliminate the parenthesis expression in the right member of (49). This elimination gives

$$(52) \quad G\bar{\zeta} - \rho\bar{\lambda}_{,i}\nu_i = \left(\frac{\gamma + 1}{2G^2}\right)\xi\lambda - 2\zeta\lambda - \rho\lambda\Omega.$$

Let us now eliminate the last two terms in (44) by the substitution (51) and also the last two terms in the right members of (45) and (46) by means of (52) and (50) respectively. This gives us the following relations

$$(53) \quad \frac{\delta\lambda}{\delta t} = \left(\frac{\gamma+1}{2\rho G}\right)\xi\lambda + G\Omega\lambda,$$

$$(54) \quad \frac{\delta\zeta}{\delta t} = \left(\frac{\gamma+1}{2G^2}\right)\xi\lambda + \rho\Omega\lambda,$$

$$(55) \quad \frac{\delta\xi}{\delta t} = \left(\frac{\gamma+1}{2}\right)\xi\lambda + \gamma p\Omega\lambda.$$

Using (17) we can write the equations (53), (54) and (55) in the forms

$$(56) \quad \frac{\delta\lambda}{\delta t} = \left(\frac{\gamma+1}{2}\right)\lambda^2 + G\Omega\lambda,$$

$$(57) \quad \frac{\delta\zeta}{\delta t} = \left(\frac{\gamma+1}{2\rho}\right)G\zeta^2 + G\Omega\zeta,$$

$$(58) \quad \frac{\delta\xi}{\delta t} = \left(\frac{\gamma+1}{2\rho G}\right)\xi^2 + G\Omega\xi.$$

The equations (56), (57) and (58) are the fundamental differential equations for the variation of the quantities λ , ζ and ξ along the normal trajectories of the family of sonic wave surfaces $\Sigma(t)$ when these surfaces are propagated into a uniform gas at rest. We recall that in these equations the quantity ρ is the constant density of the uniform gas and G is the constant velocity of propagation of the sonic wave. The mean curvature Ω will of course, in general, vary along the normal trajectories. In the remainder of this article we shall show how the equations (56), (57) and (58) can be used to predict the growth or the decay of the sonic discontinuity associated with the wave surface $\Sigma(t)$.

Let $\Sigma(t_0)$ represent the sonic wave surface at the time t_0 and denote by σ the distance measured from $\Sigma(t_0)$ along the normal trajectories to the family of surfaces $\Sigma(t)$ in the direction of propagation. Then $\sigma = G(t - t_0)$ and the above scalars λ , ζ , ξ can be regarded as functions of the distance σ along each normal trajectory. Hence we can write

$$\frac{\delta\lambda}{\delta t} = G \frac{d\lambda}{d\sigma}; \quad \frac{\delta\zeta}{\delta t} = G \frac{d\zeta}{d\sigma}; \quad \frac{\delta\xi}{\delta t} = G \frac{d\xi}{d\sigma}.$$

Making these substitutions the equations (56), (57) and (58) become

$$(59) \quad \frac{d\lambda}{d\sigma} = \left(\frac{\gamma+1}{2G}\right)\lambda^2 + \Omega\lambda,$$

$$(60) \quad \frac{d\zeta}{d\sigma} = \left(\frac{\gamma+1}{2\rho}\right)\zeta^2 + \Omega\zeta,$$

$$(61) \quad \frac{d\xi}{d\sigma} = \left(\frac{\gamma+1}{2\rho G^2}\right)\xi^2 + \Omega\xi.$$

We shall find it convenient to use the forms (59), (60) and (61) of the fundamental differential equations in the following discussion.

8. Plane and spherical waves. For plane waves $\Sigma(t)$ the mean curvature $\Omega = 0$ and hence (59), (60) and (61) become

$$(62) \quad \frac{d\lambda}{d\sigma} = \left(\frac{\gamma+1}{2G}\right)\lambda^2; \quad \frac{d\zeta}{d\sigma} = \left(\frac{\gamma+1}{2}\right)\zeta^2; \quad \frac{d\xi}{d\sigma} = \left(\frac{\gamma+1}{2\rho G^2}\right)\xi^2.$$

Integrating the equations (62) we have

$$(63) \quad \lambda = \frac{\lambda_0}{1 - \left(\frac{\gamma+1}{2G}\right)\lambda_0\sigma},$$

$$(64) \quad \zeta = \frac{\zeta_0}{1 - \left(\frac{\gamma+1}{2\rho}\right)\zeta_0\sigma},$$

$$(65) \quad \xi = \frac{\xi_0}{1 - \left(\frac{\gamma+1}{2\rho G^2}\right)\xi_0\sigma},$$

where λ_0 , ζ_0 and ξ_0 denote the values of the scalars λ , ζ and ξ at points of the surface $\Sigma(t_0)$ where $\sigma = 0$ by construction. Equations (63), (64) and (65) permit us to state the following result. *If λ_0 , ζ_0 and ξ_0 are negative then the scalars λ , ζ and ξ will approach zero monotonically as the distance $\sigma \rightarrow \infty$ while if λ_0 , ζ_0 and ξ_0 are positive these scalars will increase monotonically and become infinite for*

$$(66) \quad \sigma = \frac{2G}{(\gamma+1)\lambda_0} = \frac{2\rho}{(\gamma+1)\zeta_0} = \frac{2\rho G^2}{(\gamma+1)\xi_0}.$$

Actually there is no inconsistency in the above statement since if one of the quantities λ_0 , ζ_0 or ξ_0 is negative (or positive) the other two will likewise be negative (or positive) on account of the relations (17). Also it follows from (17) that the three ratios in (66) must have equal values. In the former of these two situations the sonic discontinuities will decay or be damped out while in the latter they will grow until the wave finally terminates in a shock for the value of σ given by (66).

If the sonic wave surfaces $\Sigma(t)$ consist of a family of concentric spheres the mean curvature Ω is $-1/R$ where R denotes the radii of the spheres of the family⁷ provided it is assumed that R increases with the time t . Evidently we can replace the above distance σ by the radius R . Hence the equations (59), (60) and (61) become

$$(67) \quad \frac{d\lambda}{dR} = \left(\frac{\gamma+1}{2G}\right)\lambda^2 - \frac{\lambda}{R},$$

⁷We have chosen the unit vector ν to have the direction of the propagation of the wave surface. Hence the vector ν is the outwardly directed normal to the above spherical surfaces and in consequence of this fact the mean curvature Ω has the negative value $-1/R$ rather than the usual positive value $1/R$.

$$(68) \quad \frac{d\zeta}{dR} = \left(\frac{\gamma + 1}{2\rho} \right) \zeta^2 - \frac{\zeta}{R},$$

$$(69) \quad \frac{d\xi}{dR} = \left(\frac{\gamma + 1}{2\rho G^2} \right) \xi^2 - \frac{\xi}{R}.$$

Integrating (67), (68) and (69) we find

$$(70) \quad \frac{1}{R\lambda} = \frac{1}{R_0\lambda_0} - \left(\frac{\gamma + 1}{2G} \right) \log \frac{R}{R_0},$$

$$(71) \quad \frac{1}{R\zeta} = \frac{1}{R_0\zeta_0} - \left(\frac{\gamma + 1}{2\rho} \right) \log \frac{R}{R_0},$$

$$(72) \quad \frac{1}{R\xi} = \frac{1}{R_0\xi_0} - \left(\frac{\gamma + 1}{2\rho G^2} \right) \log \frac{R}{R_0},$$

where the constants of integration have been chosen so that λ , ζ , ξ equal λ_0 , ζ_0 , ξ_0 respectively for $R = R_0 > 0$. Corresponding to the above italicized statement we can now say that *the sonic discontinuities associated with the spherical waves damp out as $R \rightarrow \infty$ if the values λ_0 , ζ_0 and ξ_0 are negative. If λ_0 , ζ_0 and ξ_0 are positive the discontinuities become indefinitely large as R approaches the value given by*

$$\log \frac{R}{R_0} = \frac{2G}{(\gamma + 1)\lambda_0 R_0}.$$

For this value of R the sonic wave must degenerate into a spherical shock.

9. Growth and decay of sonic discontinuities in general. To treat the general case we make use of the fact that the mean curvature Ω is given as a function of the distance σ , introduced at the end of §7, by the equation⁸

$$(73) \quad \Omega = \frac{\Omega_0 - K_0\sigma}{1 - 2\Omega_0\sigma + K_0\sigma^2},$$

where K_0 and Ω_0 are the Gaussian and mean curvatures of the surface $\Sigma(t_0)$. Hence equation (60) becomes

$$(74) \quad \frac{d\zeta}{d\sigma} = \left(\frac{\gamma + 1}{2\rho} \right) \zeta^2 + \frac{(\Omega_0 - K_0\sigma)\zeta}{1 - 2\Omega_0\sigma + K_0\sigma^2}.$$

We shall use the equation (74) as the basis of the discussion in this section since the corresponding equations obtainable from (59), (61) and (73) yield identical results regarding the growth and decay of the sonic discontinuities.

The differential equation (74) can be written in the form

$$(75) \quad \frac{d}{d\sigma} \left(\frac{e^{\int \Omega d\sigma}}{\zeta} \right) = - \left(\frac{\gamma + 1}{2\rho} \right) e^{\int \Omega d\sigma}.$$

⁸E. P. Lane, *loc. cit.*, p. 209.

But

$$\int \Omega d\sigma = \int \frac{(\Omega_0 - K_0\sigma) d\sigma}{1 - 2\Omega_0\sigma + K_0\sigma^2} = -\log \phi(\sigma),$$

where

$$\phi(\sigma) = \sqrt{1 - 2\Omega_0\sigma + K_0\sigma^2}.$$

Hence, integrating (75) between the limits $\sigma = 0$ and $\sigma = \sigma$, we have

$$(76) \quad \frac{1}{\zeta} = \phi(\sigma)J(\sigma), \quad J(\sigma) = \frac{1}{\zeta_0} - \left(\frac{\gamma + 1}{2\rho}\right) \int_0^\sigma \frac{d\sigma}{\phi(\sigma)}.$$

Our discussion will be based on four cases depending on the values of the curvatures K_0 and Ω_0 at points of the surface $\Sigma(t_0)$. Excluding the case of plane waves, treated in §8, for which $K_0 = \Omega_0 = 0$ on $\Sigma(t_0)$, these cases will include all possibilities subject to the restriction that the curvatures K_0 and Ω_0 are finite. In Case III and Case IV reference will be made to the two radii of normal curvature of the surface $\Sigma(t_0)$. Denoting these radii by R_1 and R_2 we have the relations

$$(77) \quad K_0 = \frac{1}{R_1 R_2}; \quad \Omega_0 = \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

Using (77) it follows immediately that

$$(78) \quad \phi^2(\sigma) = 1 - 2\Omega_0\sigma + K_0\sigma^2 = \frac{(R_1 - \sigma)(R_2 - \sigma)}{R_1 R_2}.$$

The results obtained for each case will depend essentially on whether $\zeta_0 < 0$ or $\zeta_0 > 0$. We cannot have $\zeta_0 = 0$ since then $\lambda_0 = \xi_0 = 0$ from the relations (17) and the sonic wave surface $\Sigma(t_0)$ would not be singular of order 1 as postulated.

Case I

$$(\mathbf{K}_0 \geq 0, \Omega_0 < 0)$$

$\zeta_0 < 0$. Here $\phi(\sigma) > 0$ for $\sigma \geq 0$ and $\phi(\sigma) \rightarrow \infty$ as $\sigma \rightarrow \infty$. Also $J(\sigma) < 0$ for $\sigma \geq 0$ and $J(\sigma) \rightarrow -\infty$ as $\sigma \rightarrow \infty$. Hence from (76) it follows that $1/\zeta \rightarrow -\infty$ or $\zeta \rightarrow 0$ as $\sigma \rightarrow \infty$.

$\zeta_0 > 0$. We can now find a value $\sigma^* > 0$ such that $J(\sigma^*) = 0$ and $J(\sigma) > 0$ for $0 \leq \sigma < \sigma^*$. Then, since $\phi(\sigma) \geq 1$ for $0 \leq \sigma \leq \sigma^*$ and $\phi(\sigma^*)$ is finite, we have $1/\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$ as $\sigma \rightarrow \sigma^*$.

Case II

$$(\mathbf{K}_0 \leq 0, \Omega_0 > 0) \text{ or } (\mathbf{K}_0 < 0, \Omega_0 = 0)$$

For this case we can find a value σ^* of σ such that $\phi(\sigma^*) = 0$ with $\phi(\sigma) > 0$ for $0 \leq \sigma < \sigma^*$.

$\zeta_0 < 0$. Now $J(\sigma) < 0$ for $0 \leq \sigma \leq \sigma^*$ and $J(\sigma^*)$ is finite. Hence $1/\zeta \rightarrow 0$ or $\zeta \rightarrow -\infty$ as $\sigma \rightarrow \sigma^*$.

$\zeta_0 > 0$. If $J(\sigma^*) \geq 0$ we have $1/\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$ as $\sigma \rightarrow \sigma^*$. If $J(\sigma^*) < 0$ we can find a value $\sigma' < \sigma^*$ such that $J(\sigma') = 0$ with $J(\sigma) > 0$ for $0 \leq \sigma < \sigma'$ and $\phi(\sigma) > 0$ for $0 \leq \sigma \leq \sigma'$. Hence $1/\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$ as $\sigma \rightarrow \sigma'$.

Case III

$$(\mathbf{K}_0 > 0, \Omega_0 > 0)$$

Under the conditions of this case we see from (77) that R_1 and R_2 are positive. Hence we can define a value $\sigma^* > 0$ as follows. Take $\sigma^* = R_1 = R_2$ if R_1 and R_2 are equal. Take σ^* to be the smaller of the two values R_1 and R_2 if R_1 and R_2 are distinct. Then we see from (78) that $\phi(\sigma^*) = 0$ and $\phi(\sigma) > 0$ for $0 \leq \sigma < \sigma^*$.

$\zeta_0 < 0$. If the radii R_1 and R_2 are distinct the quantity $J(\sigma^*)$ is finite and $J(\sigma) < 0$ for $0 \leq \sigma \leq \sigma^*$. Hence $1/\zeta \rightarrow 0$ or $\zeta \rightarrow -\infty$ as $\sigma \rightarrow \sigma^*$.

In the special case $R_1 = R_2$ we have $J(\sigma) \rightarrow -\infty$ as $\sigma \rightarrow \sigma^*$. Hence the product $J(\sigma)\phi(\sigma)$ has the indeterminate form $-\infty \cdot 0$ at $\sigma = \sigma^*$. But

$$\lim_{\sigma \rightarrow \sigma^*} \phi(\sigma)J(\sigma) = \lim_{\sigma \rightarrow \sigma^*} \frac{J(\sigma)}{1/\phi(\sigma)} = -\lim_{\sigma \rightarrow \sigma^*} \frac{\phi^2(\sigma)J'(\sigma)}{\phi'(\sigma)},$$

where the prime denotes differentiation with respect to σ . Carrying out the differentiation indicated in the last limit we readily deduce that $1/\zeta \rightarrow 0$ or $\zeta \rightarrow -\infty$ as $\sigma \rightarrow \sigma^*$. Hence we have the same limiting value as for the case of distinct radii as would be expected.

$\zeta_0 > 0$. If $J(\sigma^*) \geq 0$ then $1/\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$ as $\sigma \rightarrow \sigma^*$. If $J(\sigma^*) < 0$ we can find a value $\sigma' < \sigma^*$ such that $J(\sigma') = 0$ with $J(\sigma) > 0$ for $0 \leq \sigma < \sigma'$. Hence $1/\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$ as $\sigma \rightarrow \sigma'$.

Case IV

$$(\mathbf{K}_0 < 0, \Omega_0 < 0)$$

Now R_1 and R_2 have opposite algebraic signs. Assuming $R_1 > 0$ we choose $\sigma^* = R_1$. Then $\phi(\sigma^*) = 0$ and $\phi(\sigma) > 0$ for $0 \leq \sigma < \sigma^*$.

$\zeta_0 < 0$. We have $J(\sigma) < 0$ and finite for $0 \leq \sigma \leq \sigma^*$. Hence $1/\zeta \rightarrow 0$ or $\zeta \rightarrow -\infty$ as $\sigma \rightarrow \sigma^*$.

$\zeta_0 > 0$. The quantity $J(\sigma^*)$ is finite. If $J(\sigma^*) \geq 0$ we must have $J(\sigma) > 0$ for $0 \leq \sigma < \sigma^*$. Hence $1/\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$ as $\sigma \rightarrow \sigma^*$. If $J(\sigma^*) < 0$ we can choose $\sigma' < \sigma^*$ such that $J(\sigma') = 0$ and $J(\sigma) > 0$ for $0 \leq \sigma < \sigma'$. Then $1/\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$ as $\sigma \rightarrow \sigma'$.

Except for the case of plane waves which are discussed fully in §8 the results of the above Cases I—IV give all points on the individual normal trajectories

of the family of sonic surfaces $\Sigma(t)$ where the scalar ζ becomes zero (decay of the sonic discontinuity) or infinite (transition to shock). Those points which are determined by the above limiting values σ^* and σ' will be in a finite region of space and will unite to form a *limiting surface* S (which may be degenerate) over which the sonic discontinuity will be damped out or the sonic wave will terminate in a shock. In general the surface S will not be one of the parallel sonic wave surfaces $\Sigma(t)$ since σ^* and σ' will have different values for the various normal directions which issue from the surface $\Sigma(t_0)$.

By recourse to the relations (17) it is seen that each of the three equations (59), (60) and (61) can be transformed into the other two equations of this set. Hence the limiting values of λ , ζ and ξ , which will be related by (17), will be realized at the same points of space and the same limiting surface S will be obtained regardless of which of the equations (59), (60) and (61) is selected for discussion.

10. Pressure, density and velocity increments. From the relation (6) it follows that

$$[p, \cdot] \nu_i = \left[\frac{dp}{dn} \right] = \xi,$$

where n denotes distance measured in the direction of the unit normal ν to $\Sigma(t)$. Since $p = \text{const.}$ on the side 2 of $\Sigma(t)$, in which we assume the gas to be uniform and at rest, the above relation gives $dp/dn = -\xi$ where the derivative is evaluated on the side 1. Now if s denotes distance measured in the direction of the normal ν' , oppositely directed from the normal ν at the points of Σ , the above equation becomes $dp/ds = \xi$. Hence ξds , where $ds = PQ > 0$, will be the

Fig. 1

Fig. 2

difference in the pressure at the points P and Q (see Fig. 1). This is illustrated more fully in Fig. 2 in which the pressure difference is represented by QR .

Now suppose that the pressure configuration PQR in Fig. 2 passes into the configuration $P'PR'$ where $PQ = PP'$ when the wave moves in the direction of the normal ν from point P to point P' . If this displacement takes place in time dt then the above pressure difference will be $\xi G dt$. Neglecting the effect of decay or growth of the sonic discontinuity in the time dt , which we assume to be arbitrarily small, the pressure difference $\xi G dt$ will also be represented by PR' in Fig. 2 and hence *the relative increase in the pressure at the point P as the wave moves from P to P' in the time dt will be given by*

$$(79) \qquad Q dt, \qquad Q = \frac{\xi G}{p_0}$$

where we have used p_0 to denote the constant pressure in the uniform gas at rest.

We shall call the non-dimensional quantity Qdt , defined by (79), the *pressure increment at P* for want of a more suggestive terminology. Density and velocity increments can of course be defined in a corresponding manner. We note that the equations of the preceding sections directly determine the decay or growth of these increments.

Graduate Institute for Mathematics and Mechanics
Indiana University, Bloomington, Indiana