The Green's Function for Periodic Structures in Diffraction Theory with an Application to Parallel Plate Media, I

ALBERT E. HEINS

Communicated by H. LEWY

1. Introduction. In 1946, J. F. CARLSON & A. E. HEINS [1] studied the effect of a plane electromagnetic wave incident upon an infinite set of equally spaced, parallel plates of constant angle of stagger. At that time, the problem was formulated in terms of a finite number of plates, and then the limit was taken to produce the infinite structure. While, for the case of a finite number of plates, the edges act as scatterers, in the infinite case, the edges act as a grating. Hence, in this latter case, we may expect the usual specularly reflected wave, as well as the higher order diffracted plane waves (if they are present). Accordingly, we should start directly with the infinite structure and take into account the periodicity in the structure from the outset. If it were possible to find an appropriate Green's function, this could be accomplished. Such a Green's function may be constructed, and it is wave-guide-like in character, that is, it is defined in an infinite duct of finite width. The boundary conditions on the walls of this duct-like region are dictated by those in the original periodic structure. The radiation condition in a duct may be given without trouble, and we find that the formulation of the problem is much more direct and poses fewer mathematical difficulties.

The possibility of employing the "wave-guide" or rather "parallel plate" region from the start is due to two major features. First, we recognize that we have a periodic structure, that is, the set of equally spaced parallel plates. In the second place, because of the plane wave excitation, there is a phase change in the field due to this uniform spacing between the plates, the angle of stagger of the plates and the angle of incidence of the plane wave. With this clue, we observe that we are in a position to construct a Green's function with the appropriate type of periodicity and hence form an integral equation for the

The work on this paper was sponsored by the Office of Ordnance Research, U. S. Army, Contract No. DA-36-061-ORD-490.

current density on any one plate, rather than the system of equations we had in [1]. Incidentally, we are not confronted with some of the mathematical difficulties which were encountered in our earlier work concerning the solution of the integral equation. The wave-number k is real throughout our analysis.

We derive here a solution of the problem which we have described, as well as the various reflection and transmission coefficients within the wave-length band that supports the dominant mode in the parallel plate region. The Green's function, which we derive here, will be discussed elsewhere in connection with general periodic structures.

2. Formal Derivation of the Green's Function. In the study of the boundary value problems of diffraction theory, one is led to seek an appropriate representation which provides an integral equation incorporating the required boundary conditions. In the problem which we shall formulate in Section 4, we shall be concerned with the solution of the partial differential equation

$$(2.1) \quad \phi_{\xi\xi} + \phi_{\eta\eta} + k^2\phi = 0,$$

where the subscripts denote partial differentiation with respect to these variables. A useful representation for solutions of Eq. (2.1) can be provided by Green's theorem with the aid of an appropriate Green's function. Such a Green's function satisfies the partial differential equation

$$(2.2) \quad G_{\xi\xi} + G_{\eta\eta} + k^2G = 0,$$

save at the point $P(\xi, \eta) = P'(\xi', \eta')$ where it has a logarithmic singularity. In addition to this requirement, we prescribe the periodic boundary conditions

$$(2.3) \quad G(\xi, -\beta, \xi', \eta') = e^{-ik\rho}G(\xi, c - \beta, \xi', \eta')$$

and

$$(2.4) \quad G_{\eta}(\xi, \eta, \xi', \eta')|_{\eta=-\beta} = e^{-ik\rho}G_{\eta}(\xi, \eta, \xi', \eta')|_{\eta=c-\beta},$$

where ρ and β are real parameters. We omit the conditions at infinity for the time being, that is, the behavior of G for $|\xi|$ or $|\xi'| \rightarrow \infty$. The motivation for this particular Green's function will appear later in this paper. Let it suffice to say, at this point, that the G described by Eqs. (2.2), (2.3) and (2.4) provides a natural setting for one class of boundary value problems in diffraction theory which involve periodic structures.

Let us assume for the present that $\xi' = 0$ and that the bilateral Fourier transform of $G(\xi, \eta, 0, \eta')$ with respect to ξ exists. That is, if we write

$$g(\eta, \eta', w) = \int_{-\infty}^{\infty} G(\xi, \eta, 0, \eta') e^{-i w \xi} d\xi,$$

the Fourier transform of Eq. (2.2) is

$$(2.2a) \quad g_{\eta\eta} + \sigma^2 g = 0, \quad \eta \neq \eta',$$

where

$$\sigma^2 = k^2 - w^2.$$

This equation is subject to the boundary conditions

$$(2.3a) \quad g(-\beta, \eta', w) = e^{-ik\rho} g(c - \beta, \eta', w)$$

and

$$(2.4a) \quad g_\eta|_{\eta=-\beta} = e^{-ik\rho} g_\eta|_{\eta=c-\beta}.$$

Upon solving Eq. (2.2a) we get

$$g = A \cos \sigma(\eta - c + \beta) + B \sin \sigma(\eta - c + \beta), \quad \eta > \eta',$$

and

$$g = C \cos \sigma(\eta + \beta) + D \sin \sigma(\eta + \beta), \quad \eta < \eta'.$$

The coefficients A , B , C and D are determined by the conditions (2.3a), (2.4a), the continuity condition on g at $\eta = \eta'$ and the jump condition

$$g_\eta \begin{cases} \eta = \eta' + 0 \\ \eta = \eta' - 0 \end{cases} = -1.$$

These conditions lead to the following relations,

$$C = A e^{-ik\rho}, \quad D = B e^{-ik\rho},$$

$$\begin{aligned} A \cos \sigma(\eta' - c + \beta) + B \sin \sigma(\eta' - c + \beta) \\ = C \cos \sigma(\eta' + \beta) + D \sin \sigma(\eta' + \beta) \end{aligned}$$

and

$$\begin{aligned} -A \sigma \sin \sigma(\eta' - c + \beta) + B \sigma \cos \sigma(\eta' - c + \beta) + \\ C \sigma \sin \sigma(\eta' + \beta) - D \cos \sigma(\eta' + \beta) = -1. \end{aligned}$$

Upon solving this system of equations, we find that g assumes the following form. For $\eta > \eta'$

$$g(\eta, \eta', w) = \frac{e^{ik\rho} \sin \sigma(\eta' - \eta) - \sin \sigma(\eta' - \eta + c)}{2\sigma[\cos k\rho - \cos \sigma c]}$$

and for $\eta < \eta'$

$$g(\eta, \eta', w) = \frac{\sin \sigma(\eta' - \eta - c) - e^{-ik\rho} \sin \sigma(\eta' - \eta)}{2\sigma[\cos k\rho - \cos \sigma c]}.$$

Let us note, in passing, that $g(\eta, \eta', w)$ is independent of β and therefore $G(\xi, \eta, \xi', \eta')$ will also be independent of β .

The determination of $G(\xi, \eta, 0, \eta')$ is effected by the inverse Fourier transform

$$(2.5) \quad \frac{1}{2\pi} \int_P e^{i w \xi} g(\eta, \eta', w) dw,$$

where P is a path parallel to the real axis in the w plane. The path P will be specified more precisely later in this section. For the time being, we note that the evaluation of the integral (2.5) requires a knowledge of the zeros of $\cos k\rho - \cos \sigma c = 0$, that is, the poles of the integrand. If $c > 0$, the zeros satisfy the equations (if $0 < k < kc$)

$$(2.6a) \quad k\rho + 2n\pi = c\sqrt{k^2 - w^2}, \quad n = 0, 1, \dots,$$

and

$$(2.6b) \quad -k\rho + 2n\pi = c\sqrt{k^2 - w^2}, \quad n = 1, 2, \dots,$$

where now we have chosen the branch of $\sqrt{k^2 - w^2}$ which reduces to k at $w = 0$. Eqs. (2.6) provide two families of roots; from Eq. (2.6a) we get

$$wc = \pm \sqrt{c^2 k^2 - (2n\pi + k\rho)^2}$$

while from Eq. (2.6b) we get

$$w'c = \pm \sqrt{c^2 k^2 - (2n\pi - k\rho)^2}.$$

These roots will be real if the inequalities

$$kc > 2n\pi + k\rho, \quad 0 < n \leq M$$

and

$$kc > 2n\pi - k\rho, \quad 0 < n \leq N$$

hold. Since kc will be assigned, there are only a finite number of real roots, the remainder being imaginary. Henceforth we write

$$\sqrt{k^2 c^2 - (k\rho + 2n\pi)^2} = i\Delta_n, \quad n \geq M + 1$$

and

$$\sqrt{k^2 c^2 - (2n\pi - k\rho)^2} = i\Delta'_n, \quad n \geq N + 1$$

for the cases in which the square roots are imaginary and assume that Δ_n, Δ'_n, w and w' never vanish for a given k . The integers M and N are defined as the greatest positive integers less than or equal to the quotients

$$\frac{kc - k\rho}{2\pi} \quad \text{and} \quad \frac{kc + k\rho}{2\pi},$$

respectively. If M and N are zero, there are no real w 's for $n > 0$. Clearly then the imaginary roots distribute themselves symmetrically about the real axis while the real roots do the same thing about the imaginary axis. For n large compared to $k\rho/2\pi$, $\Delta_n = 2n\pi + k\rho + O(1/n)$, while $\Delta'_n = 2n\pi - k\rho + O(1/n)$. In addition to the roots we have just found for $n = 1, 2, \dots$, there are two roots which satisfy Eq. (2.6a) for $n = 0$. We shall be interested in the case $c \geq \rho$ in this paper, and in this case we have

$$cw = \pm k\sqrt{c^2 - \rho^2}.$$

We now choose P so that it is a line drawn parallel to and below the real axis, but above the complex root, with smallest numerical part, of the equation $\cos k\rho - \cos \sigma c = 0$. We close P by a sequence of semi-circles whose centers are on the intersection of P and the imaginary axis and which pass between the poles of the integrand of Eq. (2.5). The details of this procedure are standard and we omit them. The limit when the radius becomes infinite provides us with two infinite sums due to the residues at the poles. For $\xi > 0$, $\eta \geq \eta'$ we have

$$\begin{aligned}
 (2.7a) \quad G(\xi, \eta, 0, \xi') = & \sum_{(a)} \frac{\exp \{i\sigma(\eta - \eta') - \Delta_n \xi/c\}}{2\Delta_n} \\
 & + \sum_{(b)} \frac{\exp \{i\sigma(\eta' - \eta) - \Delta'_n \xi/c\}}{2\Delta'_n} \\
 & - \sum_{(a)} \exp \{i\sigma(\eta - \eta')\} \frac{\sin w\xi}{wc} - \sum_{(b)} \exp \{i\sigma(\eta' - \eta)\} \frac{\sin w'\xi}{w'c} \\
 & - \frac{\exp \{ik\rho(\eta - \eta')/c\} \sin [k\sqrt{c^2 - \rho^2}\xi/c]}{k\sqrt{c^2 - \rho^2}}.
 \end{aligned}$$

The index (a) indicates summation over the roots of Eq. (2.6a), while (b) has a similar significance. The first two sums are over imaginary roots, while the second two are over real roots. If $\xi < 0$, we close the path P in the lower half plane in a fashion similar to the procedure which we described for $\xi > 0$. Here we get

$$\begin{aligned}
 (2.7b) \quad G(\xi, \eta, 0, \eta) = & \sum_{(a)} \frac{\exp \{i\sigma(\eta - \eta') + \Delta_n \xi/c\}}{2\Delta_n} \\
 & + \sum_{(b)} \frac{\exp \{i\sigma(\eta' - \eta) + \Delta'_n \xi/c\}}{2\Delta'_n}.
 \end{aligned}$$

We observe that if these sums for $\xi \geq 0$ are convergent (a fact which we shall prove in the next section) we have

$$\begin{aligned}
 G(\xi, \eta, 0, \eta') &= O(1) && \text{as } \xi \rightarrow \infty \\
 &= O(e^{\epsilon\xi}), && \epsilon > 0 \text{ as } \xi \rightarrow -\infty,
 \end{aligned}$$

where ϵ is the smaller of Δ_n or Δ'_n . Had the path P been drawn above the real axis but below the imaginary root of $\cos k\rho - \cos \sigma c = 0$ with smallest imaginary part, the bounded terms in (2.7a) would be transferred to (2.7b). In this case

$$\begin{aligned}
 G(\xi, \eta, 0, \xi') &= O(e^{-\epsilon_1\xi}), && \epsilon_1 > 0 \text{ as } \xi \rightarrow \infty \\
 &= O(1) && \text{as } \xi \rightarrow -\infty,
 \end{aligned}$$

where once again ϵ_1 is the smaller of Δ_n or Δ'_n . The ξ' dependence can be restored by replacing ξ by $\xi - \xi'$. This replaces the inequalities $\xi > 0$ and $\xi < 0$ in (2.7a) and (2.7b) by $\xi > \xi'$ and $\xi < \xi'$, respectively.

3. The Verification of $G(\xi, \eta, \xi', \eta')$. We have seen that a formal application of the Fourier transform theorem to $G(\xi, \eta, \xi', \eta')$ provided us with expansions for this function which differed mainly by their orders at infinity. For the case in which $G(\xi, \eta, 0, \xi') = O(1)$ for $\xi \rightarrow \infty$ and $O(e^{\epsilon\xi})$ for $\xi \rightarrow -\infty$ and $\epsilon > 0$, we observe that this bilateral Fourier transform makes sense if $-\epsilon < \text{Im } w < 0$, at least as far as the existence of the integrals from $-\infty$ to $-L$ and L_1 to ∞ are concerned ($L, L_1 > 0$). We choose L and L_1 sufficiently large so that we may invoke the asymptotic forms of G . We should, of course, investigate the convergence of these expansions. For example, an application of the ratio test, for $\xi > \xi'$, gives us for the sum with index (a)

$$\frac{\Delta_n}{\Delta_{n+1}} \exp \left\{ \frac{2\pi i}{c} (\eta - \eta') - \frac{(\Delta_{n+1} - \Delta_n)}{c} (\xi - \xi') \right\}.$$

In the limit, as $n \rightarrow \infty$, this ratio is of the order $\exp \{-2\pi(\xi - \xi')/c\}$. That is, the infinite series in (2.7a) converge uniformly and absolutely for $\xi > \xi'$. In fact, the same remarks hold for $\xi < \xi'$ and the expansions in (2.7b) as well as the ξ, ξ', η and η' derivatives of G . It follows for $\xi \neq \xi', \eta \neq \eta'$ that G satisfies Eq. (2.2) and the boundary conditions (2.3) and (2.4).

The behavior of G in the neighborhood of $\xi = \xi'$ and $\eta = \eta'$ can be determined by observing that, for n sufficiently large, Δ_n is asymptotic to $2n\pi + k\rho$ as $n \rightarrow \infty$. $G(\xi, \eta, \xi', \eta')$ then has the same behavior as the series

$$\begin{aligned} \sum_N \exp \left\{ [i(\eta - \eta') - (\xi - \xi')] \frac{[2n\pi + k\rho]}{c} \right\} \frac{1}{2n\pi} \\ + \sum_N \exp \left\{ [i(\eta' - \eta) - (\xi - \xi')] \frac{[2n\pi - k\rho]}{c} \right\} \frac{1}{2n\pi} \end{aligned}$$

for $\xi > \xi'$ and N sufficiently large to permit the use of the asymptotic forms of Δ_n and Δ'_n . In the neighborhood of $\xi = \xi'$ and $\eta = \eta'$, G has the behavior

$$-\frac{1}{4\pi} \log [(\xi - \xi')^2 + (\eta - \eta')^2],$$

and thus G has an integrable singularity. A similar result holds for $\xi < \xi'$. The logarithmic singularity of G at $\xi = \xi'$ and $\eta = \eta'$ makes possible the Green's theorem representation which we shall use in the next section. Let us observe that the boundary conditions on the variable η' are somewhat different from those on the variable η . In fact

$$G(\xi, \eta, \xi', -\beta)e^{-ik\rho} = G(\xi, \eta, \xi', c - \beta)$$

with a similar condition on the derivative of G with respect to η' .

4. Statement and Formulation of the Problem. With the aid of the Green's function which we have derived in Section 2, we are in a position to state and formulate the problem which we described in Section 1. Suppose a plane monochromatic wave, whose propagation normal makes an angle θ with respect to

the positive x -axis of a two dimensional rectangular coordinate system where $\alpha - \pi < \theta < \alpha$, is incident upon an infinite set of equally spaced parallel plates. The functional form of the incident plane wave is $\exp \{ik(x \cos \theta + y \sin \theta)\}$. The plates are assumed to be perfect conductors of zero thickness, are spaced b units apart and are parallel to the x -axis of the (x, y) coordinate system. The trace of the edges of these plates make an angle α ($0 < \alpha \leq \frac{1}{2}\pi$) with respect to the positive x -axis. (See Figure 1 for a side view.)

FIGURE 1.

We seek a solution of the two-dimensional steady-state wave equation

$$(4.1) \quad \phi_{xx} + \phi_{yy} + k^2\phi = 0$$

subject to the boundary condition that ϕ vanishes on the conductors. In addition to this, we assume that all the excited waves, in the region to the left of the trace of the edges, are outward going. That is, in a (ξ, η) coordinate system given by the transformation

$$\xi = x \sin \alpha - y \cos \alpha$$

$$\eta = x \cos \alpha + y \sin \alpha$$

the waves in the region $\xi < 0$ are either propagating in the negative direction or are attenuated for $\xi \rightarrow -\infty$. The use of this (ξ, η) coordinate system enables us to describe what happens for $\xi < 0$ with relative ease, while the (x, y) coordinate system is convenient in the ducts. The above problem is one in electromagnetic diffraction; it is equivalent to a plane wave, whose electric vector ϕ is parallel to the z -axis of an (x, y, z) coordinate system, being incident upon the structure given in Figure 1.

Figure 1 is periodic, for, if we write $a = c \cos \alpha$ and $b = c \sin \alpha$, we can pass to corresponding points in the figure by increasing x by na and y by nb . Furthermore

$$(4.2) \quad \phi(x + a, y + b) = \exp \{ik(a \cos \theta + b \sin \theta)\} \phi(x, y).$$

That is, in passing from a point (x, y) to a point $(x + a, y + b)$, ϕ changes by a phase factor whose angle is $ck \cos(\alpha - \theta) = k(a \cos \theta + b \sin \theta)$. There is a similar relation for the y derivative of ϕ , provided the point with the coordinates (x, y) is not on a plate. As we have remarked, ϕ is the z component of the electric field and as such is continuous everywhere, vanishing on the plates. On the other hand ϕ_y is proportional to the x component of the magnetic field and suffers a discontinuity on crossing any plate. We interpret the periodicity condition for ϕ_y to be

$$(4.2a) \quad \phi_y(x + a, y + b) = \exp \{ik(a \cos \theta + b \sin \theta)\} \phi_y(x, y)$$

if (x, y) is not on a point on a plate. If (x, y) is on a plate, the condition (4.2a) is to be interpreted as the limit as $y \rightarrow 0^-$ or as $y \rightarrow 0^+$ as the case may be. The difference of these two limits will merely state that the current densities on consecutive plates differ by the same phase factor. The point $x = y = 0$ is excluded since ϕ_y is singular at this point [2]. ϕ_x , which is proportional to the y component of the magnetic field, satisfies a periodicity condition of the type (4.2) save at the point $x = y = 0$, where it is also singular. In spite of the singular character of ϕ_x and ϕ_y at $x = y = 0$, both functions are integrable in the neighborhood of this point and indeed are $O(r^{-1/2})$, $r \rightarrow 0$, if r measures the distance from $(0, 0)$, [2]. This singularity is to be found at all of the points (na, nb) .

Since $\phi(x, y)$ and its x and y derivatives undergo the phase change which we have just described, it is not necessary to consider the entire structure in Figure 1. We shall take advantage of the periodic character of the geometry and consider a duct-like region. (See Figure 2.) We now apply Green's Theorem

FIGURE 2.

to the region in Figure 2 over the path which is made up of the line segments I, II, III, IV, V and VI taken in the counter-clockwise sense. The points 0 and A, at which we have a confluence of two boundary conditions are avoided by employing semicircles to join the line segments IV and III as well as VI and I. These circles are clearly oriented in the clockwise sense. For a point $P(x, y)$ interior to the parallelogram which we have just described, we have

$$(4.3) \quad \phi(x, y) = \int [G' \phi_n - \phi G_n'] ds',$$

where the closed path is made up of the six line segments and the two semi-circles. Here G' is a Green's function whose properties we shall describe in the next paragraph.

In view of the integrability of ϕ , ϕ_n and G' in the neighborhood of the points O and A for $P(x, y)$ interior to the parallelogram, we may let $r \rightarrow 0$ without obtaining a contribution. On the segments I and III, ϕ vanishes, while $\phi_{n'} = -\phi_{v'}$ for $x < 0$ and $y \rightarrow 0^+$ and $\phi_{n'} = \phi_{v'}$ for $x > a$ and $y \rightarrow b^-$. Furthermore, we have the periodicity condition (4.2a), and if we choose G' such that it satisfies the periodic boundary condition

$$(4.4) \quad G'(x, y, x' + a, b) \exp \{ik(a \cos \theta + b \sin \theta)\} = G'(x, y, x', 0),$$

we have

$$\int_{\text{I+III}} = \int_0^L G'(x, y, x', 0) I(x') dx'$$

where

$$I(x') = \phi_{v'} \begin{cases} y' = 0^+ \\ y' = 0^- \end{cases}.$$

The contributions on the segments IV and VI annihilate each other in view of the periodicity conditions we have for ϕ , ϕ_v , G' and G'_v . Hence

$$\phi(x, y) = \int_0^L G'(x, y, x', 0) I(x') dx' + \int_{\text{II}} + \int_{\text{V}}.$$

A method of constructing the Green's function G' has been indicated in Section 2. Let us note that typical terms of $G(\xi, \eta, \xi', \eta')$ are

$$\exp \{i\sigma(\eta - \eta') - \Delta_n(\xi - \xi')/c\}, \xi > \xi'.$$

If we rotate the (ξ, η) coordinate system through an angle $\frac{1}{2}\pi - \alpha$, this becomes $\exp \{i\sigma(x - x') \cos \alpha + i\sigma(y - y') \sin \alpha$

$$- \frac{\Delta_n}{c} (x - x') \sin \alpha + \frac{\Delta_n}{c} (y - y') \cos \alpha\}.$$

Let us compare this expression on the line $y' = 0$ with the same expression on the line $y' = b$ with x' increased by a . We find that these two terms are the same up to a phase factor $e^{-ik\rho}$. A similar remark holds for the remaining terms. Hence, if we write $G(\xi, \eta, \xi', \eta') = G'(x, y, x', y')$, we see that G' will satisfy the wave equation since this equation is invariant under rotation of coordinates. Finally, if we choose $\rho = a \cos \theta + b \sin \theta$, we have a Green's function which satisfies the boundary condition (4.4) as well as a similar one involving the y derivatives. We should observe that G' depends on $x - x'$. The contribution on segment II may be appraised, if we note that for $\xi' - \xi \rightarrow \infty$, and therefore for $x' - x \rightarrow \infty$, $G'(x, y, x', y')$ and its x' and y' derivatives vanish exponentially.

The parallel plate region, for x' sufficiently large and positive, can only support bounded solutions. That is, for x' large and positive, $\phi(x', y')$ and its x' and y' derivatives are either bounded (the duct sustains propagating modes) or vanish exponentially (the duct does not sustain propagating modes). Hence, for $x' - x$ sufficiently large and positive, the integral over the segment II is $O(e^{-\epsilon(x'-x)})$ for some $\epsilon > 0$ and hence vanishes for $x' - x \rightarrow \infty$. We shall see, with the solution of this problem which we are now formulating, that we can assume excitation from within the parallel plate region, as well as excitation from free space, provided that the fields within the ducts have the correct change in phase. This will then enable us to study this structure as a receiving or transmitting antenna, and we can determine as many transmission or reflection coefficients as we desire.

We are now left with the integral over segment V, and to discuss this we should describe $\phi(x', y')$ for $x' \rightarrow -\infty$. In the first place, we have assumed that a plane wave of the form

$$\phi(x, y) = A \exp \{ik(x \cos \theta + y \sin \theta)\}$$

is incident from the region $\xi < 0$. Any secondary excitation should obey the requirement that it is outward going, that is, for $\xi \rightarrow -\infty$, ϕ should be composed of waves whose propagation normals have components in the direction of the negative ξ -axis or which attenuate for ξ negative. How can we describe such waves? We have seen, with the construction of the Green's function, that there are a finite number of propagating modes, the remainder being attenuated. Of these propagating modes only a portion will have the required outward going character. In order to find them, we introduce the transformation from (x, y) coordinates to (ξ, η) coordinates and examine the various bounded solutions. In the (x, y) coordinates, the functional form of ϕ is different from that of ϕ in the (ξ, η) coordinate system. In order to examine the possible propagating modes, we see that in the (ξ, η) coordinates, we have the possibilities

$$(4.4) \quad \exp \left\{ \pm \frac{ik}{c} \xi \sqrt{c^2 - \rho^2} + \frac{ik\rho\eta}{c} \right\}$$

and

$$(4.5) \quad \exp \{ \pm iw\xi + i\sigma\eta \} \quad \text{or} \quad \exp \{ \pm iw'\xi - i\sigma\eta \}$$

where

$$(4.6) \quad wc = [c^2k^2 - (2n\pi - k\rho)^2]^{\frac{1}{2}}, \quad 0 < n \leq M$$

and

$$(4.7) \quad w'c = [c^2k^2 - (2n\pi - k\rho)^2]^{\frac{1}{2}}, \quad 0 < n \leq N$$

with wc and $w'c$ real. Accordingly, (4.4) may be viewed as an incident and reflected wave, the plus sign giving the incident wave and the minus sign the

reflected one if $\alpha > \theta$. Since $\alpha - \pi < \theta < \alpha$, it follows that $0 < \alpha - \theta < \pi$, and therefore $(c^2 - \rho^2)^{\frac{1}{2}}$ is positive in this interval. (4.4) reduces to

$$\exp \{ik(x \cos \theta + y \sin \theta)\}$$

for the incident wave and

$$\exp \{ik(x \cos (2\alpha - \theta) + y \sin (2\alpha - \theta))\}$$

for the reflected wave. The terms of the type (4.5) represent reflected waves if we choose the negative sign before w or w' . Hence for $\xi \rightarrow -\infty$

$$\begin{aligned} \phi(\xi, \eta) = & A \exp \left\{ \frac{ik\xi}{c} (c^2 - \rho^2)^{\frac{1}{2}} + \frac{ik\rho\eta}{c} \right\} \\ & + B \exp \left\{ -\frac{ik\xi}{c} (c^2 - \rho^2)^{\frac{1}{2}} + \frac{ik\rho\eta}{c} \right\} \\ (4.8) \quad & + \sum_n D_n \exp \{i(-w\xi + \sigma\eta)\} + \sum_n E_n \exp \{-i(w'\xi + \sigma'\eta)\} \\ & + O(e^{\epsilon\xi}), \quad \epsilon > 0, \end{aligned}$$

where the sums are carried out over the real roots (4.6) and (4.7). (4.8) is the form of the field for $\xi \rightarrow -\infty$ since it describes the incident wave, a series of reflected waves and exponentially attenuated waves. Upon inserting this expression for ϕ into the integral over the left boundary of the parallelogram, we find upon evaluating the integral that it is equal to (4.8) with the order term replaced by $O(e^{-\epsilon L_1})$ for some $\epsilon > 0$. Hence, in the limit as L and $L_1 \rightarrow \infty$, we have

$$\begin{aligned} \phi(x, y) = & A \exp \left\{ \frac{ik\xi}{c} (c^2 - \rho^2)^{\frac{1}{2}} + \frac{ik\rho\eta}{c} \right\} \\ (4.9) \quad & + B \exp \left\{ -\frac{ik\xi}{c} (c^2 - \rho^2)^{\frac{1}{2}} + \frac{ik\rho\eta}{c} \right\} \\ & + \sum F_n \exp \{i(-w\xi + \sigma\eta)\} + \int_0^\infty G'(x, y, x', 0) I(x') dx'. \end{aligned}$$

The sum is now taken over positive and negative indices for which w and w' are real, observing that $w(-n) = w'(n)$. This means that $F_n = D_n$ for $n > 0$ and $F_n = E_n$ for n negative. Upon converting the (ξ, η) coordinates to the (x, y) coordinates, we get

$$\begin{aligned} \phi(x, y) = & A \exp \{ik(x \cos \theta + y \sin \theta)\} \\ (4.9) \quad & + B \exp \{ik[x \cos (2\alpha - \theta) + y \sin (2\alpha - \theta)]\} \\ & + \sum_n F_n \exp \{ix\gamma + iy\delta\} + \int_0^\infty G'(x, y, x', 0) I(x') dx' \end{aligned}$$

where

$$\gamma = -\frac{\sin \alpha}{c} [c^2 k^2 - (2n\pi + k\rho)^2]^{\frac{1}{2}} + \frac{\cos \alpha}{c} (2n\pi + k\rho),$$

$$\delta = \frac{\cos \alpha}{c} [c^2 k^2 - (2n\pi + k\rho)^2]^{\frac{1}{2}} + \frac{\sin \alpha}{c} (2n\pi + k\rho)$$

and $\gamma^2 + \delta^2 = k^2$. For $x > 0, y = 0, \phi$ vanishes and Eq. (4.9) becomes an integral equation of the Wiener-Hopf type for $I(x)$ since the Green's function $G'(x, 0, x'; 0)$ is a function of $x - x'$ and the limits of integration run from zero to infinity.

5. The Solution of the Integral Equation (4.9). The integral equation (4.9), being one of the Wiener-Hopf type, can best be solved with the complex form of the Fourier transform theorem. Many of the details have been supplied in [1], but there are a number which require modification due to the possibility of having more than one reflected wave in $\xi < 0$. In order to prepare equation (4.9) for the Fourier transform theorem we write it as

$$(5.1) \quad \begin{aligned} \phi(x, 0) = & A \exp \{ikx \cos \theta\} + B \exp \{ik \cos (2\alpha - \theta)\} \\ & + \sum F_n \exp \{i\gamma x\} + \int_0^\infty G'(x, 0, x', 0) I(x') dx'. \end{aligned}$$

For $x > 0, \phi(x, 0)$ vanishes identically, while for $x < 0, I(x)$ also does. Hence we can rewrite equation (5.1) in the form

$$(5.2) \quad \begin{aligned} \phi(x, 0) = & A \exp \{ikx \cos \theta\} + B \exp \{ik \cos (2\alpha - \theta)\} \\ & + \sum F_n \exp \{i\gamma x\} + \int_{-\infty}^\infty G'(x, 0, x', 0) I(x') dx'. \end{aligned}$$

In this latter form we cannot apply the bilateral Fourier transform since there are terms which remain bounded for all x , namely the incident wave term and all of the reflected wave terms. In order to circumvent this difficulty, we rewrite equation (5.2) as

$$(5.3) \quad \phi_2(x) = \phi_1(x) + \int_{-\infty}^\infty G(x, 0, x', 0) I(x') dx'$$

where

$$\begin{aligned} I(x) &\equiv 0, & x < 0 \\ \phi_1(x) &\equiv A \exp \{ikx \cos \theta\} + B \exp \{ikx \cos (2\alpha - \theta)\} \\ &\quad + \sum F_n \exp \{i\gamma x\}, & x > 0 \\ &\equiv 0, & x < 0 \end{aligned}$$

and

$$\phi_1(x) \equiv 0, \quad x > 0.$$

In this form, equation (5.3) contains the original equation (5.1) but does not suffer from the presence of terms which are bounded for $x \rightarrow -\infty$. From equation (5.3) we now see that

$$\phi_2(x) = O(e^{\epsilon x}), \quad x \rightarrow -\infty,$$

where ϵ is the smaller of Δ_n or Δ'_n multiplied by $\sin \alpha/c$. $I(x)$, which is proportional to the surface current density on any plate, is $O(e^{i\kappa x})$ for $x \rightarrow \infty$, where $\kappa = (k^2 - \pi^2/b^2)^{\frac{1}{2}} > 0$ (if we assume an excitation due in part to the external plane wave and in part to a parallel plate mode which is properly phased from one parallel plate region to the next one). Suppose we now assume that only one mode can propagate in the parallel plate region. We have then a further restriction on bk , that is, $\pi < bk < 2\pi$.

The integral equation (5.3) has the Fourier integral representation [3]

$$\int_P e^{iwx} \Phi_2(w) dw = \int_P e^{iwx} \Phi_1(w) dw + \int_P e^{iwx} g'(0, 0, w) J(w) dw,$$

where now $\Phi_1(w)$, $\Phi_2(w)$ and $J(w)$ are the unilateral Fourier transforms of $\phi_1(x)$, $\phi_2(x)$ and $I(x)$, respectively. That is,

$$\Phi_1(w) = \int_0^\infty e^{-iwx} \phi_1(x) dx,$$

$$\Phi_2(w) = \int_{-\infty}^0 e^{-iwx} \phi_2(x) dx$$

and

$$J(w) = \int_0^\infty e^{-iwx} I(x) dx.$$

Since $\phi_1(x)$ is bounded for all positive x , the integral defining $\Phi_1(w)$ will exist for all $w = u + iv$ whose imaginary part is negative, that is in the lower half plane $v < 0$. In fact, $\Phi_1(w)$ will be analytic in this lower half plane. Now since $\phi_2(x)$ is integrable for $x \leq 0$ and is indeed $O(e^{\epsilon x})$, $x \rightarrow -\infty$, $\Phi_2(w)$ will exist if $\epsilon + v > 0$, that is $v > -\epsilon$.

In a similar fashion we see that $J(w)$ is analytic in the lower half plane $v < 0$. The function $g'(y, y', w)$ is the bilateral Fourier transform of $G'(x, y, 0, y')$, that is

$$g'(y, y', w) = \int_{-\infty}^\infty e^{-iwx} G'(x, y, 0, y') dx.$$

But $g'(y, y', w)$ is essentially the Fourier transform of $G(\xi, \eta, 0, \xi')$, save for the fact that c is replaced by b and $k\rho$ by $k\rho - wa$. Hence

$$\begin{aligned} & g'(y, y', w) \\ &= \frac{\exp \{i(k\rho - wa)\} \sin [\sigma(y' - y)] - \sin [\sigma(y' - y + b)]}{2\sigma[\cos(k\rho - wa) - \cos(\sigma b)]}, \quad y > y', \end{aligned}$$

and

$$= \frac{\sin[\sigma(y' - y - b)] - \exp\{-i(k\rho - wa)\} \sin[\sigma(y' - y)]}{2\sigma[\cos(k\rho - wa) - \cos(\sigma b)]}, \quad y < y',$$

and, because of the manner in which we have chosen P , that is, in the strip $-\epsilon < \text{Im } w < 0$, $g'(y, y', w)$ is also regular in this strip. It follows that $g'(0, 0, w)$, $\Phi_1(w)$, $\Phi_2(w)$ and $J(w)$ have a common strip of regularity and therefore

$$\Phi_2(w) = \Phi_1(w) - J(w)K(w)$$

where

$$K(w) = \frac{\sin(\sigma b)}{2\sigma[\cos(k\rho - wa) - \cos(\sigma b)]}.$$

If we could write $K(w) = K_-(w)/K_+(w)$ where $K_-(w)$ is free of zeros and poles in the lower half plane $\text{Im } w < 0$ and $K_+(w)$ is free of zeros in the upper half plane $\text{Im } w > -\epsilon$, the functions $J(w)$ and $\Phi_2(w)$ could be obtained as we observed in [1]. In the present formulation, we can not only perform the factoring for k real, but we can perform it in a manner which brings into evidence each of the successive diffracted waves. In the first place

$$\frac{\sin \sigma b}{\sigma} = \frac{b^3}{\pi^2} (w^2 - \kappa^2) \prod_2^\infty \left[1 - \frac{b^2 k^2}{n^2 \pi^2} + \frac{b^2 w^2}{n^2 \pi^2} \right]$$

where $\kappa^2 = k^2 - \pi^2/a^2$. It follows that

$$L_-(w) = \frac{b^3}{\pi^2} (w^2 - \kappa^2) \prod_2^\infty [\sqrt{1 - (bk/\pi n)^2} + ibw/\pi n] \exp\{-ibw/\pi n\}$$

is free of zeros in the lower half plane $\text{Im } w < 0$ while

$$L_+(w) = \prod_2^\infty [\sqrt{1 - (bk/\pi n)^2} - ibw/\pi n] \exp\{ibw/\pi n\}$$

is free of zeros in the upper half plane

$$\text{Im } w > -\sqrt{(4\pi^2/b^2) - k^2}.$$

The factors $\exp\{\pm ibw/\pi n\}$ have been inserted in each infinite product to guarantee their absolute convergence. Further

$$\begin{aligned} \cos \sigma b - \cos(k\rho - wa) &= \frac{1}{2} (a^2 + b^2)(w - \sigma_1)(w - \sigma_2) \\ (5.4) \quad &\cdot \prod_1^\infty \left[\left\{ 1 - \frac{k\rho - wa}{2n\pi} \right\}^2 - b^2 \frac{(k^2 - w^2)}{4n^2 \pi^2} \right] \exp\{(k\rho - wa)/n\pi\} \\ &\cdot \prod_1^\infty \left[\left\{ 1 + \frac{k\rho - wa}{2n\pi} \right\}^2 - b^2 \frac{(k^2 - w^2)}{4n^2 \pi^2} \right] \exp\{(wa - k\rho)/n\pi\}. \end{aligned}$$

If diffracted plane waves exist, the infinite products may individually or collec-

tively contain factors which vanish for real w . Hence we will rewrite this last expression as

$$\frac{(a^2 + b^2)}{2} (w - \sigma_1)(w - \sigma_2) \prod_a \prod_b \prod_c$$

where $\prod_a$ contains those factors of the infinite products in (5.4) which have real roots (and therefore correspond to propagating modes in the region $\xi < 0$), while $\prod_b$ and $\prod_c$ are infinite products all of whose factors vanish for complex w . Accordingly, we may rewrite (5.4) so that

$$\begin{aligned} M_-(w) = & \frac{(a^2 + b^2)}{2} (w - \sigma_1)(w - \sigma_2) \prod_a \\ & \cdot \prod_{n=-\infty}^{\infty} [\Gamma_n - i\Psi_n] \exp \{(wa - k\rho + wbi)/2\pi n + i(\pi/2 - \alpha)\} \\ & \cdot \prod_{n=-\infty}^{\infty} [\Gamma'_n + i\Psi'_n] \exp \{(k\rho - wa - wbi)/2\pi n - i(\pi/2 - \alpha)\} \end{aligned}$$

is free of zeros in the lower half plane $\text{Im } w < 0$, while

$$\begin{aligned} M_+(w) = & \prod_{n=-\infty}^{\infty} [\Gamma_n + i\Psi_n] \exp \{(wa - k\rho + wbi)/2\pi n - i(\pi/2 - \alpha)\} \\ & \cdot \prod_{n=-\infty}^{\infty} [\Gamma'_n - i\Psi'_n] \exp \{(k\rho - wa + wbi)/2\pi n + i(\pi/2 - \alpha)\} \end{aligned}$$

is free of zeros in the upper half plane $\text{Im } w > -\epsilon$, where ϵ is defined by the root of one of the factors in the products whose imaginary part has the smallest numerical value. Here

$$\Gamma_n = [(\sin^2 \alpha)(1 + k\rho/2\pi n)^2 - b^2 k^2/4\pi^2 n^2]^{\frac{1}{2}} > 0,$$

$$\Gamma'_n = [(\sin^2 \alpha)(1 - k\rho/2\pi n)^2 - b^2 k^2/4\pi^2 n^2]^{\frac{1}{2}} > 0,$$

$$\Psi_n = (\cos \alpha)(1 + k\rho/2\pi n) - wb(\csc \alpha)/2\pi n$$

and

$$\Psi'_n = (\cos \alpha)(1 - k\rho/2\pi n) + wb(\csc \alpha)/2\pi n.$$

Once again, the occurrence of the factors $\exp \{(k\rho - wa)/n\pi\}$ is to guarantee the convergence of the individual products in (5.4). The factors $\exp \{\pm i(\frac{1}{2}\pi - \alpha)\}$ in $M_+(w)$ and $M_-(w)$ arise because the dominant term in each factor of the products is not unity but rather $\exp \{\pm i(\frac{1}{2}\pi - \alpha)\}$.

It follows, therefore, that

$$2K_-(w) = L_-(w) e^{\chi(w)}/M_-(w)$$

is free of zeros and poles in the lower half plane $\text{Im } w < 0$, while

$$1/K_+(w) = L_+(w) e^{\chi(w)}/M_+(w)$$

is free of zeros and poles in the appropriate upper half plane. The factor $\chi(w)$ has been inserted to provide $K_-(w)$ and $K_+(w)$ with algebraic growth in the

appropriate half plane as $|w| \rightarrow \infty$, and this will enable us to use the Liouville theorem to determine an entire function which arises at a later stage.

Suppose that the only excited wave in the region $\xi < 0$ is the specularly reflected one. Then $K_-(w) = O(w^{-\frac{1}{2}})$, $|w| \rightarrow \infty$, $\text{Im } w < 0$, with $\chi(w)$ chosen as

$$-\frac{i bw}{\pi} [(\alpha - \pi/2) \cot \alpha - (\ln 2) \sin \alpha].$$

This asymptotic form is due to the fact that $K_-(w)$ is independent of kb for $|w| \rightarrow \infty$, $\text{Im } w < 0$, and hence we are basically calculating the dominant term in the gamma function [1]. If one diffracted wave propagates, $K_-(w) = O(w^{-\frac{1}{2}})$, $|w| \rightarrow \infty$, $\text{Im } w < 0$, and $\chi(w)$ will be modified by adding or subtracting a term of the type $(k\rho - wa)/2\pi$, depending on whether the wave corresponds to a w or w' root. For p diffracted waves, $K_-(w) = O(w^{-\frac{1}{2}(2p+1)})$ will, for $|w| \rightarrow \infty$, $\text{Im } w < 0$, be changed from the basic case (no diffracted waves) to meet the needs of each special situation. Let us note that for w real

$$\begin{aligned} & \frac{1}{|K_+(w)|^2} \\ &= \frac{\prod_2^\infty \left[1 - b^2 \frac{k^2 - w^2}{n^2 \pi^2} \right] \exp \{-2 \text{Re } \chi\}}{\prod^\infty [\Gamma^2 + \Psi_n^2] \exp \{(wa - k\rho)/n\pi\} \prod^\infty [\Gamma'^2 + \Psi_n'^2] \exp \{(k\rho - wa)/n\pi\}} \\ &= \frac{\pi^2 (\sin b\sigma)(a^2 + b^2)(w - \sigma_1)(w - \sigma_2) \prod \left[\left(1 + \frac{k\rho - wa^2}{2n\pi} \right) - b^2 \frac{k^2 - w^2}{4n^2 \pi^2} \right]}{2b^3(w^2 - \kappa^2)\sigma[\cos \sigma b - \cos(k\rho - wa)]}, \end{aligned}$$

where the product has only a finite number of terms which vanish at the real roots which have been described earlier in this section. Since we only propose to calculate the magnitudes of the various reflection and transmission coefficients, we give here $|K_+(w)|^2$ for κ , $-\kappa$, σ_1 , σ_2 and γ_n .

$$\begin{aligned} |K_+(\kappa)|^{-2} &= -\frac{(a^2 + b^2)(\kappa - \sigma_1)(\kappa - \sigma_2)}{4[\cos(k\rho - \kappa a) + 1]} \prod \left[\left\{ 1 + \frac{k\rho - \kappa a}{2n\pi} \right\}^2 - \frac{1}{4n^2} \right], \\ |K_+(-\kappa)|^{-2} &= -\frac{(a^2 + b^2)(\kappa + \sigma_1)(\kappa + \sigma_2)}{4[\cos(k\rho + \kappa a) + 1]} \prod \left[\left\{ 1 + \frac{k\rho + \kappa a}{2n\pi} \right\}^2 - \frac{1}{4n^2} \right], \\ |K_+(\sigma_1)|^{-2} &= \frac{\pi^2 \prod [1 + kb(\sin \theta)/\pi n]}{b^2(\sigma_1^2 - \kappa^2)}, \\ |K_+(\sigma_2)|^{-2} &= \frac{\pi^2 \prod [1 + kb\{\sin(2\alpha - \theta)\}/\pi n]}{b^2(\sigma_2^2 - \kappa^2)}, \\ |K_+(\gamma_n)|^{-2} &= \frac{c^2(\gamma_n - \sigma_1)(\gamma_n - \sigma_2) \prod_a^{(n)}(\gamma_n)}{4n^2 b^2(\gamma_n^2 - \kappa^2)}. \end{aligned}$$

Here $\prod_a^{(n)}(w) = \prod_a(w)$ with the index n omitted.

The Fourier transform equation

$$\Phi_2(w) = \Phi_1(w) - K(w)J(w)$$

may now be rewritten as

$$\Phi_2(w) = \Phi_1(w) - \frac{K_-(w)}{K_+(w)}J(w)$$

or

$$K_+(w)\Phi_2(w) = K_+(w)\Phi_1(w) - K_-(w)J(w).$$

The first term is regular in the upper half plane $v > -\epsilon$, while the last term is regular in the lower half plane $v < 0$. We use the term regular to imply that the function as well as its reciprocal is analytic in the half plane in question. The middle term is analytic in the strip $-\epsilon < v < 0$ and, under normal circumstances, we would require an application of Cauchy's theorem to perform the additive decomposition. However, in this case, we have the advantage of knowing $\Phi_1(w)$ explicitly. That is, since

$$\Phi_1(w) = \frac{A}{i(w - \sigma_1)} + \frac{B}{i(w - \sigma_2)} + \sum \frac{F_n}{i(w - \gamma_n)},$$

we can effect the required decomposition by noting that

$$\begin{aligned} K_+(w)\Phi_2(w) &= \left[\frac{AK_+(\sigma_1)}{i(w - \sigma_1)} + \frac{BK_+(\sigma_2)}{i(w - \sigma_2)} + \sum \frac{F_n K_+(\gamma_n)}{i(w - \gamma_n)} \right] \\ &+ \frac{A\{K_+(w) - K_+(\sigma_1)\}}{i(w - \sigma_1)} + \frac{B\{K_+(w) - K_+(\sigma_2)\}}{i(w - \sigma_2)} \\ &+ \sum \frac{F_n\{K_+(w) - K_+(\gamma_n)\}}{i(w - \gamma_n)} \Big] - K_-(w)J(w). \end{aligned}$$

The terms in the first square brackets are regular in the lower half plane $v < 0$, while the terms in the second are regular in the upper half plane $v > -\epsilon$, since the poles $w = \sigma_1, \sigma_2, \dots, \gamma_n$ have been removed.

Hence

$$\begin{aligned} (5.5) \quad K_+(w)\Phi_2(w) &= \left[\frac{A\{K_+(w) - K_+(\sigma_1)\}}{i(w - \sigma_1)} + \frac{B\{K_+(w) - K_+(\sigma_2)\}}{i(w - \sigma_2)} \right. \\ &\quad \left. + \sum \frac{F_n\{K_+(w) - K_+(\gamma_n)\}}{i(w - \gamma_n)} \right] \\ &= \frac{AK_+(\sigma_1)}{i(w - \sigma_1)} + \frac{BK_+(\sigma_2)}{i(w - \sigma_2)} + \sum \frac{F_n K_+(\gamma_n)}{i(w - \gamma_n)} - K_-(w)J(w). \end{aligned}$$

But the left side of equation (5.5) is analytic in the upper half plane $v > -\epsilon$, while the right side of this equation is analytic in the lower half plane $v < 0$. Both sides of this equation are analytic in the strip $-\epsilon < v < 0$. Hence the left side is the analytic continuation of the right side and therefore analytic everywhere, that is, an entire function $E(w)$.

The determination of $E(w)$ is carried out by a simple application of Liouville's theorem on polynomials. Let us note that $I(x) = O(x^{-\frac{1}{2}})$, $x \rightarrow 0^+$ and therefore $J(w) = O(w^{-\frac{1}{2}})$, $|w| \rightarrow \infty$, $\text{Im } w < 0$. Furthermore $K_-(w) = O(w^{-(p+\frac{1}{2})})$, $|w| \rightarrow \infty$, $\text{Im } w < 0$, where p is the number of diffracted waves. Accordingly, $E(w) = O(w^{-1})$, $|w| \rightarrow \infty$ in the lower half plane $\text{Im } w < 0$. Now $K_+(w)$ has a similar asymptotic form, the first term of which is given by the order condition $K_+(w) = O(w^{-(p-\frac{1}{2})})$, $|w| \rightarrow \infty$, $\text{Im } w > -\epsilon$. Hence since $\phi(x, 0) = O(x^{\frac{1}{2}})$, $x \rightarrow 0^-$, $\Phi_2(w) = O(w^{-\frac{3}{2}})$ and $E(w) = O(w^{-(\frac{1}{2}+p)})$ in the half plane $v > \epsilon$ as $|w| \rightarrow \infty$. But $E(w)$, being of algebraic order for $|w| \rightarrow \infty$ in the entire plane, can only be a polynomial, and since the polynomial vanishes at infinity, it vanishes everywhere. That is, $E(w)$ vanishes identically. Observe that had we not inserted the factor $e^{x(w)}$ in the definition of $K_-(w)$ and $K_+(w)$, the Liouville theorem would have been inapplicable.

We have then that the Fourier transform of $J(x)$ is

$$J(w) = \left[\frac{AK_+(\sigma_1)}{i(w - \sigma_1)} + \frac{BK_+(\sigma_2)}{i(w - \sigma_2)} + \sum \frac{F_n K_+(\gamma_n)}{i(w - \gamma_n)} \right] \frac{1}{K_-(w)}.$$

Let us note that the coefficients $A, B, F_1, \dots$ are not linearly independent. For $J(w) = O(w^{-\frac{1}{2}})$, $v < 0$, $|w| \rightarrow \infty$. On the other hand $K_-(w) = O(w^{-(p+\frac{1}{2})})$, $|w| \rightarrow \infty$, $v < 0$. Hence only in the case that $p = 0$ (that is $F_n = 0$, for all n) does $J(w)$ obey the Riemann-Lebesgue lemma. If for example $F_1 \neq 0$, $F_2, \dots = 0$, then $J(w) = O(w^{\frac{1}{2}})$, $|w| \rightarrow \infty$, $v < 0$, thus violating the lemma. It may be satisfied if we expand the square bracket in powers of $1/w$ for w sufficiently large and note the vanishing of the first coefficient.

$$AK_+(\sigma_1) + BK_+(\sigma_2) + F_1 K_+(\gamma_1) = 0$$

will supply us with the appropriate order condition on $J(w)$ for $|w| \rightarrow \infty$, $v < 0$. If, for example, the waves with amplitudes F_1 and F_{-1} propagate, we then obtain in a similar manner

$$AK_+(\sigma_1) + BK_+(\sigma_2) + F_1 K_+(\sigma_1) + F_{-1} K_+(\gamma_{-1}) = 0$$

and

$$A\sigma_1 K_+(\sigma_1) + B\sigma_2 K_+(\sigma_2) + F_1 \gamma_1 K_+(\gamma_1) + F_{-1} \gamma_{-1} K_+(\gamma_{-1}) = 0.$$

The general situation is obvious. We shall find presently two further linear relations among the coefficients $A, B, F_1, \dots$ and this will enable us to determine the various reflection and transmission coefficients.

Before we turn to the task of finding these remaining conditions between the

coefficients, let us find more explicit information concerning the propagation of the diffracted waves. The cut-offs for the diffracted waves occur at

$$ck = kc \cos (\theta - \alpha) + 2n\pi, \quad n = 1, 2, \dots$$

or

$$ck = -kc \cos (\theta - \alpha) + 2n\pi, \quad n = 1, 2, \dots$$

Let us write

$$\frac{1}{2}\pi - (\alpha - \theta) = \nu.$$

Then

$$ck(1 - \sin \nu) = 2n\pi, \quad -\frac{1}{2}\pi < \nu < \frac{1}{2}\pi$$

or

$$ck(1 + \sin \nu) = 2n\pi, \quad -\frac{1}{2}\pi < \nu < \frac{1}{2}\pi.$$

A plot of these two families of curves is given in Figure 3.

FIGURE 3.

Under the curves (i) and (a), specular reflection may take place for $\xi < 0$. Above the curve (i) and below the curve (ii), a single diffracted beam whose propagation constant in the ξ direction is γ_1 can propagate so long as we stay to the left of the curve (a). If we pass to the right of (a) but to the left of (b) and between (i) and (ii), two diffracted waves can propagate. The plot in Figure 3 may thus be used to determine the regions in which one or more diffracted waves can propagate.

6. The Determination of the Reflection and Transmission Coefficients. In order to determine the reflection and transmission coefficients, we require two further relations between A , B , F_n , $\dots$. These are obtained by studying the asymptotic form of $\phi(x, y)$ for $x \rightarrow \infty$. The necessary relations may be obtained

by returning to Eq. (4.9). It may now be written for $0 < y < b$ as

$$\begin{aligned}
 \Phi(x, y) = & A \exp \{ik(x \cos \theta + y \sin \theta)\} \\
 & + B \exp \{ik[x \cos (2\alpha - \theta) + y \sin (2\alpha - \theta)]\} \\
 & + \sum F_n \exp \{i(x\gamma_n + y\delta_n)\} \\
 & - \frac{1}{4\pi} \int_P \frac{\exp \{ik\rho - iwa\} \sin(\sigma y) + \sin[\sigma(b - y)]}{\sigma [\cos(k\rho - wa) - \cos(\sigma b)]} I(w) \exp \{iwx\} dw \\
 = & A \exp \{ik(\cos \theta + y \sin \theta)\} + B \exp \{ik[x \cos (2\alpha - \theta) + y \sin (2\alpha - \theta)]\} \\
 & + \sum F_n \exp \{i(x\gamma_n + y\delta_n)\} \\
 & - \frac{1}{2\pi} \int_P \frac{\exp \{ik\rho - iwa\} \sin(\sigma y) + \sin[\sigma(b - y)]}{K_+(w) \sin(\sigma b)} \exp \{iwx\} \\
 & \cdot \left[\frac{AK_+(\sigma_1)}{i(w - \sigma_1)} + \frac{BK_+(\sigma_2)}{i(w - \sigma_2)} + \sum \frac{F_n K_+(\gamma_n)}{i(w - \gamma_n)} \right] dw
 \end{aligned}$$

For $x \rightarrow \infty$, we find that

$$\begin{aligned}
 \phi(x, y) = & \frac{\pi \left(\sin \frac{\pi y}{b} \right) [\exp \{ik\rho - ika\} + 1]}{K_+(\kappa) b^2 \kappa} f(\kappa) \exp \{ikx\} \\
 & - \frac{\pi \left(\sin \frac{\pi y}{b} \right) [\exp \{ik\rho + ika\} + 1]}{K_+(-\kappa) b^2 \kappa} f(-\kappa) \exp \{-ikx\} \\
 & + O \left[\exp \left\{ -x \sqrt{\frac{4\pi^2}{b^2} - k^2} \right\} \right]
 \end{aligned}$$

where

$$f(w) = \frac{AK_+(\sigma_1)}{i(w - \sigma_1)} + \frac{BK_+(\sigma_2)}{i(w - \sigma_2)} + \sum \frac{F_n K_+(\gamma_n)}{i(w - \gamma_n)}.$$

Accordingly, the amplitude of the wave traveling to the right in the duct $x > a$, $0 < y < b$ is

$$\frac{\pi(\exp \{ik\rho - ika\} + 1)f(\kappa)}{b^2 \kappa K_+(\kappa)} = D_0,$$

and the one traveling to the left is

$$\frac{-\pi(\exp \{ik\rho + ika\} + 1)f(-\kappa)}{b^2 \kappa K_+(-\kappa)} = D_1.$$

Two major cases can now arise. First $A \neq 0$ and $D_1 = 0$. In this case, there is a plane wave incident from the region $x < 0$, but there is no duct mode incident from the region $x > a$, $0 < y < b$. The other case is $A = 0$ and $D_1 \neq 0$, in which case the excitation is reversed. We observe that, between the two relations

we just derived and the homogeneous linear conditions we found in Section 5, we have enough equations to find the ratios of B , F_n and D_0 to A or B , F_n and D_0 to D_1 , since the determinants involved in solving either system do not vanish in general. These ratios are not, in general, the reflection or transmission coefficients, since they are not properly normalized. In order to obtain the proper normalization, we have need of an energy conservation relation.

To this end, we consider a function $\phi(x, y)$ which satisfies the differential equation (4.1) and the boundary condition $\phi = 0$ on the line $x > 0, y = 0$. In addition, ϕ satisfies the periodic boundary conditions described in Section 4, as well as the asymptotic conditions

$$(6.1) \quad \begin{aligned} \phi(x, y) = & [D_0 \exp \{ikx\} + D_1 \exp \{-ikx\}] \sin \frac{\pi y}{b} \\ & + O\left(\exp\left\{-\sqrt{\frac{4\pi^2}{b^2} - k^2}x\right\}\right), x \rightarrow \infty \end{aligned}$$

and

$$\begin{aligned} \phi(x, y) = & A \exp \{ik(x \cos \theta + y \sin \theta)\} \\ & + B \exp \{ik[x \cos (2\alpha - \theta) + y \sin (2\alpha - \theta)]\} \\ & + \sum F_n \exp \{i(\gamma_n x + \delta_n y)\} + O(\epsilon x), x \rightarrow -\infty, \epsilon > 0. \end{aligned}$$

We now apply Green's theorem to the function ϕ and its conjugate ϕ^* over the path indicated in Figure 4.

FIGURE 4.

Then

$$\int \left[\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right] ds = 0$$

since ϕ and ϕ^* are source free and ϕ as well as $\partial \phi / \partial n$ are integrable in the neighborhood of $(0, 0)$. In view of the periodicity of ϕ , the integral along the upper and lower sides of the parallelogram cancel each other and we are left with the integral over the two end sides. The integral on the left side reduces to

$$2i[k|A|^2 c \sin(\alpha - \theta) - k|B|^2 c \sin(\alpha - \theta) - \sum w_n c |F_n|^2] + O(e^{-\epsilon L}),$$

where $\epsilon > 0$ and L is sufficiently large and positive to permit the use of the asymptotic form (6.1). The integral on the right side may be replaced by the path given in the figure below.

FIGURE 5.

The left sides of the paths in the last figure annihilate the right side of the paths in Figure 5, and the integrals along the line $y = 0$ vanish since ϕ vanishes there. The integral along the segment from $(L, -\beta)$ to $(L + a, -\beta)$ is $O(e^{-\epsilon_1 L})$ where $\epsilon_1 > 0$ and L is sufficiently large. This leaves us with an integral along the line $x = L + a$ which reduces to

$$-ib\kappa |D_0|^2 + ib\kappa |D_1|^2 + O(e^{-\epsilon_1 L}).$$

From this we conclude that, for $L \rightarrow \infty$, we have

$$kc |A|^2 \sin(\alpha - \theta) - kc |B|^2 \sin(\alpha - \theta) - \sum w_n |F_n|^2 = \frac{b\kappa}{2} [|D_0|^2 - |D_1|^2].$$

If $D_1 = 0$, that is there is only excitation in the region $\xi < 0$, then

$$1 = \left| \frac{B}{A} \right|^2 + \sum \frac{w_n |F_n|^2}{k[\sin(\alpha - \theta)] |A|^2} + \frac{b\kappa}{2kc \sin(\alpha - \theta)} \left| \frac{D_0}{A} \right|^2.$$

From this we see that $|B/A|$ is the reflection coefficient of the specularly reflected wave, while

$$\left[\frac{w_n |F_n|^2}{k[\sin(\alpha - \theta)] |A|^2} \right]^{\frac{1}{2}} \quad n = \pm 1, \pm 2, \dots,$$

are the reflection coefficients for the diffracted waves. Further

$$\left[\frac{b\kappa}{2kc \sin(\alpha - \theta)} \left| \frac{D_0}{A} \right|^2 \right]^{\frac{1}{2}}$$

is the transmission coefficient of the wave travelling down the duct. On the other hand, if $A = 0$, then D_0/D_1 is the reflection coefficient in the duct, while

$$\left[\frac{2kc}{b\kappa} \left| \frac{B}{D_1} \right|^2 \sin(\alpha - \theta) \right]^{\frac{1}{2}} \quad \text{and} \quad \left[\frac{w_n c}{b\kappa} \left| \frac{F_n}{D_1} \right|^2 \right]^{\frac{1}{2}}$$

are the transmission coefficients of the diffracted waves in free space, that is $\xi < 0$.

The solution of the system of linear equations for the unknowns $B, F_1, \dots$ and D_0 in terms of A (when $D_1 = 0$) and $B, F_1, \dots$ and D_0 in terms of D_1 (when $A = 0$) is quite direct. An examination of the two cases which we have enumerated will show us that the determinants are essentially of the van der Monde type and can be evaluated in much the same manner. Accordingly, the details of this evaluation will be omitted.

If $D_1 = 0$,

$$\frac{B}{A} = -\frac{(\kappa + \sigma_2)}{(\kappa + \sigma_1)} \frac{\prod (\gamma_i - \sigma_1)}{\prod (\gamma_i - \sigma_2)} \frac{K_+(\sigma_1)}{K_+(\sigma_2)}, \quad i \neq 0,$$

$$\frac{F_p}{A} = -\frac{(\kappa + \gamma_p)}{(\kappa + \sigma_1)} \frac{\prod (\gamma_i - \sigma_1)}{\prod (\gamma_i - \gamma_p)} \frac{K_+(\sigma_1)}{K_+(\gamma_p)}, \quad i \neq p$$

and

$$\frac{D_0}{A} = \frac{2\pi}{ib^2} \frac{\prod (\gamma_i - \sigma_1) [\exp \{ik\rho - ika\} + 1] K_+(\sigma_1)}{\prod (\kappa - \gamma_i)(\kappa_2 - \sigma_1^2) K_+(\kappa)}$$

where we have written $\gamma_0 = \sigma_2$. These products are taken over all indices for which the γ 's are real, save for the notation beside each term. From these ratios we can form the magnitudes of the reflection and transmission coefficients. Indeed, since B/A is the reflection coefficient of the specularly reflected wave, we have (upon using the identities at the end of Section 5) that

$$\left| \frac{B}{A} \right|^2 = |R_0|^2 = \frac{(\kappa - \sigma_1)(\kappa + \sigma_2)}{(\kappa + \sigma_1)(\kappa - \sigma_2)} \frac{\prod (\gamma_i - \sigma_1)^2}{\prod (\gamma_i - \sigma_2)^2} \frac{\prod [1 + kb \{\sin(2\alpha - \theta)\}/\pi n]}{\prod [1 + kb \{\sin \theta\}/\pi n]},$$

$i \neq 0, n \neq 0.$

If we introduce the notation

$$\tilde{\gamma}_i = \frac{\sin \alpha}{c} [c^2 k^2 - (2i\pi + k\rho)^2]^{\frac{1}{2}} + \frac{\cos \alpha}{c} (2i\pi + k\rho),$$

the expression for $|R_0|^2$ reduces to

$$\frac{\kappa - \sigma_1}{\kappa + \sigma_1} \frac{\kappa + \sigma_2}{\kappa - \sigma_2} \prod \frac{(\gamma_i - \sigma_1)(\tilde{\gamma}_i - \sigma_1)}{(\gamma_i - \sigma_2)(\tilde{\gamma}_i - \sigma_1)}, \quad i \neq 0.$$

It appears that the denominator of this expression has zeros, but an examination of the fraction as a whole will reveal that $\gamma_i - \sigma_1$ vanishes when $1 + kb \sin \theta / i\pi = 0$. That is, since kb is positive, θ and i have opposite signs. Further, $\kappa + \sigma_1$ vanishes at $kb \sin \theta = \pm \pi$, if $\alpha - \pi < \theta < -\frac{1}{2}\pi$. A little algebra will reveal that the ratio $\prod (\gamma_i - \sigma_1) / (\kappa + \sigma_1)$ is always finite, and similar remarks apply to the other terms.

The higher reflection coefficients are formed from the ratio F_p/A . Indeed, upon normalizing this ratio we obtain

$$\begin{aligned} |R_p|^2 &= |F_p/A|^2 w_p/k \sin(\alpha - \theta) \\ &= \frac{4(\kappa - \sigma_1)(\kappa + \gamma_p)}{(\kappa + \sigma_1)(\kappa - \gamma_p)} \frac{k[\sin(\alpha - \theta)]w_p \sin^2 \alpha}{(\gamma_p - \sigma_2)(\tilde{\gamma}_p - \sigma_1)} \prod \frac{(\gamma_i - \sigma_1)(\tilde{\gamma}_i - \gamma_p)}{(\gamma_i - \sigma_1)(\gamma_i - \gamma_p)}, \\ &\quad i \neq 0 \text{ or } p. \end{aligned}$$

Finally, the transmission coefficient T is formed from the ratio D_0/A upon proper normalization. In this case we get

$$|T_0|^2 = \frac{4k\kappa(\sin \alpha) \sin(\alpha - \theta)}{(\kappa + \sigma_1)(\kappa - \sigma_2)} \prod \frac{(\gamma_i - \sigma_1)(\kappa - \tilde{\gamma}_i)}{(\tilde{\gamma}_i - \sigma_1)(\kappa - \gamma_i)}, i \neq 0.$$

The phase angles are computed directly from $\arg K_+(w)$ for w real. We recall once again that the products in these coefficients are taken over all the real γ 's save for any special indications we have made.

If we excite the ducts from within and the field satisfies the same phase relation as it did with excitation from free space, we find the following in the limited wave length region we have considered. First we note that now $A = 0$ and $D_1 \neq 0$. In each duct there will be an incident and reflected wave guide mode, while in free space there will be a series of transmitted plane waves. The plane waves have the same propagation normals which the reflected waves had in the case $A \neq 0$. We shall call the transmission coefficients of these waves T'_n , while R'_0 will be the reflection coefficient in the parallel plate region. In this case we find that

$$|R'_0|^2 = \frac{(\kappa + \sigma_2)(\kappa - \sigma_1)}{(\kappa - \sigma_2)(\kappa + \sigma_1)} \prod' \frac{(\gamma_i + \kappa)(\tilde{\gamma}_i - \kappa)}{(\tilde{\gamma}_i + \kappa)(\gamma_i - \kappa)}, i \neq 0$$

while

$$|T'_0|^2 = \frac{4k\kappa(\sin \alpha) \sin(\alpha - \theta)}{(\kappa - \sigma_2)(\kappa + \sigma_1)} \prod' \frac{(\gamma_i + \kappa)(\tilde{\gamma}_i - \sigma_2)}{(\tilde{\gamma}_i + \kappa)(\gamma_i - \sigma_2)}, i \neq 0$$

and

$$|T'_n|^2 = \frac{bc\kappa w_n}{2n^2} \frac{(\gamma_n - \sigma_1)(\gamma_n - \sigma_2)}{(\kappa - \gamma_n)(\kappa + \gamma_n)} \frac{(\kappa + \sigma_2)}{(\kappa + \sigma_1)} \prod_i \frac{\kappa + \gamma_i}{\kappa + \tilde{\gamma}_i} \prod_i' \frac{\tilde{\gamma}_i - \gamma_n}{\gamma_i - \gamma_n}, i \neq n.$$

Actually, the reflection and transmission coefficients for the two cases $A \neq 0$ and $D_1 \neq 0$ are not independent. There is a reciprocity relation between some of these coefficients which forms, among other things, a basis for a check. In order to derive these relations, we consider excitation in the region $\xi < 0$ by a plane wave whose propagation normal makes an angle $\pi + 2\alpha - \theta$ with respect to the positive x -axis. This excitation serves to replace ρ by $-\rho$, σ_1 by $-\sigma_2$ and σ_2 by $-\sigma_1$ throughout the solution we have just found. Hence such a solution has the form

$$\begin{aligned} \phi_0 = & A \exp \{i\xi k \sin (\alpha - \theta) - i\eta k \cos (\alpha - \theta)\} + B \exp \{-i\xi k \sin (\alpha - \theta) \\ & - i\eta k \cos (\alpha - \theta)\} \\ & + \sum D_n \exp \left\{ -i \frac{\xi}{c} \sqrt{c^2 k^2 - (2n\pi - k\rho)^2} + i\eta \frac{(2n\pi - k\rho)}{c} \right\} + O(e^{\epsilon\xi}), \\ & \xi \rightarrow -\infty \end{aligned}$$

and

$$\phi_0 = T e^{i\kappa x} \sin \frac{\pi y}{b} + O(e^{-\epsilon\xi}), \quad x \rightarrow \infty, \quad \epsilon > 0$$

where the T will depend on the duct number. For excitation within the ducts we have found that

$$\phi_1 = [D_0 e^{i\kappa x} + D_1 e^{-i\kappa x}] \sin \frac{\pi y}{b} + O(e^{-\epsilon\xi}), \quad x \rightarrow \infty$$

and

$$\begin{aligned} \phi_1 = & F_0 \exp \left\{ -\frac{ik}{c} \left[\xi \sin (\alpha - \theta) - \eta \cos (\alpha - \theta) \right] \right\} \\ & + \sum F_n \exp \left\{ -i \frac{\xi}{c} \sqrt{c^2 k^2 - (2n\pi + k\rho)^2} + i\eta \frac{(2n\pi + k\rho)}{c} \right\} \\ & + O(e^{\epsilon\xi}), \quad \xi \rightarrow -\infty. \end{aligned}$$

Upon forming the integral

$$\int \left[\phi_1 \frac{\partial \phi_0}{\partial n} - \phi_0 \frac{\partial \phi_1}{\partial n} \right] ds = 0$$

over the same path which we employed to find the relation between the squares of the magnitudes of the various coefficients, we find in the limit as $\xi \rightarrow -\infty$, $x \rightarrow \infty$ that

$$b_k T D_1 = 2A c k F_0 \sin (\alpha - \theta).$$

Upon forming magnitudes we get

$$\frac{b_k}{2kc \sin (\alpha - \theta)} \left| \frac{T}{A} \right|^2 = \frac{2ck}{a\kappa} [\sin (\alpha - \theta)] \left| \frac{F_0}{D_1} \right|^2.$$

That is,

$$|T'_0|^2 = |T_0|^2,$$

or to be more specific

$$|T'_0(-\rho)|^2 = |T_0(\rho)|^2$$

where T'_0 is the transmission coefficient of the mode propagating in the duct

and T_0 is the transmission coefficient of the plane wave in $\xi < 0$. It is easy to verify that $|T'_0|^2 = |T_0|^2$.

For the diffracted waves (higher modes) we are required to assume an excitation of the form $\exp \{i(\xi/c) \sqrt{c^2 k^2 - (2p\pi - k\rho)^2} + i(\eta/c) (2\rho\pi - kp)\}$ which will produce a set of reflected plane waves in the region $\xi < 0$ and transmit a parallel-plate mode in the duct of the form $T_p \exp \{ixk\} \sin \pi y/b$. Once again we form the line integral as we did in the previous paragraph. Unfortunately, this calculation will relate the transmission coefficient in the duct (due to a higher mode excitation in free space) to the corresponding plane wave in free space. Since we have not studied this particular excitation in free space, we shall not pursue this matter further.

BIBLIOGRAPHY

- [1] J. F. CARLSON & A. E. HEINS, The Reflection of an Electromagnetic Plane Wave by an Infinite Set of Plates, I, *Quarterly of Applied Mathematics* 4 (1947) pp. 313-329.
- [2] A. E. HEINS & S. SILVER, The Edge Conditions and Field Representation Theorems in the Theory of Electromagnetic Diffraction, *Proceedings of the Cambridge Philosophical Society* 51 (1955) pp. 149-161.
- [3] E. C. TITCHMARSH, *Introduction to the Theory Fourier Integrals*, Oxford, 1948, Chapter XI.

Carnegie Institute of Technology
Pittsburgh, Pennsylvania